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Bifurcations of Limit Cycles From Infinity in Quadratic Systems

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Abstract. We investigate the bifurcation of limit cycles in one-parameter unfoldings of quadractic differential systems in the plane having a degenerate critical point at infinity. It is shown that there are three types of quadratic systems possessing an elliptic critical point which bifurcates from infinity together with eventual limit cycles around it. We establish that these limit cycles can be studied by performing a degenerate transformation which brings the system to a small perturbation of certain well-known reversible systems having a center. The corresponding displacement function is then expanded in a Puiseux series with respect to the small parameter and its coefficients are expressed in terms of Abelian integrals. Finally, we investigate in more detail four of the cases, among them the elliptic case (Bogdanov-Takens system) and the isochronous center $\mathcal{S}_3$. We show that in each of these cases the corresponding vector space of bifurcation functions has the Chebishev property: the number of the zeros of each function is less than the dimension of the vector space. To prove this we construct the bifurcation diagram of zeros of certain Abelian integrals in a complex domain.

1 Introduction

The problem about the bifurcation of limit cycles under small quadratic perturbations in planar quadratic differential systems with a center has received in recent years a great deal of attention. Most of the known results are concerned with the bifurcation of limit cycles in the finite part of the plane [1, 20, 6, 15, 8, 5, 19, 2]. In this paper we study limit cycles bifurcating from infinity. Since a limit cycle of a quadratic system surrounds a unique equilibrium point which is a focus [16], such a kind of bifurcation is possible in the following two cases. The first one is when a limit cycle bifurcates from a unbounded separatrix contour (compound cycle) having a monodromy map when considered on the Poincaré sphere, and contains *a finite focus* in its interior. A typical example is the contour that surrounds a unbounded period annulus of a center placed in the finite plane. An unbounded contour having a monodromy map and going through a degenerate finite point serves as another example (see [14, Figure 2.3]). The second case appears when the unperturbed integrable system has a degenerate point at infinity which after a perturbation could produce a focus in the finite plane coming from infinity (together with eventual limit cycles around it). In the present paper we shall study the appearance of limit cycles around this *focus near infinity*. It should be pointed out that the study of a limit cycle bifurcation from infinity is an important part of the so called weakened (or infinitesimal) 16-th Hilbert problem which asks for the number and distribution of limit cycles in systems close to the integrable (*i.e.*, having a center) ones.

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In a general setting, the problem on bifurcation of limit cycles from infinity in quadratic systems can be stated as follows. Consider a planar quadratic system depending on a parameter $\lambda \in \mathbb{R}^n$:

$$(1_\lambda) \qquad \begin{aligned} \dot{x} &= P(x, y, \lambda), \\ \dot{y} &= Q(x, y, \lambda). \end{aligned}$$

The question is: when (1_λ) can have a limit cycle γ_λ which tends to infinity as $\lambda \to 0$ and how many such limit cycles can exist? It is said that a limit cycle γ_λ tends to infinity as $\lambda \to 0$ if for any compact $K \subset \mathbb{R}^2$ there is C_K so that $\gamma_\lambda \not\subset K$ for $|\lambda| < C_K$.

In the present paper we will consider a simplified version of the problem assuming that:

1) the quadratic system depends analytically on a single parameter $\varepsilon \in \mathbb{R}^+$;

2) for $\varepsilon = 0$, the system has an elliptic critical point at the origin (a nondegenerate focus or center) and a degenerate critical point at infinity.

In that context, our plan consists in the following:

a) To identify systems having a focus ζ in the finite plane which tends to infinity as $\varepsilon \to 0$.

b) To show that, under certain genericity condition, any such a system reduces to a small perturbation of a quadratic reversible system with a center.

c) Using this fact, to estimate the number of limit cycles around ζ (for some cases).

More precisely, we assume that ζ tends to infinity when $\varepsilon \to 0$ as fast as ε^{-1}, which is the common case (and our main genericity condition throughout the paper). Then, we are able to study the limit cycles around ζ obeying the asymptotics $O(\varepsilon^{-1})$ at each their point. All the limit cycles bifurcating from the center or the period annulus of the reduced reversible system are among them (see Definition 3.1 and Theorem 3.3 for the exact formulations). Thus, the only limit cycles still to study are the ones tending in the reduced system to separatrix contours other than saddle loops. It is important to note that the initial system is not supposed to be close to an integrable one.

To begin with, we give the list of quadratic systems which can acquire, after a small quadratic perturbation, a focus coming from infinity (Theorem 2.1). At this step the condition in 2) concerning the existence of an elliptic equilibrium point at the origin turns out to be inessential. Next, in Section 3 we study the bifurcations of limit cycles around foci appearing near infinity as a result of a perturbation satisfying our genericity assumption. Performing suitable translation and rescaling, we prove that any such a system becomes a small quadratic perturbation of a special reversible quadratic system with a center at the origin (Theorem 3.1 and 3.2). The bifurcating limit cycles we investigate will then surround the finite focus (placed at the origin).

In the final Section 4 we apply our method to four famous examples of quadratic systems with a center. We show that the bifurcations of limit cycles from infinity are governed (as is often the case) by the bifurcations of zeros of suitable complete Abelian integrals associated to curves of genus 1 or 0. In all these cases the corresponding vector space of functions has the following non-oscillation property: the

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number of the zeros of each function is less than the dimension of the vector space. This provides new interesting Chebyshev spaces (Theorem 4.2 and 4.3). To prove this we construct, following [3], the bifurcation diagrams of zeros of certain Abelian integrals in a complex domain.

2 Systems With a Focus Escaping to Infinity

Let us first recall that if a quadratic system has at least one limit cycle, then this cycle surrounds exactly one equilibrium point which is a focus [16]. The limit cycles bifurcating from infinity surround therefore either a finite or infinite (*i.e.* bifurcating from infinity) focus. In the present paper we study only the second possibility. As already mentioned, we will consider a more general class to be perturbed than the class of quadratic integrable systems. Namely, we shall suppose that the system under consideration has a finite elliptic equilibrium point which we place at the origin. Then the system can be written in the form of a single complex equation for $z = x+iy \in \mathbb{C}$

$$\dot z = \alpha z + Az^2 + Bz\bar z + C\bar z^2, \tag{1}$$

where α, A, B, C take complex values. Without any loss of generality, it is assumed that $|A| + |B| + |C| \neq 0$, $\operatorname{Im}\alpha = -1$. The critical points of (1) $\zeta = re^{i\theta}$ outside the origin are determined by the equation

$$\alpha e^{i\theta} + r(Ae^{2i\theta} + B + Ce^{-2i\theta}) = 0.$$

Hence (1) has a critical point escaped to infinity provided that

$$Ae^{2i\theta} + B + Ce^{-2i\theta} = 0 \tag{2}$$

for certain $\theta \in [0, 2\pi]$, which is tantamount to

$$|A\bar B - B\bar C| = \big|\, |A|^2 - |C|^2 \big|, \quad |A| + |C| \geq |B|. \tag{3}$$

Assume now that the coefficients of (1) depend analytically on a small parameter ε

$$\alpha = \sum_{i=0}^{\infty} \alpha_i\varepsilon^i, \quad A = \sum_{i=0}^{\infty} A_i\varepsilon^i, \quad B = \sum_{i=0}^{\infty} B_i\varepsilon^i, \quad C = \sum_{i=0}^{\infty} C_i\varepsilon^i. \tag{4}$$

Let us take for definiteness $\varepsilon \geq 0$. Assume further that as ε tends to 0 a critical point ζ of system (1) tends to infinity. Taking a second power of both sides of equation (3), one can then rewrite it as

$$p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \cdots = 0 \tag{5}$$

where p_k are polynomials of the coefficients A_k, B_k, C_k and their complex-conjugate. Equation (5) determines the perturbations of the system

$$\dot z = \alpha_0 z + A_0 z^2 + B_0 z\bar z + C_0\bar z^2, \quad \operatorname{Im}\alpha_0 = -1, \quad |A_0| + |B_0| + |C_0| \neq 0 \tag{6}$$

under which the critical point will remain at infinity. For this, an infinite sequence of conditions should be satisfied: $p_1 = p_2 = p_3 = \cdots = 0$. The above consideration shows that when a system (6) with a critical point at infinity undergoes a small perturbation, we need to calculate the successive coefficients in (5) until we find out that $p_k \neq 0$ for some $k \geq 1$. If so, an additional critical point ζ in the finite part of the plane will appear, coming from infinity. To determine the conditions that this critical point near infinity be a focus, we denote by $\Phi(z, \bar{z})$ the right-hand side of (1). Then $\Phi(\zeta, \bar{\zeta}) = 0$ and a translation $z \to z + \zeta$ in equation (1) leads to

$$\dot{z} = \Phi_z(\zeta, \bar{\zeta})z + \Phi_{\bar{z}}(\zeta, \bar{\zeta})\bar{z} + Az^2 + Bz\bar{z} + C\bar{z}^2. \tag{7}$$

The characteristic equation corresponding to the linear part of (7) is

$$\omega^2 - 2\omega \operatorname{Re} \Phi_z(\zeta, \bar{\zeta}) + |\Phi_z(\zeta, \bar{\zeta})|^2 - |\Phi_{\bar{z}}(\zeta, \bar{\zeta})|^2 = 0.$$

Therefore ζ is a focus provided that

$$-\Delta = \left(\operatorname{Im} \Phi_z(\zeta, \bar{\zeta})\right)^2 - |\Phi_{\bar{z}}(\zeta, \bar{\zeta})|^2 > 0.$$

