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# Polygons with Prescribed Gauss Map in Hadamard Spaces and Euclidean Buildings

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*Abstract.* We show that given a stable weighted configuration on the asymptotic boundary of a locally compact Hadamard space, there is a polygon with Gauss map prescribed by the given weighted configuration. Moreover, the same result holds for semistable configurations on arbitrary Euclidean buildings.

In the first section, we recall some background material on Hadamard spaces and Euclidean buildings, and we introduce the concepts needed to state and prove our theorems. In particular, we define stability for weighted configurations on the boundary at infinity of a Hadamard space. In the second section, we introduce ultralimits and the special cases ultraproducts and asymptotic tubes which we use in our proofs. In the third section, we prove our results.

***Main Theorem*** *Let $X$ be a Euclidean building and $c$ a semistable weighted configuration on its boundary at infinity, or let $X$ be a locally compact Hadamard space and $c$ a stable weighted configuration on its boundary at infinity. Then the associated weak contraction $\Phi_c$ has a fixed point. In particular, there exists a polygon $p$ in $X$ such that $c$ is a Gauss map for $p$.*

For a slightly more general statement in the case of a Hadamard space, see Corollary 3.9.

As an immediate consequence, we can formulate the following classification of configurations which can occur as Gauss maps on Euclidean buildings and symmetric spaces:

***Corollary*** *Let $X$ be a symmetric space of non-compact type or a Euclidean building, and let $c$ be a weighted configuration on its boundary at infinity. Then there exists a polygon having this configuration as a Gauss map if and only if the configuration is semistable in the building case and nice semistable in the case of a symmetric space.*

Necessity of semistability, as well as the Theorem and the Corollary in the case where $X$ is a symmetric space or a locally compact Euclidean building were shown in [KLM1, KLM2]. We extend their ideas by suitable use of ultralimits. Along the way, we discuss how rays project to subspaces, and obtain the following result of independent interest:

***Proposition*** *Let $X'$ be a Hadamard space and $X \subset X'$ a closed convex subset. Let $o \in X$, $\rho := \overline{o\eta}$ a ray in $X$ (with $\eta \in \partial_\infty X'$), and $\pi\colon X' \to X$ be the nearest point projection.*

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*If $\angle(\eta, \partial_\infty X) < \frac{\pi}{2}$, then the segments $\overline{o\ (\pi \circ \rho(t))}$ converge to the ray $\overline{o\xi}$ (in the cone topology), where $\xi \in \partial_\infty X$ is the unique point with $\angle(\eta, \xi) = \angle(\eta, \partial_\infty X)$.*

*If the projection of the ray $\overline{o\eta}$ to $X$ is bounded, then there exists a point $p \in X$ such that $\pi \circ \overline{p\eta}(t) = p$ for all $t > 0$.*

In the last section, we discuss relations of the above results to algebra.

# 1 Introduction

## 1.1 Hadamard spaces

We will use the language of non-positively curved metric spaces, as developed in [Bal95].

Throughout, let $X$ be a Hadamard space, unless otherwise stated.

Recall that $X$ has a *boundary at infinity* $\partial_\infty X$, which is given by equivalence classes of rays, where two (unit-speed) rays are equivalent if their distance is bounded.

In particular, we will use Busemann functions $b_\eta$ associated to an asymptotic boundary point $\eta \in \partial_\infty X$. A Busemann function measures (relative) distance from a point at infinity, and is determined up to an additive constant only. Busemann functions are convex (along any geodesic) and 1-Lipschitz.

Geodesics, rays, and geodesic segments are always assumed to be parametrized by unit speed (*i.e.,* they are isometric embeddings).

For a line $l$ in $X$, there is the space $P_l$ of parallel lines. $P_l$ splits as a product $P_l \cong l \times CS(l)$, where $CS(l)$ is a Hadamard space again.

For points $x, \xi$ with $x \in X$, $\xi \in X \cup \partial_\infty X$, and $t \geq 0$ (if $\xi \in X$, let $t \leq d(x, \xi)$), we let $\overline{x\xi}(t)$ denote the point on the segment/ray $\overline{x\xi}$ at distance $t$ from $x$. When we denote a ray by $\overline{o\eta}$, we order the points such that $o \in X$ and $\eta \in \partial_\infty X$.

***Definition 1.1*** For $\xi \in \partial_\infty X$ and $t \geq 0$, we define the map $\phi_{\xi,t} \colon X \to X$ defined by $\phi_{\xi,t}(x) := \overline{x\xi}(t)$. Observe that $\phi_{\xi,t}$ is a 1-Lipschitz map by convexity of a non-positively curved metric.

Let $o \in X$ be a point in a Hadamard space, and let $\eta, \xi \in \partial_\infty X$. Let $c, c'$ be the rays $\overline{o\eta}, \overline{o\xi}$. For points $c(t), c'(t')$, one can consider the *Euclidean comparison triangle* corresponding to the points $o, c(t), c'(t')$, *i.e.,* the Euclidean triangle with side-lengths $d(o, c(t)), d(c(t), c'(t')), d(c'(t'), o)$ (which is well defined up to isometries of the Euclidean plane). The *comparison angle* between $c(t)$ and $c'(t')$ at $o$ is the angle of the comparison triangle at the point corresponding to $o$. It is denoted by $\widetilde{\angle}_o(c(t), c'(t'))$.

We have the following monotonicity property:

$$0 < t \leq s \text{ and } 0 < t' \leq s' \text{ implies } \widetilde{\angle}_o(c(t), c'(t')) \leq \widetilde{\angle}_o(c(s), c'(s')).$$

From this, one can deduce a notion of angle between geodesic segments and rays:

$$\angle_o(\eta, \xi) := \lim_{t,t' \to 0} \widetilde{\angle}_o(c(t), c'(t')) \in [0, \pi],$$

and an "angle at infinity", the *Tits angle* between boundary points

$$\angle(\eta, \xi) := \angle_{\mathrm{Tits}}(\eta, \xi) := \lim_{t,t' \to \infty} \widetilde{\angle}_o(c(t), c'(t')) \in [0, \pi].$$

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It is easy to see that the Tits angle between $\eta, \xi$ does not depend on the chosen basepoint $o$. The length metric induced on $\partial_\infty X$ by $\angle$ is called *Tits distance* Td, and makes $\partial_\infty X$ a CAT(1) space. If the Tits angle (between $\eta, \xi$) is less than $\pi$, there is a unique geodesic $\overline{\eta\xi} \subset \partial_\infty X$ connecting them.