We can use the obvious identity

$$\zeta\Phi_z(\zeta, \bar{\zeta}) + \bar{\zeta}\Phi_{\bar{z}}(\zeta, \bar{\zeta}) = -\alpha\zeta \Leftrightarrow \Phi_z(\zeta, \bar{\zeta}) + e^{-2i\theta}\Phi_{\bar{z}}(\zeta, \bar{\zeta}) = -\alpha$$

to rewrite the above condition as

$$-\Delta = -1 + 2 \operatorname{Im} \Phi_z(\zeta, \bar{\zeta}) - \left(\operatorname{Re}\left(\Phi_z(\zeta, \bar{\zeta}) + \alpha\right)\right)^2 > 0.$$

The last inequality implies that the leading coefficient in $\operatorname{Re} \Phi_z$ should vanish and the first nonzero coefficient in $\operatorname{Im} \Phi_z$ should be positive. We write the first of these conditions in the form $(2A + \bar{B})e^{i\theta} + (2\bar{A} + B)e^{-i\theta} = o(1)$ as $\varepsilon \to 0$ or, equivalently,

$$(2A_0 + \bar{B}_0)\eta_0^2 + (2\bar{A}_0 + B_0) = 0 \Leftrightarrow \operatorname{Re}(2A_0\eta_0 + B_0\bar{\eta}_0) = 0 \tag{8}$$

where we set $e^{i\theta} = \eta_0 + o(1)$. These relations combined with (2) and (3) lead to

$$|A_0\bar{B}_0 - B_0\bar{C}_0| = \big|\, |A_0|^2 - |C_0|^2 \big|, \quad |A_0| + |C_0| \geq |B_0|,$$
$$C_0(2A_0 + \bar{B}_0)^2 = (2\bar{A}_0 + B_0)(A_0B_0 + |B_0|^2 - 2|A_0|^2).$$

We have either $B_0 = 0$, or $B_0 \neq 0$. In the first case we can take without loss of generality $A_0 = 1$, and in the second one, $B_0 = 2$. Thus we have proved the following

Theorem 2.1 *Suppose that the quadratic system (6) can acquire, after a small quadratic perturbation, a focus coming from infinity. Then it can be taken in one of the following normal forms*

(I) $\dot{z} = \alpha_0 z + z^2 - \bar{z}^2,$

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(II) $\dot{z} = \alpha_0 z - z^2 + 2z\bar{z} + C_0\bar{z}^2$, $2|C_0 + 1| = |C_0|^2 - 1$,
(III) $\dot{z} = \alpha_0 z + A_0 z^2 + 2z\bar{z} + C_0\bar{z}^2$, $C_0 = \frac{(\bar{A}_0+1)(2+A_0-|A_0|^2)}{(A_0+1)^2}$, $A_0 \neq -1$.

It should be noted that some well-known systems are included in (II), (III). Let us take α_0 purely imaginary. When $C_0 = -1$ in (II), one obtains the Bogdanov-Takens system, defining the Hamiltonian vector field that has been used in the 1970's to study unfoldings of the cusp singularity in the so called Bogdanov-Takens bifurcation. Systems (III) with $A_0 = 2$ and $A_0 = 5$ correspond to the isochronous centers $\mathcal{S}_2$ and $\mathcal{S}_3$ respectively, see [1]. System (III) with $A_0 = 4$ presents one of the two intersection points of two strata in the quadratic center manifold, the reversible Q_3^R and the codimension four Q_4 ones [20, 8].

The restriction that the system has an elliptic critical point at the origin is not essential in the above considerations. It is easy to see that a similar analysis is applicable to the general quadratic system

$$\dot{z} = \gamma + \alpha z + \beta\bar{z} + Az^2 + Bz\bar{z} + C\bar{z}^2$$

as well. Thus, the list of all quadratic systems which can acquire a focus coming from infinity as a result of a small perturbation is the following:

$$\dot{z} = \gamma_0 + \alpha_0 z + \beta_0\bar{z} + z^2 - \bar{z}^2,$$
$$\dot{z} = \gamma_0 + \alpha_0 z + \beta_0\bar{z} - z^2 + 2z\bar{z} + C_0\bar{z}^2, \quad 2|C_0 + 1| = |C_0|^2 - 1,$$
$$\dot{z} = \gamma_0 + \alpha_0 z + \beta_0\bar{z} + A_0 z^2 + 2z\bar{z} + C_0\bar{z}^2, \quad C_0 = \frac{(\bar{A}_0+1)(2+A_0-|A_0|^2)}{(A_0+1)^2}, \; A_0 \neq -1.$$

For simplicity, below we are not going to deal with the general case. In the next section we will study small analytic one-parameter perturbations of the systems from Theorem 2.1. Consider system (1) where α, A, B, C depend analytically on ε as in (4), and the initial coefficients α_0, A_0, B_0, C_0 of (6) are as in (I), (II) or (III) above. Our main genericity assumption on the perturbation is that $\varepsilon r \to r_0 \neq 0$ as $\varepsilon \to 0$, namely:

(H1) $|\zeta|$ goes to infinity when $\varepsilon \to 0$ as rapidly as a constant multiplier of ε^{-1}.

In what follows, the perturbations satisfying (H1) are called *transversal to infinity*, while the others are called *tangential to infinity*. The latter can have any order k of tangency to infinity, thus $k = \infty$ corresponding to perturbations that leave the critical point ζ to stay at infinity. It seems to us that the study of limit cycles around foci coming from infinity as a result of quadratic perturbations tangential to infinity becomes more difficult when k is growing. For this reason, we will consider below only perturbations satisfying (H1) which is the common class. This will be the first step towards the solution of the general problem.

3 Reduction to a Perturbation of a Reversible Integrable System

We rescale time in (7) $t \to t\sqrt{-\Delta}$ and introduce there a new variable Z through

$$z = iZ + \delta\bar{Z}, \quad Z = (iz + \delta\bar{z})/(|\delta|^2 - 1)$$

where

$$\delta = \frac{\Phi_{\bar{z}}(\zeta, \bar{\zeta})}{\operatorname{Im} \Phi_z(\zeta, \bar{\zeta}) - \sqrt{-\Delta}} \quad \text{and} \quad |\delta|^2 - 1 = \frac{2\sqrt{-\Delta}}{\operatorname{Im} \Phi_z(\zeta, \bar{\zeta}) - \sqrt{-\Delta}} \neq 0.$$

System (7) becomes

$$\dot{Z} = (\Omega - i)Z + \mathcal{A}Z^2 + \mathcal{B}Z\bar{Z} + \mathcal{C}\bar{Z}^2 \tag{9}$$

where

$$\Omega = \frac{\operatorname{Re} \Phi_z(\zeta, \bar{\zeta})}{\sqrt{-\Delta}} = \frac{\omega}{\sqrt{-\Delta}} + i$$

$$\mathcal{A} = \frac{\bar{\delta}^2(\delta\bar{A} + iC) + i\bar{\delta}(\delta\bar{B} + iB) - (\delta\bar{C} + iA)}{(|\delta|^2 - 1)\sqrt{-\Delta}}$$

$$\mathcal{B} = \frac{2i\delta(\delta\bar{C} + iA) - 2i\bar{\delta}(\delta\bar{A} + iC) + (|\delta|^2 + 1)(\delta\bar{B} + iB)}{(|\delta|^2 - 1)\sqrt{-\Delta}}$$

$$\mathcal{C} = \frac{\delta^2(\delta\bar{C} + iA) - i\delta(\delta\bar{B} + iB) - (\delta\bar{A} + iC)}{(|\delta|^2 - 1)\sqrt{-\Delta}}$$

It should be noted that the change of the variables $z = iZ + \delta\bar{Z}$ we used is *asymptotic* one in the sense that it degenerates for $\varepsilon = 0$ but not for $\varepsilon > 0$. The change degenerates for $\varepsilon = 0$ because the matrix of the linear change $z \to Z$ (considered in $\mathbb{R}^2$) has a determinant $1 - |\delta|^2$ which possesses (as we shall see below) an expansion of the form (10). Therefore the change is singular for $\epsilon = 0$ and regular for small nonzero ε.

It is easily seen that system (9) is a small quadratic perturbation of the linear center $\dot{Z} = -iZ$. Below we are going to show that, after additional rescaling and rotation, system (9) becomes a small perturbation of a reversible quadratic system having a center at the origin. For this purpose we have to calculate explicitly the first nontrivial terms in the expansions of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, Ω. It is easy to see that these functions can be expanded in Puiseux series

$$\sum_{k=1}^{\infty} a_k (\varepsilon^{1/p})^k \tag{10}$$

where $p = 2$ or $p = 4$. We assume first that $B_0^2 \neq 4A_0C_0$. This condition allows one to use the implicit function theorem for solving the equations for the coordinates r, θ of the focus ζ near infinity. Thus, under this condition, θ is an analytic function of

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ε, while r in the transversal case is expanded in a Laurent series beginning with term ε^{-1}. The case $B_0^2 = 4A_0C_0$ corresponds to the situation when two critical points appear simultaneously in the finite plane coming from the same infinity direction. This special bifurcation occurs in (II) for $C_0 = -1$ (which is the Bogdanov-Takens case) and in (III) provided that $|A_0| = 1$. Now, r and θ have Puiseux expansions with $p = 2$. So we let first exclude these cases from our consideration.

3.1 The General Case $B_0^2 \neq 4A_0C_0$

This case corresponds to a single critical point escaped to infinity. Our main result in this subsection is:

Theorem 3.1 *Suppose that* $B_0^2 \neq 4A_0C_0$ *and* ζ *is a focus satisfying* (H1). *Then the change of variables*

$$z = \zeta + \frac{2i}{\bar{\mathcal{B}}} Z + \frac{2\delta}{\mathcal{B}} \bar{Z} \tag{11}$$

transforms system (1) *into a system*

$$\dot{Z} = (\Omega - i)Z + \frac{2\mathcal{A}}{\bar{\mathcal{B}}} Z^2 + 2Z\bar{Z} + \frac{2\bar{\mathcal{B}}\mathcal{C}}{\mathcal{B}^2} \bar{Z}^2 \tag{12}$$

which is a small perturbation of:

(i) *the reversible quadratic-like-linear system*

$$\dot{Z} = -iZ - 2Z^2 + 2Z\bar{Z} \quad \text{in case I;}$$

(ii) *the reversible Hamiltonian system with a non-Morsean singularity*

$$\dot{Z} = -iZ - Z^2 + 2Z\bar{Z} + \frac{1}{3}\bar{Z}^2 \quad \text{in case II;}$$

(iii) *the reversible system*

$$\dot{Z} = -iZ + \frac{-1 + \operatorname{Re} A_0 + 2|A_0|^2}{2 + \operatorname{Re} A_0 - |A_0|^2} Z^2 + 2Z\bar{Z} + \frac{1 + \operatorname{Re} A_0}{2 + \operatorname{Re} A_0 - |A_0|^2} \bar{Z}^2 \quad \text{in case III,}$$

when $2 + \operatorname{Re} A_0 - |A_0|^2 \neq 0$.
The change of variables

$$z = \zeta + \frac{i}{\mathcal{A}} Z + \frac{\delta}{\bar{\mathcal{A}}} \bar{Z} \tag{13}$$

transforms system (1) *into a system*

$$\dot{Z} = (\Omega - i)Z + Z^2 + \frac{\mathcal{B}}{\bar{\mathcal{A}}} Z\bar{Z} + \frac{\mathcal{A}\mathcal{C}}{\bar{\mathcal{A}}^2} \bar{Z}^2 \tag{14}$$

which is a small perturbation of:

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(iv) *the reversible Lotka-Volterra system*

$$\dot{Z} = -iZ + Z^2 + \frac{1}{3}\bar{Z}^2 \quad \textit{in case III},$$

when $2 + \operatorname{Re} A_0 - |A_0|^2 = 0$.

All the systems in (i)–(iv) *have a center at the origin.*

Proof The proof follows by asymptotic calculations. We begin with examining in more detail the transversality condition in (H1). Take $e^{i\theta} = \eta_0 + \varepsilon\eta_1 + \mathrm{O}(\varepsilon^2)$. Then $|\eta_0| = 1$, $\operatorname{Re} \eta_0\bar{\eta}_1 = 0$, hence one can write $e^{i\theta} = \eta_0\big(1 + \varepsilon ik + \mathrm{O}(\varepsilon^2)\big)$ where k is real. The function $e^{i\theta}$ is determined from the requirement that

$$r = -\frac{\alpha e^{i\theta}}{Ae^{2i\theta} + B + Ce^{-2i\theta}} \tag{15}$$

be real and positive. Using straightforward calculations we obtain that

$$e^{i\theta} = \eta_0\left[1 + \frac{\varepsilon i \operatorname{Im} \alpha_0\eta_0(\bar{A}_1\bar{\eta}_0^2 + \bar{B}_1 + \bar{C}_1\eta_0^2)}{2 \operatorname{Re} \alpha_0\eta_0(\bar{A}_0\bar{\eta}_0^2 - \bar{C}_0\eta_0^2)} + \mathrm{O}(\varepsilon^2)\right],$$

$$\begin{aligned} r &= \frac{-\alpha_0\eta_0 + \mathrm{O}(\varepsilon)}{\varepsilon[A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2 + 2ik(A_0\eta_0^2 - C_0\bar{\eta}_0^2)] + \mathrm{O}(\varepsilon^2)} \\ &= \frac{1}{\varepsilon \operatorname{Im} \bar{\eta}_0(A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2) + \mathrm{O}(\varepsilon^2)}. \end{aligned}$$

Therefore in the case we consider, (H1) reduces to

$$\operatorname{Im} \bar{\eta}_0(A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2) > 0. \tag{16}$$

One can easily verify that the denominator in the above formula of $e^{i\theta}$ does not vanish. Indeed, by equation

$$A_0\eta_0^2 + B_0 + C_0\bar{\eta}_0^2 = 0 \tag{17}$$

we have

$$D_0 = \eta_0(\bar{A}_0\bar{\eta}_0^2 - \bar{C}_0\eta_0^2) = 2\bar{A}_0\bar{\eta}_0 + \bar{B}_0\eta_0 = 4i \operatorname{Im} \eta_0$$

in case (II) and

$$D_0 = \eta_0(2\bar{A}_0\bar{\eta}_0^2 + \bar{B}_0) = \pm i\frac{4|A_0|^2 - |B_0|^2}{|2A_0 + \bar{B}_0|}$$

in cases (I), (III) (we make use of (8) here). Hence, the denominator is not zero unless $C_0 = -1$ in (II) and $|A_0| = 1$ in (III), which is not the case we deal with. Condition (16) yields that $r = \mathrm{O}(\varepsilon^{-1})$. Thus, $\operatorname{Re} \Phi_z(\zeta, \bar{\zeta}) = \mathrm{O}(1)$, hence

$$-\Delta \sim 2 \operatorname{Im} \Phi_z(\zeta, \bar{\zeta}) \sim -2r \operatorname{Im} D_0 > 0$$

and $\Omega = O(\varepsilon^{1/2})$. Now, conditions (8), (17) along with $\operatorname{Im} D_0 < 0$ allow one to calculate η_0. One obtains

$$\eta_0 = i \quad \text{for (I)},$$

$$\eta_0 = -i\frac{|C_0|^2 + 2C_0 + 1}{\big|\,|C_0|^2 + 2C_0 + 1\big|} \quad \text{for (II)},$$

$$\eta_0 = \begin{cases} i\frac{1+\bar{A}_0}{|1+A_0|} & \text{if } |A_0| > 1, \\ -i\frac{1+\bar{A}_0}{|1+A_0|} & \text{if } |A_0| < 1 \end{cases} \quad \text{for (III)},$$

respectively.

Below we calculate the first nonzero terms in the expansions in ε of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. We first obtain that

$$\begin{aligned}
\xi &= \frac{\Phi_{\bar{z}}(\zeta,\bar{\zeta})}{\operatorname{Im}\Phi_z(\zeta,\bar{\zeta})} = \frac{Be^{i\theta} + 2Ce^{-i\theta}}{-r^{-1} + \operatorname{Im}(2Ae^{i\theta} + Be^{-i\theta})} \\
&= \frac{(B_0\eta_0 + 2C_0\bar{\eta}_0) + \varepsilon[B_1\eta_0 + 2C_1\bar{\eta}_0 + ik(B_0\eta_0 - 2C_0\bar{\eta}_0)] + O(\varepsilon^2)}{\operatorname{Im}(2A_0\eta_0 + B_0\bar{\eta}_0) + \varepsilon\operatorname{Im}[A_1\eta_0 - C_1\bar{\eta}_0^3 + ik(2A_0\eta_0 - B_0\bar{\eta}_0)] + O(\varepsilon^2)} \\
&= -i\eta_0^2\bigg[1 - \varepsilon\frac{\operatorname{Im}\bar{\eta}_0(A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2)}{\operatorname{Im}(2A_0\eta_0 + B_0\bar{\eta}_0)} \\
&\qquad + \varepsilon i\frac{\operatorname{Re}[B_1\bar{\eta}_0 + 2C_1\bar{\eta}_0^3 + ik(2A_0\eta_0 + 3B_0\bar{\eta}_0)]}{\operatorname{Im}(2A_0\eta_0 + B_0\bar{\eta}_0)} + O(\varepsilon^2)\bigg].
\end{aligned}$$

On the other hand, with $\xi = -i\eta_0^2\big(1 - \mu\varepsilon + O(\varepsilon^2)\big)$ where $\operatorname{Re}\mu > 0$, we have

$$\delta = \frac{\xi}{1 - \sqrt{1 - |\xi|^2}} = -i\eta_0^2(1 + \sqrt{2\operatorname{Re}\mu}\,\varepsilon^{1/2} + \bar{\mu}\varepsilon + O\big(\varepsilon^{3/2})\big).$$

After replacing the above expansion formula of δ in the coefficients of (9) we get the following expressions for the coefficients:

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\varepsilon\bar{\eta}_0\operatorname{Im}\bar{\eta}_0[A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2 + \mu(2A_0\eta_0^2 - C_0\bar{\eta}_0^2 - \bar{C}_0\eta_0^4)] + O(\varepsilon^{3/2}) \\
\mathcal{B} &= \varepsilon\eta_0\operatorname{Im}\bar{\eta}_0[-(A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2) + \mu(A_0\eta_0^2 - \bar{C}_0\eta_0^4)] + O(\varepsilon^{3/2}) \\
\mathcal{C} &= \frac{1}{2}\varepsilon\eta_0^3\operatorname{Im}\bar{\eta}_0[A_1\eta_0^2 + B_1 + C_1\bar{\eta}_0^2 + \mu(A_0\eta_0^2 + \bar{A}_0 - 2\bar{C}_0\eta_0^4)] + O(\varepsilon^{3/2})
\end{aligned}$$