Similarly, the space of directions $S_o$, *i.e.,* the completion of the space of starting directions of geodesic segments initiating in $o$ (modulo the equivalence of directions enclosing a zero angle), can be regarded as a CAT(1) space.

We state Lemma [BH99, II.8.3], since it will be of fundamental importance in the proof of Lemma 2.4. It says that given one geodesic ray $\overline{o\eta}$ and another point $y \in X$, the ray $\overline{y\eta}$ can be approximated by segments $\overline{y\rho(t)}$ for $t$ large enough, and the approximation can be controlled independently from the Hadamard space $X$.

***Lemma 1.2*** *Given $\varepsilon > 0, m > 0$ and $c > 0$, there is a constant $K = K(\varepsilon, m, c) > 0$ such that: for every ray $\rho = \overline{o\eta}$ in a Hadamard-space X, if $y \in X$ satisfies $d(y, o) \leq m$, then we have*

$$d\big(\overline{y\eta}(c), \overline{y\rho(K)}(c)\big) < \varepsilon.$$

## 1.2 Euclidean Buildings

We will also need some Euclidean building geometry. For an introduction, we refer to [KL97, §4]. A brief introduction of the notation we use can be found in [KLM2, §2.4]. Note that in particular, a Euclidean building is a Hadamard space.

The boundary at infinity of a Euclidean building $X$ of rank $n$ is a spherical building of dimension $n - 1$; we refer to [KL97, §3] for an introduction.

We will use that a spherical building is a spherical simplicial complex, where all the simplices are isometric to a spherical polytope $\Delta$ (in particular, $\Delta$ tesselates $S^{n-1}$), which is the *spherical Weyl chamber* of the building. Apartments (*i.e.,* isometrically embedded copies $S^{n-1}$) intersect in (unions of) Weyl chambers.

We prove some elementary lemmas which we will use later:

***Lemma 1.3*** *Let X be a Euclidean building, l a line in X with $l(\infty) = \eta \in \partial_\infty X$, and c a ray asymptotic to $\eta$. Then c eventually coincides with a line parallel to l.*

**Proof** Pick an apartment $A' \supset c$, and an apartment $A$ containing $\eta^- := l(-\infty)$ in its boundary, which has the property that $\partial A = \partial A'$ near $\eta$ (*i.e.,* let $S \subset \partial_\infty A'$ be the subset of $\partial_\infty A'$ consisting of the union of Weyl chambers containing $\eta$, and let $A$ be an apartment containing $S$ and $\eta^-$ in its boundary).

We want to show that $c(t) \in A$ for large $t$, which finishes the proof.

We may assume that $\eta$ is singular, since otherwise $c(t) \in A$ for large $t$ by [KL97, Lemma 4.6.3].

Pick regular points $\xi_i \in S$ ($i \in \{1, 2\}$) such that $\eta$ is the midpoint of $\overline{\xi_1\xi_2}$ (and $\angle_{\text{Tits}}(\xi_1, \xi_2) < \pi$).

Let $c_i$ be the ray $\overline{c(0)\xi_i} \subset A'$. For some $t_0$, both $c_i(t_0) \in A \cap A'$ (again by [KL97, Lemma 4.6.3]). Then the midpoint of $\overline{c_1(t_0)c_2(t_0)}$ is also in $A \cap A'$; this midpoint is $c(T)$ for some $T$ (since $c_1, c_2$ span a flat sector in $A'$), implying that $c(t)$ is in $A \cap A'$ for $t \geq T$. ∎

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Observe that this shows in particular that the space of strong asymptote classes of rays asymptotic to $\eta$ is isometric to $CS(l)$ (see [Kar67], [Lee97, §2.1.3]).

***Lemma 1.4*** *Consider a ray $\rho = \overline{o\xi}$ and a segment $\overline{op'}$ in a Euclidean building $X$. Then there is an apartment containing $\rho$ and an initial part of $\overline{op'}$.*

**Proof** The claim is clear if $\rho$ and $\overline{op'}$ initially coincide or their initial directions are antipodal. So we assume not. By [KL97, Lemma 4.1.2], there is a point $p \in \overline{op'}$, such that the triangle $D := \Delta(o, p, \xi)$ is flat (*i.e.*, a flat half-strip). If $X$ is discrete, the claim follows from [BL04, Proposition 1.3, Remark 1.4]. We give a direct argument for our special situation here.

We show that $D$ is contained in a half-plane: let $H$ be a flat half-strip containing $D$ with $\partial H \supset \overline{op}$; assume that $H$ cannot be enlarged under these conditions, and is not a half-plane. Since $X$ is complete, we see that $H$ is closed, *i.e.*, of the form $\Delta(p_1, p_2, \xi)$. Now $S_{p_1}H$ is a geodesic segment, which can be prolonged to a geodesic of length $\pi$ in the spherical building $S_{p_1}X$. By [KL97, Lemma 4.1.2], this yields a direction in which we can glue another flat half-strip to $H$, so $H$ was not maximal.

Thus, $D$ is contained in a half-plane, and this half-plane is contained in a plane by [Lee97, Lemma 5.2]. Finally, every plane in $X$ is contained in an apartment by [Lee97, Corollary 5.4]. ■

## 1.3 Weighted Configurations at Infinity

In this subsection, we recall some notions from [KLM1, KLM2] needed to discuss the relationship of configurations on $\partial_\infty X$ and polygons in $X$.

***Definition 1.5*** Let $X$ be a Hadamard space. A *weighted configuration $c$ on $\partial_\infty X$* is an $n$-tuple of points $(\xi_1, \dots, \xi_n)$ in $\partial_\infty X$ together with a weight function

$$m \colon \{1, \dots, n\} \to \mathbb{R}_{>0}.$$

There is a *weighted Busemann function* associated to a weighted configuration $c$. It is given by

$$b_c := \sum_{i=1}^{n} m_i b_{\xi_i}.$$

Weighted Busemann functions are convex, asymptotically linear, Lipschitz-continuous, and well defined up to an additive constant. As for any convex, asymptotically linear Lipschitz-function on a Hadamard space, we can associate a function $\mathrm{slope}_{b_c} \colon \partial_\infty X \to \mathbb{R}$ to a weighted Busemann function, which is given by assigning the asymptotic slope of $b_c$ on a ray $\overline{o\xi}$ to the point $\xi$. Since two rays asymptotic to the same boundary point have bounded distance and $b_c$ is Lipschitz, the slope does not depend on the choice of $o$, so $\mathrm{slope}_{b_c}$ is well defined (see also [KLM1, §3]).