(note that the denominator in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ satisfies $(|\delta|^2 - 1)\sqrt{-\Delta} \sim 4$). One can easily verify that the imaginary part of μ does have no impact in the values of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ as given above. Using this fact, we ultimately obtain the expressions

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\varepsilon\bar{\eta}_0\operatorname{Re}\mu\operatorname{Im}\eta_0(4A_0 - \bar{B}_0) + O(\varepsilon^{3/2}) \\
\mathcal{B} &= 2\varepsilon\eta_0\operatorname{Re}\mu\operatorname{Im}\eta_0(\bar{B}_0 - A_0) + O(\varepsilon^{3/2}) \\
\mathcal{C} &= \frac{1}{2}\varepsilon\eta_0^3\operatorname{Re}\mu\operatorname{Im}\eta_0\bar{B}_0 + O(\varepsilon^{3/2}).
\end{aligned}$$

Assume first that the leading term in $\mathcal{B}$ does not vanish. Then we can perform an appropriate rotation and rescaling in (9) which transforms the system into (12). Replacing in (12) the values we just calculated, we get that this system is a small perturbation of a reversible quadratic system with a center at the origin as described in (i)–(iii).

Assume now that the leading term in the above expansion of $\mathcal{B}$ vanishes. This can happen only in case (III) when A_0 lies on the circle $2 + \operatorname{Re} A_0 - |A_0|^2 = 0$. In this situation, the leading term in $\mathcal{A}$ is not zero and we can perform another linear transformation reducing (9) to (14). Replacing, we get that in this case (14) is a small perturbation of the system in (iv). The theorem is proved.

Remark 3.1 Each one among the systems in (i)–(iv) is remarkable in some sense. The system in (i) presents the unique "quadratic-like-linear center" since its right-hand side can be factorized into $-iZ(1 + 4\operatorname{Im} Z)$. The system in (ii) presents the unique quadratic Hamiltonian center having a triple singularity (a degenerate saddle at $Z = -\frac{3}{8}i$). The system in (iii) forms a codimension-one subset in the two-dimensional space Q_3^R of reversible quadratic systems with a center. It presents the vector fields in Q_3^R with a multiple singularity (at $Z = \frac{2+\operatorname{Re} A_0 - |A_0|^2}{4(|A_0|^2-1)}i$). Taking the real a and b, the coefficients at Z^2 and $\bar{Z}^2$ respectively, be the parameters in Q_3^R, this subset is defined by $a - 3b + 2 = 0$, $a \neq 1$. The system in (iv) presents the unique center in the Lotka-Volterra stratum Q_3^{LV} having a triple singularity (at $Z = \frac{3}{4}i$). See [8, 20] for more details.

3.2 The Special Case $B_0^2 = 4A_0C_0$

This is the case when the original system has a double critical point at infinity which can produce either two different critical points (saddle and focus) or one double critical point near infinity under a small perturbation. Clearly, in this case an additional genericity condition is needed to unfold the eventual double singularity coming from infinity.

We can set without loss of generality $|A_0| = 1$, $B_0 = B = 2$, $C_0 = \bar{A}_0$. Take $e^{i\theta} = \eta_0 + \varepsilon^{1/2}\eta_1 + \varepsilon\eta_2 + \varepsilon^{3/2}\eta_3 + O(\varepsilon^2)$ where $|\eta_0| = 1$, $\operatorname{Re} \bar{\eta}_0\eta_1 = 0$, $\eta_0\bar{\eta}_2 + \eta_1\bar{\eta}_1 + \bar{\eta}_0\eta_2 = 0$, $\operatorname{Re}(\eta_0\bar{\eta}_3 + \eta_1\bar{\eta}_2) = 0$. By (17), in the considered case $A_0\eta_0^2 + 1 = 0$. As above, we take $\eta_1 = ik\eta_0$, with $k \in \mathbb{R}$. We use formula (15) to express $-\Delta$ in the form

$$-\Delta = \frac{|\alpha|^2\left[\left(\operatorname{Im}(Ae^{i\theta} + \bar{C}e^{3i\theta})\right)^2 - |2 + 2Ce^{-2i\theta}|^2\right]}{|Ae^{2i\theta} + 2 + Ce^{-2i\theta}|^2}.$$

Inspecting the sign of the leading term in $-\Delta$ as given above, we see via easy calculation which we omit here that the critical point ζ could be a focus only provided that either a) η_0 is real or b) $k = 0$. An expansion of $e^{i\theta}$ as the one in case b) is possible only for a special kind of perturbations satisfying the restriction $\operatorname{Im} \bar{\alpha}_0\bar{\eta}_0(A_1\eta_0^2 + C_1\bar{\eta}_0^2) = 0$. To avoid the possibility of degeneracy that occurs in case b), we need a second genericity condition on the perturbation:

(H2) $$-\Delta \text{ tends to infinity as } \varepsilon \to 0.$$

It is easy to see that (H2) implies $k \neq 0$. So we will deal with a). Let $k \neq 0$ and η_0 be real (hence, $\eta_0 = \pm 1$). Therefore $A_0 = C_0 = -1$. Having in mind the remark following Theorem 2.1, we call this case (which occurs in (II)) the Bogdanov-Takens case. Below, we are going to show that a conclusion similar to that in Theorem 3.1 holds for this system too.

Theorem 3.2 *Suppose that* $B_0^2 = 4A_0C_0$ *and* ζ *is a focus satisfying* (H1) *and* (H2). *Then the change of variables* (11) *transforms system* (1) *into a system* (12) *which is a small perturbation of:*

(v) *the Bogdanov-Takens Hamiltonian system*

$$\dot{Z} = -iZ - Z^2 + 2Z\bar{Z} - \bar{Z}^2. \tag{18}$$

Proof As in the general case we considered above, we get the expressions

$$k^2 = -\frac{1}{4}\operatorname{Im}\bar{\alpha}_0(A_1 + C_1),\ r \sim \frac{\eta_0}{\varepsilon \operatorname{Im}(A_1 + C_1)},$$
$$-\Delta \sim -\frac{8k}{\varepsilon^{1/2}\operatorname{Im}(A_1 + C_1)}, \quad \Omega = \mathrm{O}(\varepsilon^{\frac{1}{4}}).$$

Next, we obtain $\xi = -i\big(1 - \mu\varepsilon^{1/2} + \mathrm{O}(\varepsilon)\big)$ where $\mu = [2i\operatorname{Re}C_1 - \operatorname{Im}(A_1 + C_1)]/4k$, $\operatorname{Re}\mu > 0$, and finally $\delta = -i[1 + \sqrt{2\operatorname{Re}\mu}\,\varepsilon^{1/4} + \bar{\mu}\varepsilon^{1/2} + \cdots]$. Then

$$4\mathcal{A} = -(\delta + i)(\overline{\delta + i})^2 + \mathrm{O}(\varepsilon) = -i(2\operatorname{Re}\mu)^{3/2}\varepsilon^{3/4} + \mathrm{O}(\varepsilon)$$
$$4\mathcal{B} = 2(\delta + i)^2(\overline{\delta + i}) + \mathrm{O}(\varepsilon) = -2i(2\operatorname{Re}\mu)^{3/2}\varepsilon^{3/4} + \mathrm{O}(\varepsilon)$$
$$4\mathcal{C} = -(\delta + i)^3 + \mathrm{O}(\varepsilon) = -i(2\operatorname{Re}\mu)^{3/2}\varepsilon^{3/4} + \mathrm{O}(\varepsilon).$$

This yields immediately that after appropriate rotation and rescaling system (9) in the Bogdanov-Takens case (II), $C_0 = -1$ reduces to a small perturbation of itself.

Remark 3.2 In the Bogdanov-Takens case above, all the considerations still hold for more general perturbations of the form $\alpha = \alpha_0 + \varepsilon^{1/2}\alpha_1 + \varepsilon\alpha_2 + \cdots$, $A = A_0 + \varepsilon A_1 + \varepsilon^{3/2}A_2 + \cdots$, $C = \bar{A}_0 + \varepsilon C_1 + \varepsilon^{3/2}C_2 + \cdots$, instead of analytical ones. We shall use this remark in the next section to construct a perturbation which is transversal to infinity and has the maximum of possible limit cycles coming from infinity.

Remark 3.3 One can consider case b) too, omitting the genericity condition (H2). This is the case occurring in (III) when $|A_0| = 1$. In general, (9) is then reduced to a small perturbation of a system (having a focus at the origin)

$$\dot{Z} = (\lambda - i)Z + (\sigma Z + \bar{\sigma}\bar{Z})^2, \quad \lambda \in \mathbb{R},\ |\sigma| = 1. \tag{19}$$

Up to an affine change of variables, (19) is the system (I) in the famous Chinese classification [16]. As is well known, such a system can have at most one (hyperbolic) limit cycle. Therefore, the same conclusion holds for any small perturbation of (19). We will not consider this case in more detail here.

Remark 3.4 In the general case when $B_0^2 \neq 4A_0C_0$, condition (H2) is a consequence of (H1).

3.3 Asymptotic Behavior of the Limit Cycles

In this subsection we discuss how the limit cycles around ζ behave when $\varepsilon \to 0$. Let Γ_ε be a limit cycle of the system (12) or (14) which contains the origin in its interior. Denote by γ_ε the pre-image of Γ_ε in the z-plane according to formula (11) or (13), respectively. Clearly, γ_ε surrounds the focus near infinity ζ. Below, we denote by d a positive constant which is independent on ε for $\varepsilon \in (0, \varepsilon_0]$.

Definition 3.1 We say that the limit cycle γ_ε tends *uniformly* to infinity as $\varepsilon \to 0$ if

$$\max_{z\in\gamma_\varepsilon} |z| \leq d \min_{z\in\gamma_\varepsilon} |z|, \quad \varepsilon \in (0, \varepsilon_0]. \tag{20}$$

Evidently, (20) implies that both $\max_{z\in\gamma_\varepsilon} |z|$ and $\min_{z\in\gamma_\varepsilon} |z|$ are $O(\varepsilon^{-1})$.

Theorem 3.3 *Let ζ be a focus satisfying* (H1), (H2) *and γ_ε a limit cycle surrounding it. Then γ_ε tends uniformly to infinity as $\varepsilon \to 0$ if and only if its image Γ_ε tends to a center, to a periodic orbit or to a homoclinic loop through a hyperbolic saddle.*

Proof Let us first describe the geometry of the period annulus around the center at $Z = 0$ in (i)–(v). It is symmetric with respect to $\operatorname{Re} Z$. The period annulus is bounded in cases (i), (ii), (v) and (iii), when $|A_0| < 1$ or $|A_0 - 1| > 2$. It is unbounded, with respect to both of the directions $\operatorname{Re} Z$ and $\operatorname{Im} Z$, in (iv) and the remaining cases of (iii). In the bounded cases, the period annulus is surrounded by a homoclinic loop through a hyperbolic saddle in case (v) and through a degenerate singularity S_0 in the other cases. We will consider the above facts as known. They can easily be verified by using the explicit formulas for the first integrals of the systems in (i)–(v), see *e.g.* [20, 8] for details.

To prove the theorem, we use formulas (11), (13) and the asymptotics of $\mathcal{A}$, $\mathcal{B}$, δ, ζ derived above. The assertion in the theorem is obvious for the Bogdanov-Takens case (v). This is so because, by formula (11), $z = \zeta\big(1 + O(\varepsilon^{1/4})\big)$ for $\varepsilon \in (0, \varepsilon_0]$ on any compact subset in the Z-plane. Take now the general case $B_0^2 \neq 4A_0C_0$. We will consider system (12) (the treatment of (14) is similar). We saw above that (12) is a small perturbation of a special reversible system with a center, which we write (in order to consider cases (i), (ii), (iii) simultaneously) in a general form

$$\dot{Z} = -iZ + \frac{\operatorname{Im} \eta_0(4A_0 - \bar{B}_0)}{2 \operatorname{Im} \eta_0(\bar{B}_0 - A_0)} Z^2 + 2Z\bar{Z} + \frac{\operatorname{Im} \eta_0 \bar{B}_0}{2 \operatorname{Im} \eta_0(\bar{B}_0 - A_0)} \bar{Z}^2. \tag{21}$$

Denote by Γ_0 either the center at the origin or an oval from the period annulus around it in (21). We recall the reader that an oval is a simple closed curve which

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is free of critical points. Let Γ_ε be a limit cycle of (12) which tends to Γ_0 when $\varepsilon \to 0$. Take a point $Z \in \Gamma_\varepsilon$ having the point $Z_0 \in \Gamma_0$ as a limit when $\varepsilon \to 0$ and denote by $z \in \gamma_\varepsilon$ its pre-image according to formula (11). Using the first terms in the asymptotics of $\mathcal{A}$, $\mathcal{B}$, *etc.*, we easily calculate

$$z = \frac{2\eta_0}{\varepsilon \operatorname{Re} \mu \operatorname{Im} \eta_0(\bar{B}_0 - A_0)}[\sigma_0 - \operatorname{Im} Z_0 + \mathrm{o}(1)], \quad \sigma_0 = \frac{\operatorname{Im} \eta_0(\bar{B}_0 - A_0)}{2 \operatorname{Im}(2A_0\eta_0 + B_0\bar{\eta}_0)}.$$

Therefore, we need to prove that $\operatorname{Im} Z_0 \neq \sigma_0$ for $Z_0 \in \Gamma_0$. On the other hand, by using (21) we obtain: $\dot{Z} - \dot{\bar{Z}} = 2i(d/dt) \operatorname{Im} Z = 2i \operatorname{Re} Z(-1 + \sigma_0^{-1} \operatorname{Im} Z)$. Hence, $L = \{Z : \operatorname{Im} Z = \sigma_0\}$ is an invariant line of (21). This means that $\Gamma_0 \cap L = \varnothing$ which proves the assertion that γ_ε tends to infinity uniformly.

Assume next that Γ_0 is the separatrix contour surrounding the period annulus and Γ_ε tends to Γ_0 as $\varepsilon \to 0$. If Γ_0 is bounded, we choose a point $Z \in \Gamma_\varepsilon$ to tend to the degenerate equilibrium $S_0 \in \Gamma_0$ as $\varepsilon \to 0$. Since $S_0 = i\sigma_0$ lies on the invariant line L, by the calculations above we have $\min_{z\in\gamma_\varepsilon} |z| = \mathrm{o}(\varepsilon^{-1})$. At the same time, $\max_{z\in\gamma_\varepsilon} |z| \geq |\zeta| = \mathrm{O}(\varepsilon^{-1})$, hence γ_ε does not tend to infinity uniformly. Finally, if Γ_0 is unbounded, then we choose $Z \in \Gamma_\varepsilon$ with $|\operatorname{Im} Z| \to \infty$. Using again the calculations above, we get $\max_{z\in\gamma_\varepsilon} |z| \gg |\zeta| \geq \min_{z\in\gamma_\varepsilon} |z|$ and therefore γ_ε does not tend uniformly to infinity, too. The proof is complete.

3.4 Multiparameter Perturbations

It can easily be seen [8] that any small quadratic perturbation of (I)–(III) can be reduced respectively to

$$\dot{z} = (\lambda_1 - i)z + z^2 + (\lambda_2 + i\lambda_3)|z|^2 + (-1 + \lambda_4 + i\lambda_5)\bar{z}^2,$$

$$\dot{z} = (\lambda_1 - i)z + (-1 + \lambda_2 + i\lambda_3)z^2 + 2|z|^2 + (C_0 + \lambda_4 + i\lambda_5)\bar{z}^2,$$

$$\dot{z} = (\lambda_1 - i)z + (A_0 + \lambda_2 + i\lambda_3)z^2 + 2|z|^2 + (C_0 + \lambda_4 + i\lambda_5)\bar{z}^2, \quad A_0 \neq -1$$

where C_0 is the same as in (II), (III) respectively, λ_k are independent real parameters and $|\lambda_2|+|\lambda_3|+|\lambda_4|+|\lambda_5| \ll 1$. We can introduce a small positive parameter ε in the following way. In case (I), take $\varepsilon = (\lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2)^{1/2}$ and rewrite the equation as

$$\dot{z} = (\lambda_1 - i)z + z^2 + \varepsilon(\lambda_2 + i\lambda_3)|z|^2 + [-1 + \varepsilon(\lambda_4 + i\lambda_5)]\bar{z}^2,$$

where $\lambda = (\lambda_2, \lambda_3, \lambda_4, \lambda_5)$ lies on the unit sphere $\mathbf{S}^3$. In case (II), we can always find a nearby value $\tilde{C}_0 \in \mathbb{C}$ and a small $\tilde{\lambda}_4 \in \mathbb{R}$ so that $2|\tilde{C}_0 + 1| = |\tilde{C}_0|^2 - 1$ and either $C_0 + \lambda_4 + i\lambda_5 = \tilde{C}_0 + \tilde{\lambda}_4$ or $C_0 + \lambda_4 + i\lambda_5 = \tilde{C}_0 + i\tilde{\lambda}_4$. This fact becomes geometrically evident if one take into account the form of the curve $2|C + 1| = |C|^2 - 1$ in the complex plane; the precise analytic proof can be done by using the implicit function theorem. For example, the first possibility occurs whenever $C_0 \neq (1 \pm 3i\sqrt{3})/2$ while the second one occurs provided $C_0 \neq -1, 3, (-3 \pm i\sqrt{3})/2$. Omitting the tildes and taking $\varepsilon = (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)^{1/2}$, where $(\lambda_2, \lambda_3, \lambda_4) \in \mathbf{S}^2$, we obtain in case (II) the equation

$$\dot{z} = (\lambda_1 - i)z + [-1 + \varepsilon(\lambda_2 + i\lambda_3)]z^2 + 2|z|^2 + (C_0 + \varepsilon k\lambda_4)\bar{z}^2, \quad k = 1, i.$$

Similarly, in case (III), we take $\tilde{A}_0 = A_0 + \lambda_2 + i\lambda_3$, determine $\tilde{C}_0$ by the corresponding formula and $\tilde{\lambda}_4$, $\tilde{\lambda}_5$ from $C_0 + \lambda_4 + i\lambda_5 = \tilde{C}_0 + \tilde{\lambda}_4 + i\tilde{\lambda}_5$. Omitting the tildes and taking $\varepsilon = (\lambda_4^2 + \lambda_5^2)^{1/2}$, where $(\lambda_4, \lambda_5) \in \mathbf{S}^1$, we obtain in case (III) the equation

$$\dot{z} = (\lambda_1 - i)z + A_0 z^2 + 2|z|^2 + [C_0 + \varepsilon(\lambda_4 + i\lambda_5)]\bar{z}^2, \quad A_0 \neq -1.$$

In all cases above, for a suitable choice of the perturbation parameters λ, the system has a focus which goes to infinity for $\varepsilon = 0$. Hence, the considerations from the preceding section is applicable to the case of an arbitrary multi-parameter perturbation of the initial equation too.