We have

$$\mathrm{slope}_c(\xi) := \mathrm{slope}_{b_c}(\xi) = -\sum_{i=1}^{n} m_i \cos \angle(\xi_i, \xi).$$

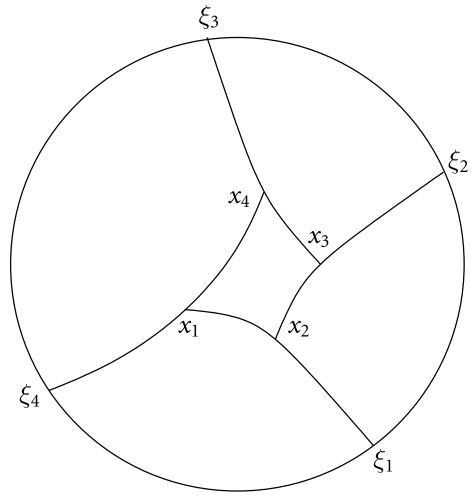


*Figure 1*: Gauss maps

The configuration $c$ is called *semistable* if $\text{slope}_c \geq 0$, and it is called *stable* if $\text{slope}_c > 0$.

Observe that (semi-)stability is defined purely in terms of the Tits-geometry of $\partial_\infty X$, without reference to $X$ itself.

Now we discuss the relation between polygons and weighted configurations: consider a polygon $p$ in $X$, which is determined by an $n$-tuple of points $(x_1, \dots, x_n)$ (with $x_i \neq x_{i+1}$ for all $i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ [1]). We can associate a set of weighted configurations $\mathcal{G}(p)$ on $\partial_\infty X$ to $p$, by choosing $\xi_i$ such that $x_{i+1} \in \overline{x_i\xi_i}$, and setting $m_i := d(x_i, x_{i+1})$. Then all $c \in \mathcal{G}(p)$ are semistable by [KLM2, Lemma 4.3] (their proof generalizes without problems). Observe that (if $X$ is not geodesically complete) it may happen that $\mathcal{G}(p) = \varnothing$.

An element $c \in \mathcal{G}(p)$ is called a *Gauss map* for $p$ (since this construction, in the case of the hyperbolic plane, was mentioned in a letter from Gauss to Bolyai, [Gau63]; see Figure 1).

On the other hand, consider a weighted configuration $c$. Let

$$\Phi_c := \phi_{\xi_n, m_n} \circ \cdots \circ \phi_{\xi_1, m_1}.$$

Since a composition of 1-Lipschitz maps is 1-Lipschitz, $\Phi_c$ is 1-Lipschitz, *i.e.*, a weak contraction. Every fixed point of $\Phi_c$ is a first vertex of a polygon $p$ with $c \in \mathcal{G}(p)$.

A more general discussion of measures on $\partial_\infty X$ (if $X$ is a symmetric space or Euclidean building) can be found in [KLM1, KLM2].

[1] For notational convenience, we consider the indices modulo $n$.

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## 2 Ultralimits, Ultraproducts, and Asymptotic Tubes

### 2.1 Ultralimits

This section introduces the notion of ultralimit, and the special cases ultraproduct and asymptotic tube, which play an important role in our proof. We keep the general discussion of ultralimits brief and refer the interested reader to [BH99, pp. 77–80] and [KL97, §2.4] for more details.

***Definition 2.1*** Let $\omega$ be a (fixed) non-principal ultrafilter[2], and let $(X_i, d_i, o_i)_i$ be a sequence of metric spaces with metrics $d_i$ and basepoints $o_i$. Then

$$X_\omega := \lim_\omega (X_i, d_i, o_i)$$

is the *ultralimit* of this sequence, a space consisting of equivalence classes of sequences $(x_i)$ with $x_i \in X_i$ and $d(x_i, o_i)$ bounded. The distance between two such sequences $(x_{i,n})_n$ (for $i \in \{1, 2\}$) is $\lim_\omega d(x_{1,n}, x_{2,n})$, the accumulation point of $(d(x_{1,n}, x_{2,n}))_n$ picked by $\omega$. The equivalence classes consist of sequences having distance zero.

If all $X_i$ are CAT(0), then their ultralimit is a Hadamard space; if all $X_i$ are (additionally) geodesically complete, then every geodesic segment, ray and line in $X_\omega$ arises as ultralimit of geodesic segments, rays, and lines, respectively [KL97, 2.4.2, 2.4.4].

If all $X_i$ are Euclidean buildings with isometric spherical Weyl chamber, then their ultralimit is also a Euclidean building with the same spherical Weyl chamber [KL97, §5.1].

Let us assume for the rest of this section that $(X_i, d_i)_i = (X, d)_i$ is a constant sequence, and $X$ is a Hadamard space, so only the basepoint varies in the construction of the ultralimit $X_\omega$.

Then there is a natural map $*\colon \partial_\infty X \to \partial_\infty X_\omega$, obtained by assigning to $\xi \in \partial_\infty X$ the equivalence class of rays in $X_\omega$, which has finite distance from the ray defined by the sequence of rays $\overline{o_i\xi}$. We denote the image of $\xi$ by $\xi_*$.

Now we can push a weighted configuration $c$ on $\partial_\infty X$ forward to a weighted configuration $c_*$ on $\partial_\infty X_\omega$ by mapping the $\xi_i$ to $\xi_{i,*}$ and keeping the weights.

***Lemma 2.2*** *Under the assumptions above, let $\Phi_*$ denote the weak contraction associated to the pushed forward configuration. Then $\Phi_*$ has the form*

$$\Phi_*\left((x_i)_i\right) = (\Phi(x_i))_i.$$

**Proof** It suffices to show that for any $\xi \in \partial_\infty X$ and a real number $m > 0$, pushing towards $\xi_*$ by $\phi_{\xi_*,m}$ has the form given above. So let $x = (x_i)_i \in X_\omega$. Recall that by definition, the distances $d(x_i, o_i)$ are bounded. Hence, the ray $\overline{x\xi_*}$ can be represented by the ultralimit of the rays $\overline{x_i\xi}$, which implies the claim. ■

[2] In our context, a non-principal ultrafilter is a means of (consistently) choosing an accumulation point for any bounded sequence of real numbers.