4 Limit Cycles Appearing From Infinity in Four Famous Examples

In this section we prove that at most two "uniform" limit cycles can appear around a focus ζ coming from infinity in four well-known systems: the Bogdanov-Takens system, the isochronous center $\mathcal{S}_3$, the reversible Lotka-Volterra center (I) and the Hamiltonian system with a center and two saddles (II), $C_0 \neq -1$.

4.1 The Bogdanov-Takens System

We already noted the interesting self-duality of the Bogdanov-Takens system (Theorem 3.2). For this case, it is well known that at most two limit cycles can appear in the finite part of the plane if the original vector field undergoes a small quadratic perturbation, and this is the exact upper bound. Then, by the general result of Zegeling and Kooij [17], the total number of limit cycles (around both of the foci in (1), placed respectively near the origin and near infinity) will not exceed two. It remains to give an example of a perturbation that produces just two limit cycles around the focus near infinity. Take a small positive ε and consider the following particular quadratic perturbation:

$$\dot{z} = (\lambda_0 - i + \lambda_1\varepsilon^{1/2})z + \left[-1 - \frac{1}{2}\lambda_0(\lambda_3 + \lambda_5)\varepsilon + i\lambda_3\varepsilon\right] z^2 + 2|z|^2 + (-1 + i\lambda_5\varepsilon)\bar{z}^2$$

where λ_j are arbitrary real parameters (independent on ε). The special form of $\operatorname{Re} A_1$ is taken because we need the annihilation of the leading term of Ω in (12) (which is $O(\varepsilon^{1/4})$ in general) in order to obtain a perturbation with two limit cycles. We next calculate the coefficients in (12). This is done by long but direct calculations which we omit here. One obtains

$$\dot{Z} = (-i+\mu_1\varepsilon^{3/4}+\cdots)Z+(-1+\mu_2\varepsilon^{1/2}+i\mu_3\varepsilon^{3/4}+\cdots)Z^2+2|Z|^2+(-1+i\mu_5\varepsilon^{1/4}+\cdots)\bar{Z}^2$$

where by dots we denote the upper order terms and μ_j are given by

$$\mu_1 = \left(\lambda_1\mu + \frac{1}{2}\lambda_3\right)(2\mu)^{-1/2},$$

$$\mu_2 = 2\lambda_0^2\mu,$$

$$\mu_3 = -2\big(\lambda_3 + \lambda_0^2(\lambda_3 + \lambda_5)\big)\,(2\mu)^{-1/2},$$
$$\mu_5 = 4\lambda_0(2\mu)^{1/2},$$
$$\mu = [-(\lambda_3 + \lambda_5)/(2\lambda_0)]^{1/2}.$$

Below we are going to prove that the displacement function related to this perturbation takes the form

$$d(h,\varepsilon) = \varepsilon^{3/4}[\mu_1 I_1(h) + \mu_3 I_2(h) + \mu_2\mu_5 I_3(h)] + \mathrm{O}(\varepsilon)$$

where $I_j(h)$, $j = 1, 2, 3$ are linearly independent Abelian integrals. Unlike the usual series in ε, in our case we need an expansion of $d(h,\varepsilon)$ in the powers of $\varepsilon^{1/4}$. For this, we can use some of the algorithms for calculating the higher order Melnikov functions M_j, see [7, 8, 14]. We rewrite the complex equation for $Z = x + iy$ as a Pfaffian system in the real plane (x, y),

$$dH = \varepsilon^{1/4}\omega_1 + \varepsilon^{1/2}\omega_2 + \varepsilon^{3/4}\omega_3 + \cdots, \quad H = \frac{1}{2}(x^2 + y^2) + \frac{4}{3}y^3$$

and then verify that coefficients $M_j(h)$ at $\varepsilon^{j/4}$ in the expansion of $d(h,\varepsilon)$ are respectively $M_1 = M_2 \equiv 0$,

$$M_3(h) = \oint_{H=h}\Big[-2\mu_1 + 4\mu_3 y + 4\mu_2\mu_5\Big(y^2 - \frac{1}{3}x^2\Big)\Big]\,x\,dy$$

(we omit the details). Denoting $J_k(h) = \oint_{H=h} y^k x\,dy$, a further calculation yields

$$M_3(h) = -\Big(2\mu_1 + \frac{24}{11}\mu_2\mu_5 h\Big)\,J_0(h) + \Big(4\mu_3 - \frac{12}{11}\mu_2\mu_5\Big)\,J_1(h), \quad h \in \Big[0, \frac{1}{96}\Big].$$

As is well known [13], this function can have two zeros for appropriate choice of the independent constants μ_j, which yields a quadratic perturbation in (12) with just two limit cycles around $Z = 0$, that is a perturbation in (1) that produces just two limit cycles around the focus coming from infinity. To summarize, the following theorem is proved.

Theorem 4.1 *For any sufficiently small quadratic perturbation of the Bogdanov-Takens system* (18), *the total number of limit cycles which bifurcate around both the finite focus and the focus near infinity ζ satisfying* (H1), (H2), *is two. There exists a quadratic perturbation which produces exactly two limit cycles coming from infinity.*

Remark 4.1 The above theorem has an interesting pre-history. First, Petrov [12, 13] proved that the open period annulus of the Bogdanov-Takens system can produce, after a small generic quadratic perturbation, at most two limit cycles. Later P. Mardešić [11] generalized this by including the center and the separatrix loop. Further Bao-yi Li and Zhi-fen Zhang [10] removed the genericity condition on the perturbation. Thus the problem was solved for the finite plane. The above theorem gives the exact upper number of the limit cycles on the *whole* plane for *generic* perturbations.

4.2 The Isochronous Center $\mathcal{S}_3$

Consider next the case of the isochronous center $\mathcal{S}_3$ which is presented by an equation $\dot{z} = -iz + 5z^2 + 2|z|^2 - 3\bar{z}^2$ in case (III) of Theorem 2.1. From the analysis in Section 3 above, the problem for the bifurcation of limit cycles that escape uniformly to infinity reduces for this case to the problem of limit cycles in the finite plane obtained from small quadratic perturbations of the following reversible center in case (iii):

$$\dot{Z} = -iZ - 3Z^2 + 2|Z|^2 - \frac{1}{3}\bar{Z}^2. \tag{22}$$

In Cartesian coordinates, system (22) has a first integral

$$H(X,Y) = X^{-1/2}\Big[\frac{1}{2}Y^2 + \frac{3}{128}(X^2 - 3X)\Big]$$

where $Z = Y + i\frac{3}{16}(X-1)$, see page 156 in [8]. The corresponding bifurcation function which zeroes determine limit cycles is given in [8], page 116, by a formula

$$I(h) = \iint_{H(X,Y)<h} X^{-3/2}(\mu_1 + \mu_2 X + \mu_3 X^{-1})\, dX\, dY, \quad h \in \Big(-\frac{3}{64}, 0\Big). \tag{23}$$

Before stating our results, we will give the problem a more convenient equivalent form. We perform in (23) a change of the variables $X = x^2$, $Y = \frac{\sqrt{3}}{8}xy$ which yields an equivalent bifurcation function

$$I(h) = \iint_{H(x,y)<h} (\mu_2 x + \mu_1 x^{-1} + \mu_3 x^{-3})\, dx\, dy, \quad h \in (-2, 0) \tag{24}$$

related to the reversible Hamiltonian $H(x,y) = x(y^2 + x^2 - 3)$. For k integer, denote $I_k(h) = \iint_{H<h} x^k\, dx\, dy$. Then we can rewrite (24) as

$$I(h) = \mu_1 I_{-1}(h) + \mu_2 I_1(h) + \mu_3 I_{-3}(h). \tag{25}$$

Using formula (1.4) in [7], we express the integral I_0 as $I_0 = (3/h)I_1 - (5h/4)I_{-3}$. On the other hand, the three integrals I_k, $k = -1, 0, 1$ satisfy a system (see (1.5) in [7])

$$\begin{aligned} 3hI'_{-1} + 6I'_0 &= I_{-1}, \\ \frac{3}{2}hI'_0 + 3I'_1 &= I_0, \\ -hI'_{-1} + hI'_1 &= I_1. \end{aligned}$$

Replacing I_0, we obtain after an easy manipulation a related system for I_k, $k = 1, -1, -3$ of the form

$$\begin{aligned} -3h^2 I'_{-3} + 12I'_{-1} &= hI_{-3}, \\ 15h^2 I'_{-3} + (3h^2 - 72)I'_{-1} &= hI_{-1}, \\ -hI'_{-1} + hI'_1 &= I_1. \end{aligned} \tag{26}$$

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Using the latter system, we next obtain

$$hI'(h) - I(h) = \frac{F(h)}{3h^2 - 12},$$

where $F(h) = [(5\mu_1 + 5\mu_2 - 4\mu_3)h^2 + 36\mu_3]I_{-3} + [(\mu_2 - 2\mu_1)h^2 + 12\mu_1 + 4\mu_3]I_{-1}$. Solving the equation with respect to I we get the representation

$$I(h) = h\int_{-2}^{h} \frac{F(s)\,ds}{3s^2(s^2 - 4)} \tag{27}$$

(we used the fact that $I(-2) = F(-2) = 0$). By (27), the number of zeroes of $I(h)$ in $(-2, 0)$ does not exceed the number of zeroes of $F(h)$ in the same interval. To study this problem, it is useful to change the variables by setting $s = h^2$, $J_k(s) = I_k(h)$, $G(s) = F(h)$ and consider the number of zeros of G in $(0, 4)$. The first two equations in (26) become

$$\begin{aligned} -6sJ'_{-3} + 24J'_{-1} &= J_{-3}, \\ 30sJ'_{-3} + (6s - 144)J'_{-1} &= J_{-1}, \end{aligned} \tag{28}$$

and the ratio $R(s) = J_{-3}(s)/J_{-1}(s)$ satisfies the Riccati equation and related system

$$6s(s-4)R' = -5sR^2 + (24 - 2s)R + 4 \qquad \begin{aligned} \dot{s} &= s(s-4) \\ \dot{R} &= -\tfrac{5}{6}sR^2 + (4 - \tfrac{1}{3}s)R + \tfrac{2}{3}. \end{aligned} \tag{29}$$

Evidently, the problem of the number of real roots of

$$G(s) = [(5\mu_1 + 5\mu_2 - 4\mu_3)s + 36\mu_3]J_{-3}(s) + [(\mu_2 - 2\mu_1)s + 12\mu_1 + 4\mu_3]J_{-1}(s) \tag{30}$$

reduces to finding the number of intersection points between the graph of the function $R(s)$ for $s \in (0, 4)$ and the hyperbola

$$r(s) = -\frac{(\mu_2 - 2\mu_1)s + 12\mu_1 + 4\mu_3}{(5\mu_1 + 5\mu_2 - 4\mu_3)s + 36\mu_3}.$$

Instead of such a kind of geometric proof, we prefer to give below a different proof following the ideas of [3]. Namely, we shall show that the function $F(h)$ can have at most three zeros in the left complex half-plane $\{h \in \mathbb{C} : \operatorname{Re} h < 0\}$ one of them being always $h = -2$. As

$$\frac{d}{dh}\left(\frac{I(h)}{h}\right) = \frac{F(h)}{h^2(3h^2 - 12)}, \quad F(-2) = 0,$$

then this would mean that the vector space of complex analytic functions

$$\frac{d}{dh}\left(\frac{I_{-3}(h)}{h}\right), \quad \frac{d}{dh}\left(\frac{I_{-1}(h)}{h}\right), \quad \frac{d}{dh}\left(\frac{I_{1}(h)}{h}\right)$$

is Chebyshev in the above half-plane. From now on we consider the variable $s = h^2$ where $\operatorname{Re} h < 0$ and put $\mathcal{D} = \mathbb{C} \setminus [0, -\infty)$, $J_k(s) = I_k(h)$. We begin with

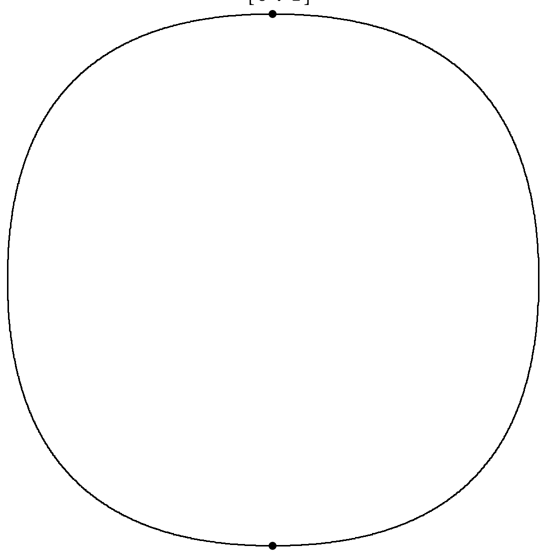

Figure 1: The projective space $\mathbb{RP}^1$ and the bifurcation set of the zeros of the function $J(s) = \mu_1 J_{-1}(s) + \mu_2 J_{-3}(s)$

Theorem 4.2 *The Abelian integral*

$$J(s) = \mu_1 J_{-1}(s) + \mu_2 J_{-3}, \quad \mu_1, \mu_2 \in \mathbb{R},\ \mu_1^2 + \mu_2^2 \neq 0$$

is an analytic function in the complex domain $\mathcal{D}$ which vanishes at $s = 4$. It has exactly two zeros if $\mu_1\mu_2 < 0$, and it has exactly one zero (at $s = 4$) if $\mu_1\mu_2 \geq 0$.

Proof It consists of three steps.

1. First, we determine the bifurcation set $\mathbf{B} \subset \mathbb{RP}^1$ (see Figure 4.2) of zeros of the function $J(s)$ in the complex domain $\mathcal{D}$.
2. Second, we determine the number of the zeros of $J(s)$ for some suitable fixed value of μ_1 and μ_2.
3. Third, we describe the bifurcation of zeros when the parameters μ_1 and μ_2 cross transversally the bifurcation set.

We begin now with Step 1. The vector (μ_1, μ_2) represents also a point (denoted $[\mu_1 : \mu_2]$) on the real projective plane $\mathbb{RP}^1$. We shall say that a point $[\mu_1^0 : \mu_2^0] \in \mathbb{RP}^1$ belongs to the bifurcation set of the zeros of $J(s)$, if a zero of this function tends to the boundary of $\mathcal{D}$ when $[\mu_1 : \mu_2] \in \mathbb{RP}^1$ tends to $[\mu_1^0 : \mu_2^0]$ in a suitable way. The