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## 2.2 Ultraproducts

***Definition 2.3*** For a metric space $X$ let the *ultraproduct* of $X$ be the ultralimit of the constant sequence $(X_i, d_i, o_i) := (X, d, o)$; *i.e.*, $X^\omega := \lim_\omega(X, d, o)$ (where we have chosen a basepoint $o$ for $X$, which has no influence on the isometry type of $X^\omega$).

There is a canonic isometric embedding $X \to X^\omega$ sending $x$ to $(x, x, \dots)$. Observe that if $X$ is proper (*e.g.*, a locally compact CAT(0)-space), the ultraproduct $X^\omega$ is isometric to $X$. For details on ultraproducts, see [Lyt04, §11].

## 2.3 Asymptotic Tubes

One of the main ideas in the proof of our main theorem is that the weak contraction $\Phi_c$ associated to a weighted configuration asymptotically moves a ray to a parallel ray.

We make this idea precise by using particular ultralimits. Throughout this section, $X$ will be a Hadamard space and $\rho = \overline{o\eta}$ will be a ray in $X$.

Let $\xi \in \partial_\infty X$. The following lemma says that pushing towards $\eta$ and $\xi$ asymptotically commutes when moving out along $\rho$.

***Lemma 2.4*** *Let $m, c > 0$ and $\xi \in \partial_\infty X$. Then*

$$\lim_{t\to\infty} d\big(\phi_{\xi,m} \circ \phi_{\eta,c} \circ \rho(t), \phi_{\eta,c} \circ \phi_{\xi,m} \circ \rho(t)\big) = 0.$$

**Proof** Let $o_t := \rho(t)$, $x_t := \phi_{\eta,c}(o_t) = \rho(t + c)$, $y_t := \phi_{\xi,m}(o_t)$, and $\hat\alpha := \angle(\eta, \xi)$. We may assume $\hat\alpha \neq 0$, since otherwise $\eta = \xi$, and there is nothing to show.

If we set $z_t := \phi_{\eta,c}(y_t)$, then the claim is $d(z_t, y_{t+c}) \xrightarrow[t\to\infty]{} 0$.

Let $\varepsilon > 0$ be given.

(i) Let $K = K(\varepsilon, m, c)$ be the constant from Lemma 1.2. We may assume $K \geq c$. Let $z'_t := \overline{y_t\rho(t + K)}(c)$. We have $d(z'_t, z_t) \leq \varepsilon$, so we try to get information about $d(z'_t, y_{t+c})$.

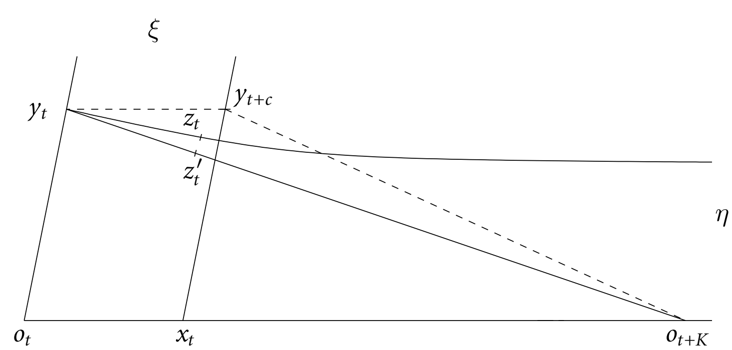


*Figure 2*: The points from the proof of Lemma 2.4

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(ii) Let $\bar{\alpha} < \hat{\alpha}$ be such that for a Euclidean triangle $ABC$ with sides $AC, AB$ of length $K - c, m$ respectively, the length of the third side varies by at most $\varepsilon$ when the angle at $A$ varies in the interval $[\bar{\alpha}, \hat{\alpha}]$.

Let $l$ be the maximal length of the third side (occurring when the angle is equal to $\hat{\alpha}$).

(iii) Observe that in a Euclidean triangle $ABC$ with sides $AC, AB$ of length $K, m$ respectively, and angle at $A$ in the interval $[\bar{\alpha}, \hat{\alpha}]$, the third side has length at least $c + (l - 2\varepsilon)$.

Since the constant $K$ from Lemma 1.2 is independent from the Hadamard space (so we may choose $X = \mathbb{R}^2$ here), the claim follows from (ii).

(iv) Finally, let $T > 0$ be such that for $t > T$, we have $\bar{\alpha} \leq \alpha_t := \angle_{o_t}(\eta, \xi) \leq \hat{\alpha}$ (observe that the second inequality is trivial).

Now we consider the triangle $\Delta(y_t y_{t+c} \rho(t + K))$ for $t > T$. Since the angle corresponding to $\alpha$ in the comparison triangle is in the interval $[\bar{\alpha}, \hat{\alpha}]$, (ii) implies $d(y_{t+c}, \rho(t + K)) \leq l$; and since $\phi_{\xi,m}$ is 1-Lipschitz, we have $d(y_t, y_{t+c}) \leq c$. On the other hand, we have $d(y_t, \rho(t + K)) \geq c + l - 2\varepsilon$ by (iii).

Considering the Euclidean comparison triangle, this shows that we have control over $d(z_t', y_{t+c})$, and this quantity becomes arbitrarily small as $\varepsilon$ goes to zero. With (i), this finishes the proof. ∎

***Definition 2.5*** In the situation described above, define the CAT(0)-space

$$X_\omega := \lim\nolimits_\omega(X, d, \rho(i)).$$

Observe that in $X_\omega$, the image of $\rho$ is a line $l$. Let $T_\eta = T_\rho := P_l$, and call this space the *asymptotic tube* of $\eta$ (it is easy to see that $T_\rho \cong T_{\rho'}$ if $\rho$ and $\rho'$ are rays asymptotic to $\eta$).

Consider the map $*\colon \partial_\infty X \to \partial_\infty X_\omega$ introduced at the end of Section 2.1.

***Lemma 2.6*** *We have* $*\colon \partial_\infty X \to \partial_\infty T_\eta$*, and for any* $\xi \in \partial_\infty X$*, we have* $\angle(\xi, \eta) = \angle(\xi_*, \eta_*)$.

**Proof** Let $\xi \in \partial_\infty X$ and $m > 0$. We claim that the map

$$l_{\xi,m}\colon t \mapsto (\phi_{\xi,m} \circ \rho(i + t))_i$$

defines a line parallel to $l$ in $X_\omega$ (for given $t$, we set the coordinates with $i + t < 0$ arbitrarily; since these are finitely many, they have no influence on the point defined in $X_\omega$). Indeed, by the Lemma above, the following equality holds in $X_\omega$ (for $t' > t$):

$$l_{\xi,m}(t') = (\phi_{\xi,m} \circ \phi_{\eta,t'-t} \circ \rho(i + t))_i = \Big(\phi_{\eta,t'-t} \circ \underbrace{\phi_{\xi,m} \circ \rho(i + t)}_{\text{defining } l_{\xi,m}(t)}\Big)_i.$$

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The right-hand side shows $d(l_{\xi,m}(t'), l_{\xi,m}(t)) = t' - t$ for $t' \geq t$; hence, $l_{\xi,m}$ is a geodesic line. Clearly, $l_{\xi,m}$ stays within bounded distance of $l$, so it is parallel to $l$ (by [BH99, II.2.13]).

For given $t$, we have

$$\angle_{\rho(i+t)}(\xi,\eta) \xrightarrow[i\to\infty]{} \angle(\xi,\eta) \quad \text{and} \quad \angle_{\rho(i+t)}(\rho(0),\xi) \to \pi - \angle(\xi,\eta)$$

(by [Bal95, Prop. 4.2]), so we find $d(l_{\xi,m}, l) = m \sin\angle(\xi,\eta)$.

It is clear that the flat strip spanned by $l_{\xi,m'}$ and $l$ contains $l_{\xi,m}$ for $m' > m > 0$, so $\xi$ determines a half-plane in $P_l$ if $\angle(\eta,\xi) \neq 0, \pi$. In the other cases, $l = l_{\xi,m}$. ■

The following observation is immediate from the previous lemma:

***Lemma 2.7*** *Let $c$ be a weighted configuration on $\partial_\infty X$, and consider the map $*\colon \partial_\infty X \to \partial_\infty T_\rho$. Then* $\operatorname{slope}_c(\eta) = \operatorname{slope}_{c_*}(\eta_*)$.

***Remark 2.8*** One can show that $*$ also has the following properties: the half-planes determined by $\xi, \xi'$ agree if the geodesic segments $\overline{\eta\xi}, \overline{\eta\xi'}$ start in the same direction. The induced map between the spaces of directions $S_\eta(\partial_\infty X) \to S_{\eta_*}(\partial_\infty T)$ is 1-Lipschitz, but not an isometric embedding in general.

We show below that in a Euclidean building, one even gets a map (with the properties we need) $*\colon \partial_\infty X \to \partial_\infty P_l$ for a line $l$ containing $\rho$. The same result holds for symmetric spaces of noncompact type.

The question arises whether in a general Hadamard space, one can get a suitable map to the boundary of $\mathbb{R} \times X_\eta$, the space of parametrized strong asymptote classes at $\eta$ (see [Kar67], [Lee97, §2.1.3], [KLM1, §3.1.2]).

However, consider the following subset of the Euclidean plane:

$$X = \{(x,y) \mid x \geq 1, y \geq \log x\}.$$

With the induced length metric, $X$ becomes a Hadamard space; the boundary at infinity is an arc of length $\frac{\pi}{2}$. Consider the boundary point $\eta$ corresponding to the ray $\rho$ in $X$ which is given by parametrizing the graph of the logarithm with unit speed. Then $X_\eta$ consists of one point only (every ray asymptotic to $\eta$ eventually lies on the graph of the logarithm), but $T_\eta$ is a half-plane.

## 2.4 Asymptotic Tubes in Euclidean buildings

In the case where $X$ is a Euclidean building or a symmetric space, the construction described above specializes to the folding map described in [KLM1, §3.2.5]. We discuss the building case:

***Lemma 2.9*** *Let $X$ be a Euclidean building, $\rho = \overline{o\eta}$ a ray in $X$, and $l$ a line extending $\rho$. Let $T$ be the asymptotic tube associated to $\rho$. Then there is a natural isometric embedding $\iota\colon P_l \to T$, and we have* $\operatorname{Im}(*) \subset \partial_\infty(\iota(P_l))$.

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**Proof** We state an explicit formula for $\iota$. We map $p \in P_l$ to $(\phi_{\eta,i}(p))_i$.

Since $\phi_{\eta,t}|_{P_l}$ is an isometry of $P_l$ for every $t \geq 0$, the first claim holds.

Let $\xi \in \partial_\infty X$ be a boundary point of $X$. For $t$ large enough, the rays $\overline{\rho(t)\eta}$ and $\overline{\rho(t)\xi}$ bound a Euclidean sector (by discreteness of the angle, see [KL97, Axiom 4.1.2.EB2]). This shows that $\phi_{\xi,m}$ eventually maps the ray $\rho$ to a parallel ray. Since this ray eventually coincides with a line parallel to $l$ by Lemma 1.3, the claim follows. ■

An immediate consequence is:

***Lemma 2.10*** *Let $c$ be a weighted configuration on the boundary of the Euclidean building $X$. Let $l$ be a line with $\lim_{t\to\pm\infty} l(t) = \xi_\pm$. Let $c_*$ denote the weighted configuration on $\partial_\infty P_l$ obtained from $c$ via Lemma 2.9. Then there exists $T > 0$ such that*

$$\forall t > T : \Phi_c \circ l(t) = \Phi_{c_*} \circ l(t).$$

**Proof** In the proof of Lemma 2.9, we showed that the definition of $\iota$ implies that the claim holds for configurations consisting of a single point, *i.e.,* for maps $\phi_{\xi,m}$.

Since $\Phi_c, \Phi_{c_*}$ are finite compositions of such maps, the lemma follows. ■

For Euclidean buildings, we obtain the following refinement of Lemma 2.7:

***Lemma 2.11*** *Let $X$ be a Euclidean building, and let $c$ be a weighted configuration on its boundary at infinity. Let $\eta \in \partial_\infty X$, and $l$ a line asymptotic to $\eta$. Consider the measure $c_*$ on $\partial_\infty P_l$ obtained via Lemma 2.9. Then*

$$\operatorname{slope}_c = \operatorname{slope}_{c_*}$$

*on a neighborhood of $\eta$.*

**Proof** Let $U$ be the neighborhood of $\eta$ consisting of points lying in a common Weyl chamber with $\eta$, and let $\xi \in U, \xi' \in \partial_\infty X$. It follows from the proof of Lemma 2.9 that $\angle(\xi,\xi') = \angle(\xi_*,\xi'_*)$, since the triangles $\xi\eta\xi'$ and $\xi_*\eta_*\xi'_*$ are isometric (both are spherical, have two sides of the same length, and have the same angle at $\eta_{(*)}$). ■

***Lemma 2.12*** *Let $X$ be a Euclidean building, and $c$ a semistable configuration on its boundary at infinity. Let $\eta \in \partial_\infty X$ be a point with $\operatorname{slope}_c(\eta) = 0$, and $l$ a line asymptotic to $\eta$. Consider the measure $c_*$ on $\partial_\infty P_l$ obtained via Lemma 2.9. Then $c_*$ is semistable on $P_l$.*

**Proof** The measure $c_*$ is supported on the product $l \times CS(l)$, and

$$\operatorname{slope}_{c_*}(\eta_*) = \operatorname{slope}_c(\eta) = 0.$$

Thus for the antipode $\eta_*^-$ of $\eta_*$, we have $\operatorname{slope}_{c_*}(\eta_*^-) = -\operatorname{slope}_{c_*}(\eta_*) = 0$.

For a point $\xi$ on $\partial_\infty P_l$ which has distance less than $\pi$ from $\eta_*$, the claim $\operatorname{slope}_{c_*}(\xi) \geq 0$ follows from (strict) convexity of the zero-sublevel set of $\operatorname{slope}_{c_*}$ [KLM1, Prop. 3.1.(ii)], together with Lemma 2.11. ■

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# 3 The Results

## 3.1 Projecting Rays to Subspaces

We examine how rays project to a subspace of a Hadamard space:

***Proposition 3.1*** *Let $X'$ be a Hadamard space and $X \subset X'$ a closed convex subset. Consider $\eta \in \partial_\infty X'$ such that $\angle(\eta, \partial_\infty X) < \frac{\pi}{2}$. Let $o \in X$, $\rho := \overline{o\eta}$, and $\pi \colon X' \to X$ be the nearest point projection. Then the segments $\overline{o\,(\pi \circ \rho(t))}$ converge to the ray $\overline{o\xi}$ (in the cone topology), where $\xi \in \partial_\infty X$ is the unique point with $\angle(\eta, \xi) = \angle(\eta, \partial_\infty X)$.*

**Proof** Observe that $\partial_\infty X$ is a closed convex subset of $\partial_\infty X'$ (it is even closed in the cone topology). Since $\angle(\eta, \partial_\infty X) < \frac{\pi}{2}$, the projection $\xi$ of $\eta$ exists and is unique [BH99, II.2.6].

Let $\bar\alpha := \angle(\eta, \xi)$, $c_t := \rho(t)$, $p_t := \pi(c_t)$, and $\alpha_t := \tilde\angle_o(c_t, p_t)$.

By considering triangles $D$ of the form $\Delta(o, c_t, \overline{o\xi}(t))$, we conclude $d(c_t, p_t) \le t \sin\bar\alpha$ (since the comparison triangle of $D$ has angle at most $\bar\alpha$ at $o$, the CAT(0)-condition gives the upper bound on $d(c_t, p_t)$); this implies that $\alpha_t \le \bar\alpha$ for all $t > 0$.

Since $d(c_t, p_t) \le t \sin\bar\alpha$, we have $d(o, p_t) \ge t(1 - \sin\bar\alpha)$. Thus, for $s(1 - \sin\bar\alpha) \ge t$, the same argument as for the boundedness of $\alpha_t$ shows $\alpha_t \le \alpha_s$ $(*)$.

Let $t_n := (1 - \sin\bar\alpha)^{-n}$ for $n \in \mathbb{N}$ (observe that $\bar\alpha \ge \alpha_t > 0$ as soon as $c_t \notin X$). By what we have shown, $\alpha_{t_n}$ is an increasing bounded sequence, which converges to some $\hat\alpha \le \bar\alpha$.

Given $\varepsilon > 0$, let $N$ be such that $\alpha_{t_N} \ge \hat\alpha - \varepsilon$. Then for $t \ge t_{N+1}$ (so $t \in [t_n, t_{n+1}]$ for some $n > N$), we have $\hat\alpha - \varepsilon \le \alpha_{t_N} \le \alpha_t \le \alpha_{t_{n+2}} \le \hat\alpha$ by $(*)$. Hence $\alpha_t \xrightarrow[t\to\infty]{} \hat\alpha$.

We will show next that $d(p_t, \overline{op_s})/t \to 0$ for $s, t$ large; since $d(p_t, o) \ge t(1 - \sin\bar\alpha)$, this implies that the segments $\overline{op_t}$ converge to a ray.

For $s(1 - \sin\bar\alpha) \ge t$, let $p_{s,t}$ be the projection of $c_t$ to the segment $\overline{op_s}$. For $\varepsilon > 0$, there exists $T$ such that $t \ge T$ implies $\sin\alpha_t \ge \sin\hat\alpha - \varepsilon$. Then for $s(1 - \sin\bar\alpha) \ge t \ge T$, we have $d(c_t, p_t) \ge t(\sin\alpha_t) \ge t(\sin\hat\alpha - \varepsilon)$ and $d(c_t, p_{s,t}) \le t \sin\alpha_s \le t \sin\hat\alpha$.

Consider the comparison triangle $\Delta(c_t, p_t, p_{s,t})$. Since $p_t$ is the projection of $c_t$ to $X$, its angle at $p_t$ is at least $\frac{\pi}{2}$. Hence for the comparison angle $\gamma_{s,t} := \tilde\angle_{c_t}(p_t, p_{s,t})$, we have $\cos\gamma_{s,t} \ge \frac{\sin\hat\alpha - \varepsilon}{\sin\hat\alpha} \xrightarrow[\varepsilon\to 0]{} 1$.

Thus

$$d(p_t, p_{s,t})/t \xrightarrow[\varepsilon\to 0, s(1-\sin\bar\alpha)\ge t\ge T_\varepsilon]{} 0.$$

This shows that the segments $\overline{op_t}$ converge to a ray $\overline{o\hat\xi}$ for some $\hat\xi \in \partial_\infty X$.

By [KL97, Lemma 2.3.1], we have $\angle(\eta, \hat\xi) \le \liminf_{t\to\infty} \tilde\angle(c_t, p_t) = \hat\alpha \le \bar\alpha$. Hence, $\hat\alpha = \bar\alpha$ and $\hat\xi = \xi$. ∎

***Proposition 3.2*** *Let $X'$ be a Hadamard space and $X \subset X'$ a closed convex subset. Consider $\eta \in \partial_\infty X'$, and assume that for some $o \in X$, the projection of the ray $\overline{o\eta}$ to $X$ is bounded,* i.e., *there is $m$ such that $d(o, \pi \circ \overline{o\eta}(t)) < m$ for all $t > 0$.*

*Then there exists a point $p \in X$ such that $\pi \circ \overline{p\eta}(t) = p$ for all $t > 0$.*

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**Proof** Let $c_t := \overline{o\eta}(t)$ and $p_t := \pi(c_t)$. Let $t_1 := 1$, and define $t_n$ inductively by $t_n := K(\frac{1}{n}, m, t_{n-1})$, where $K$ is the constant from Lemma 1.2. Observe that $t_n$ is strictly increasing and unbounded.

Observe that $\pi(\overline{p_{t_n} c_{t_n}}(t_{n-1})) = p_{t_n}$. Since $\pi$ is 1-Lipschitz, we get from Lemma 1.2 that $d(p_{t_n}, \pi(\overline{p_{t_n}\eta}(t_{n-1}))) < \frac{1}{n}$.

We consider the ultraproducts $X^\omega \subset (X')^\omega$. Let $\pi_{X^\omega} : (X')^\omega \to X^\omega$ be the projection. Note that $\pi_{X^\omega}$ can be given in the form

$$\pi_{X^\omega}(x_n)_n = (\pi(x_n))_n.$$

Then $p' := (p_{t_n})_n$ is a point in $X^\omega$ which satisfies $\pi_{X^\omega}(\overline{p'\eta}(t)) = p'$ for all $t > 0$.

Now let $p$ be the projection of $p'$ to $X$. By the above, we have $\pi_{X^\omega}|_{X'} = \pi$, so $\pi_{X^\omega}(\overline{p\eta}(t)) \in X$. On the other hand, $d(\pi_{X^\omega}(\overline{p\eta}(t)), p') \leq d(p, p') = d(p', X)$, so the projection of the ray $\overline{p\eta}$ is constant. ∎

***Remark 3.3*** Observe that a point with the properties from the lemma above is a global minimum of the Busemann function $b_\eta|_X$. Note also, that the example from Remark 2.8 shows that the assumption of the proposition above need not be fulfilled if $\angle(\eta, \partial_\infty X) \geq \frac{\pi}{2}$.

## 3.2 Persistence of Semistability

Now persistence of semistability follows easily:

***Proposition 3.4*** *Let $X \subset X'$, where $X$ is a closed convex subset of the Hadamard space $X'$, and let $c$ be a weighted configuration on the asymptotic boundary of $X$. If $c$ is semistable on $X$, then $c$ is semistable on $X'$.*

**Proof** Assume there is $\eta \in \partial_\infty X'$ with $\operatorname{slope}_c(\eta) = -c < 0$. From the formula for the slope, we conclude that there must be some $\xi_i$ in the support of $c$ which satisfies $\angle(\eta, \xi_i) < \frac{\pi}{2}$. Hence Proposition 3.1 applies.

From this point, we obtain a contradiction as in the end of the proof of [KLM1, Lemma 3.10(ii)]:

Use the notation of the proof above, and for $s \geq t$, let $\bar{p}_{s,t} := \overline{op_s}(\frac{t}{s} d(o, p_s))$. We may normalize $b_c$ such that $b_c(o) = 0$. Then, by convexity, we have $b_c(c_s) \leq -cs$. As in the proof of [KLM1, Lemma 3.10], we have $b_c \geq b_c \circ \pi$ (where $\pi$ is the projection $X' \to X$). In particular, $b(p_s) \leq -cs$.

For $s \geq t$, we conclude from convexity that $b_c(p_{s,t}) \leq -ct$. Fixing $t$ and letting $s \to \infty$, this shows $b_c(\overline{o\xi}(t \cos \hat{\alpha})) \leq -ct$, implying $\operatorname{slope}_c(\xi) \leq -c/\cos\hat{\alpha} < 0$. This is the desired contradiction. ∎

***Remark 3.5*** Observe that we cannot expect stability to be preserved under general embeddings, as one sees, *e.g.*, by embedding $X$ into $X \times \mathbb{R}$.

We will only use the above proposition for the inclusion $X \subset X^\omega$. However, we may not expect stability to be preserved in this case either, as the following example shows:

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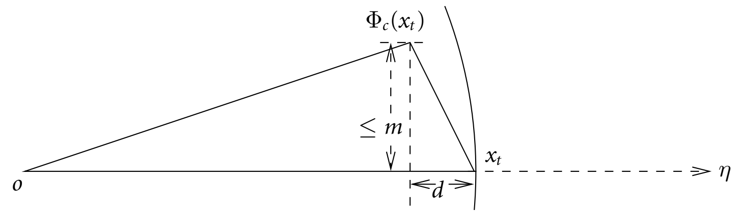


*Figure 3*: For $t$ large, $d(o, \Phi_c(x_t)) < d(o, x_t)$.

Consider the disjoint union of copies of $\mathbb{H}^2 \times [-n, n]$ for $n \in \mathbb{N}$, identified along $\mathbb{H}^2 \times \{0\}$. This is a Hadamard space by [BH99, II.11.3]. Its boundary is precisely the boundary of $\mathbb{H}^2$, but its ultraproduct contains a copy of $\mathbb{H}^2 \times \mathbb{R}$.

## 3.3 Proof of the Main Theorem

In this section we present the proof of our main theorem. We will need a lemma about fixed points of weak contractions, which we recall without proof:

***Lemma 3.6*** ([KLM2, Lemma 4.5]) *Let $X$ be a Hadamard space of finite diameter. Then every weak contraction $\Phi\colon X \to X$ has a fixed point.*

The following lemma was essentially contained in an earlier version of [KLM2]:

***Lemma 3.7*** *Let $c$ be a weighted configuration on the boundary of a Hadamard space of the form $l \times Y$, where $l$ is a line with endpoints $\eta, \eta_-$, and $Y$ is a Hadamard space.*

*If* $\operatorname{slope}_c(\eta) > 0$, *then there exists $T > 0$ such that $d(\Phi_c((l(t), y)), (l(0), y)) < t$ for all $t > T$ and $y \in Y$.*

**Proof** The configuration $c$ can be split into configurations $c_1, c_2$ on $\{\eta, \eta_-\}, \partial_\infty Y$ respectively, and this splitting is compatible with the action of $\Phi$ (see [KLM1, Lemma 3.12]). In particular, we have $(b_\eta \circ \Phi_c - b_\eta) \equiv \operatorname{slope}_c \eta =: d > 0$.

Let $o := (l(0), y)$ and $x_t := (l(t), y)$. The triangle $\Delta(o, x_t, \Phi_c(x_t))$ is Euclidean, so the claim follows from the fact that the displacement of $\Phi_c$ is bounded (by $m := \sum_{i=1}^n m_i$); see figure 3. ∎

Now we have all ingredients for the proof of our main theorem. We start with the building case:

***Theorem 3.8*** *Let $X$ be a Euclidean building, and let $c$ be a semistable weighted configuration on its boundary at infinity. Then the associated weak contraction $\Phi_c$ has a fixed point. In particular, there exists a polygon $p$ in $X$ such that $c$ is a Gauss map for $p$.*

**Proof** Fix a basepoint $o \in X$. If we find a ball $B(o, R) \subset X$ which is preserved by $\Phi$, we are done by Lemma 3.6.

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We argue by contradiction: Assume that for each $i \in \mathbb{N}$, there exists a point $x_i \in X$ such that $d(o, x_i) \geq i$ and $d(\Phi(x_i), o) \geq d(x_i, o)$ $(*)$. Observe that $(*)$ holds for each $x \in \overline{ox_i}$ since $\Phi$ is a weak contraction.

The segments $\overline{ox_i}$ define a ray $\rho = \overline{o\eta}$ in the ultraproduct $X^\omega$ (for some $\eta \in \partial_\infty X^\omega$). We have

$$\rho(t) = (\overline{ox_i}(t))_i$$

where we set $\overline{ox_i}(t) := o$ for $i < t$ (clearly, these finitely many points have no influence on the point defined in $X^\omega$).

Let $c_*$ be the configuration $c$ considered as a configuration on $\partial_\infty X^\omega$, and let $\Phi_*$ be the associated weak contraction. Now $\rho$ satisfies $d(\Phi_*(\rho(t)), o) \geq d(\rho(t), o) = t$ for all $t$, since we have

$$(\dagger) \qquad d(\Phi_*(\rho(t)), o) = \lim\nolimits_\omega \underbrace{d(\Phi(\overline{ox_i}(t)), o)}_{\geq t \text{ if } i \geq t} \geq t = d(\rho(t), o).$$

By Proposition 3.4, there are two cases to be considered:

***Case 1*** ($\operatorname{slope}_{c_*}(\eta) > 0$) We consider the asymptotic tube $T_\eta$, and the pushed forward configuration, which we denote by $c_{**}$; the associated weak contraction will be denoted by $\Phi_{**}$.

Let $l$ be the line which is obtained from $\rho$ when passing to the asymptotic tube. By Lemma 2.7 and Lemma 3.7, we have $d(\Phi_{**} \circ l(t), l(0)) < t$ for large $t$. This implies that for large $t$ and $\omega$-almost all $i$, we have $d(\Phi_* \circ \rho(i+t), \rho(i)) < t$.

By the triangle inequality, this implies $d(\Phi_* \circ \rho(i+t), o) < i + t$, in contradiction to $(\dagger)$.

***Case 2*** ($\operatorname{slope}_{c_*}(\eta) = 0$) We argue by induction on $\operatorname{rank}(X)$. Let $l$ be a line extending $\rho$; we pass to a configuration $c_{**}$ on $\partial_\infty P_l$ (via Lemma 2.9). Then $c_{**}$ is semistable by Lemma 2.12. Since $P_l = l \times CS(l)$, $c_{**}$ splits, and we obtain a semistable configuration on $\partial_\infty l$ and a semistable configuration on $\partial_\infty CS(l)$.

A semistable configuration on the boundary of a flat Euclidean space (i.p. a line) yields a constant map $\Phi$; a semistable configuration on $\partial_\infty CS(l)$ has a fixed point by the induction hypothesis.

Thus, we have a line of fixed points for $c_{**}$ in $X^\omega$. This line of fixed points yields a ray of fixed points for $\Phi_*$ by Lemma 2.10. So let $p \in X^\omega$ be a fixed point of $\Phi_*$. There is a unique point $p' \in X$ which is closest to $p$. Since $\Phi_*$ is 1-Lipschitz, it has to fix $p'$. Now the observation $\Phi_*|_X = \Phi$ finishes the proof. ■

***Corollary 3.9*** *Let $X$ be a Hadamard space, and $c$ a weighted configuration on its boundary at infinity, which is stable on $X^\omega$. Then the associated weak contraction $\Phi_c$ has a fixed point. In particular, there exists a polygon $p$ in $X$ such that $c$ is a Gauss map for $p$.*

**Proof** By assumption, Case 2 in the proof of Theorem 3.8 above does not occur; hence the proof works exactly the same (observe that building geometry was used only in the second case). ■

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In the locally compact case, $X^\omega \cong X$; hence Corollary 3.9 finishes the proof of the Main Theorem.

Observe that we cannot expect Theorem 3.8 to fully generalize to Hadamard spaces, since in the case of symmetric spaces, nice semistability of the configuration is necessary.

## 4 Relations to Algebra

Here, we discuss the relevance of our main theorem to problems from algebra. Such problems were studied in [KLM3].

In the algebraic problems, one only fixes the *type* of a configuration, *i.e.,* the projection of the points $\xi_i$ to the spherical Weyl chamber $\Delta$. Taking the weights $m_i$ into account, such a type of a configuration may be viewed as an element of $\Delta^n_{\text{euc}}$, $n$ copies of the Euclidean Weyl chamber (the Euclidean cone over the spherical Weyl chamber $\Delta$). Consider the following theorem:

***Theorem 4.1*** ([KLM2, Theorem 1.2]) *Let $X$ be a Euclidean building. For $h \in \Delta^n_{\text{euc}}$ there exists an $n$-gon in $X$ with $\Delta$-side lengths $h$ if and only if there exists a semistable weighted configuration on $\partial_\infty X$ of type $h$.*

Our main results give a natural proof, and may in fact be seen as a refinement, since the proof in [KLM2] does not provide explicit configurations for which there exists a fixed point. This indicates that there will eventually be more applications to algebra.

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