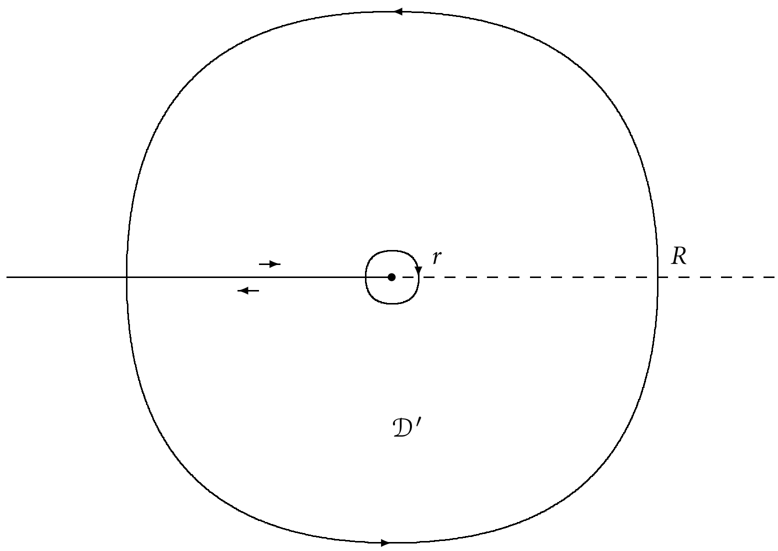

Figure 2: The complex domain $\mathcal{D}'$

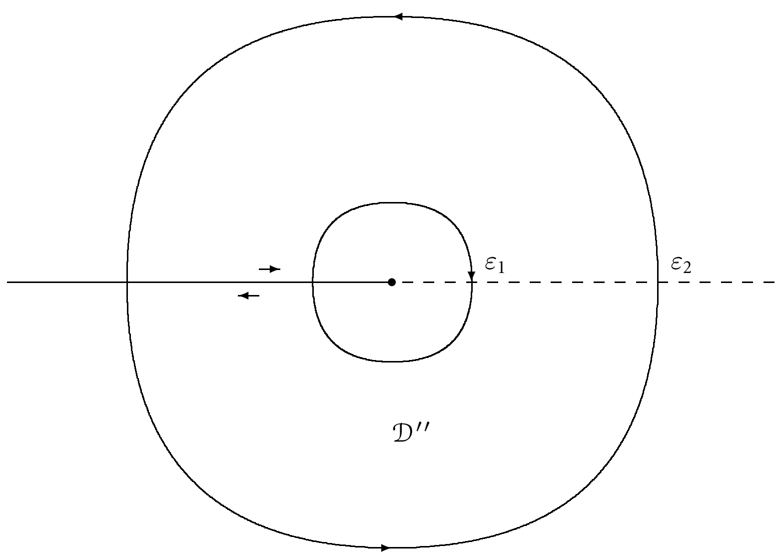

Figure 3: The complex domain $\mathcal{D}''$

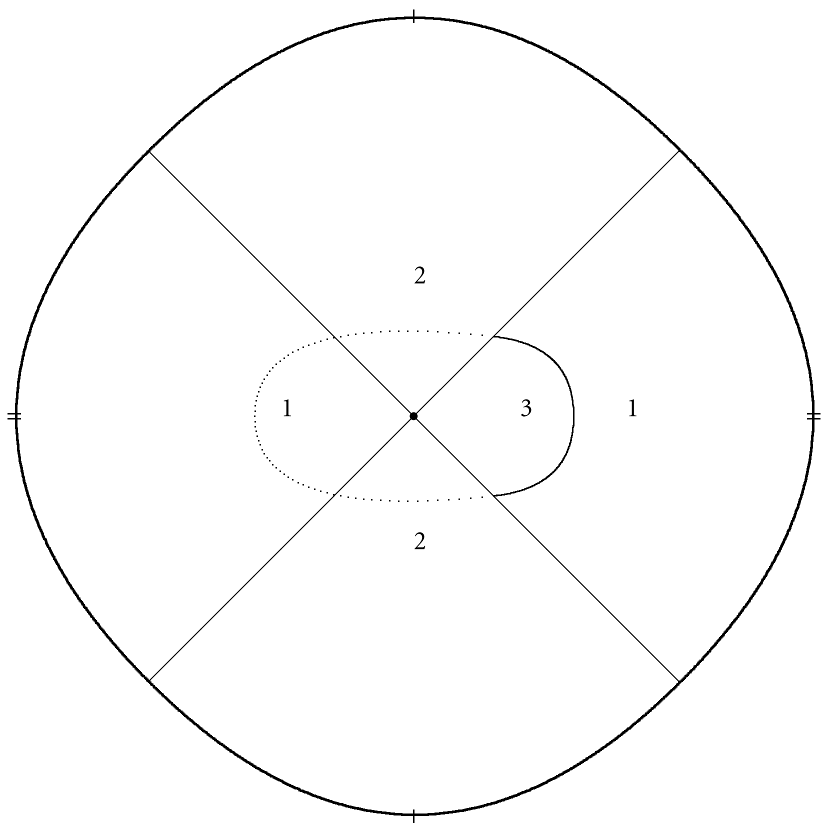

Figure 4: The complex domain $\mathcal{D}''$

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system (28) can be written in the following equivalent form

$$\frac{d}{ds}\begin{pmatrix} J_{-3} \\ J_{-1} \end{pmatrix} = \frac{A_0}{s}\begin{pmatrix} J_{-3} \\ J_{-1} \end{pmatrix} + \frac{A_4}{s-4}\begin{pmatrix} J_{-3} \\ J_{-1} \end{pmatrix} \tag{31}$$

where $A_0 = \begin{pmatrix} -1 & -\frac{1}{12} \\ 0 & 0 \end{pmatrix}$ and $A_4 = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}$. Therefore it is of Fuchs type and has three singular points $s_1 = 0, s_2 = 4, s_3 = \infty$. In a neighborhood of the singular point s_i the system has a solution $\mathbf{J}(s)$ with an asymptotic expansion

$$\mathbf{J}(s) = \mathbf{J}_0(s - s_i)^\sigma + o\big((s - s_i)^\sigma\big).$$

After substituting $\mathbf{J}(s)$ in (31) and equating the leading terms we obtain the so called *indicial equation* which must be satisfied by σ. A straightforward computation shows that the indicial equations associated to $s = 0, 4, \infty$ are

$$\sigma(\sigma+1) = 0, \quad \sigma(\sigma-1) = 0, \quad \Big(\sigma - \frac{1}{6}\Big)\Big(\sigma + \frac{1}{6}\Big) = 0,$$

respectively. From this we deduce that in a neighborhood of $s = 4$ the system (31) has an analytic solution $\mathbf{J}(s)$, where $\mathbf{J}(0) = (0, 0)$, as well a second solution with a logarithmic singularity. It follows that the vector function $\big(J_{-3}(s), J_{-1}(s)\big)^\top$, $s \in \mathcal{D}$ is, up to multiplication by a non-zero constant, the only solution of (31) which is analytic in a neighborhood of $s = 4$. Further we compute the leading terms in the expansions of this solution in a neighborhood of $s = 0$ and $s = \infty$. The system (31) implies that $y(s) = J_{-1}(4s)$ satisfies the Gauss hypergeometric equation

$$s(1-s)y'' + (1-s)y' + y/36 = 0. \tag{32}$$

As before we associate to each singular point $s = 0, 1, \infty$ an indicial equation which reads

$$\sigma^2 = 0, \quad \sigma(\sigma-1) = 0, \quad \Big(\sigma - \frac{1}{6}\Big)\Big(\sigma + \frac{1}{6}\Big) = 0,$$

respectively. According to the general theory of the Gauss hypergeometric equation [9, p. 162] we have

$$J_{-1}(s) = \text{const} \times (4-s) F\Big(\frac{5}{6}, \frac{7}{6}; 2; 1 - \frac{s}{4}\Big)$$

where $F(a, b; c; s)$ is the usual Gauss hypergeometric function. Similarly $x(s) = J_{-3}(4s)$ satisfies

$$s(1-s)x'' + 2(1-s)x' + 5x/36 = 0 \tag{33}$$

with indicial equations associated to $0, 1, \infty$

$$\sigma(\sigma+1) = 0, \quad \sigma(\sigma-1) = 0, \quad \Big(\sigma - \frac{1}{6}\Big)\Big(\sigma - \frac{5}{6}\Big) = 0,$$

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respectively, which implies

$$J_{-3}(s) = \text{const} \times (4-s) F\Big(\frac{7}{6}, \frac{11}{6}; 2; 1 - \frac{s}{4}\Big).$$

The integral formula [9, p. 196]

$$F(a,b;c;s) = \text{const} \times \int_0^1 (1-st)^{-a} t^{b-1} (1-t)^{c-b-1}\, dt$$

is valid for $c > b > a$, $b > 0$ and $s \in (-\infty, 1)$ which implies $F(a,b;c;s) \overset{\infty}{\sim} c^\infty s^{-a}$ and hence

$$J_{-1}(s) \overset{\infty}{\sim} c^\infty_{-1} s^{1/6},\ J_{-3}(s) \overset{\infty}{\sim} c^\infty_{-3} s^{-1/6}, \quad c^\infty_{-1}, c^\infty_{-3} \neq 0. \tag{34}$$

The indicial equations associated to (33), (32) show that in a neighborhood of $s = 0$ the above solutions contain logarithmic singularities (otherwise they would be single-valued on $\mathbb{C}$, and hence rational!) and therefore

$$J_{-1}(s) \overset{0}{\sim} c^0_{-1} \log(s),\ J_{-3}(s) \overset{0}{\sim} c^0_{-3} s^{-1}, \quad c^0_{-1}, c^0_{-3} \neq 0. \tag{35}$$

We conclude that if a zero of $J(s) = \mu_1 J_{-1}(s) + \mu_2 J_{-3}(s)$ bifurcates from $s = \infty$ ($s = 0$) then $\mu_1 = 0$ ($\mu_2 = 0$). Finally we note that no zero of $J(s)$ can bifurcate from a point $s \in (0, -\infty)$. Indeed, if such a point s^0 exists, then there exist $\mu^0_1, \mu^0_2 \in \mathbb{R}$, such that

$$J(s^0) = \mu^0_1 J_{-1}(s^0) + \mu^0_2 J_{-3}(s^0) = 0.$$

The imaginary and the real part of $\big(J_{-1}(s), J_{-3}(s)\big)$ restricted to $(0, -\infty)$ form, however, a fundamental system of solutions of the system (31) which shows that $\mu^0_1 = \mu^0_2 = 0$. We conclude that the bifurcation set $\mathbf{B}$ consists of two points

$$\mathbf{B} = \{[1:0]\} \cup \{[0:1]\} \subset \mathbb{RP}^1.$$

The second step is to check that the function $J(s) = J_{-1}(s)$, corresponding to the point $[1:0]$ on the bifurcation set, does not vanish in $\mathcal{D} \setminus \{4\}$. For this we use the argument principle. Let R and $1/r$ be big enough constants. Denote by $\mathcal{D}'$ the set $\mathcal{D} \cap \{|s| < R\} \cap \{|s| > r\}$ shown on Figure 4.2. Consider the increase of the argument of $J_{-1}(s)$ along the boundary of $\mathcal{D}'$. By (34) along the circle $\{|s| = R\}$ it increases by $2\pi/6$. By (35) along the circle $\{|s| = r\}$ the imaginary part of $J_{-1}(s)$ vanishes exactly once (at $s = r$). Finally on the interval $(-\infty, 0)$ the imaginary part of $J_{-1}(s)$ is the solution of (32) which is analytic in a neighborhood of $s = 0$. Therefore

$$\operatorname{Im} J_{-1}(s) = \text{const} \times F\Big(-\frac{1}{6}, \frac{1}{6}; 1; \frac{1}{4}s\Big) = \text{const} \times \int_0^1 \Big(1 - \frac{1}{4}st\Big)^{-\frac{1}{6}} t^{-\frac{5}{6}} (1-t)^{-\frac{1}{6}}\, dt$$

which does not vanish on the interval $(-\infty, 0)$. We conclude that the increment of the argument of $J_{-1}(s)$ along the boundary of $\mathcal{D}'$ is less than $2\pi + 2\pi/6$ and hence $J_{-1}(s)$ has exactly one zero in $\mathcal{D}$ (at $s = 4$).

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The last step will be to analyze the bifurcation of zeros of $J(s) = \mu_1 J_{-1}(s) + \mu_2 J_{-3}(s)$ in the complex domain $\mathcal{D}$, as the parameter $[\mu_1 : \mu_2] \in \mathbb{RP}^1$ crosses with non-zero velocity the point $[1, 0]$. According to Step 1 such a zero can bifurcate only from $s = 0$ and in a neighborhood of this point we have

$$J_{-1}(s) + \varepsilon J_{-3}(s) = \varepsilon \frac{c^0_{-3}}{s} + (c^0_{-1} + \varepsilon \tilde{c}^0_{-3}) \log(s) + \mathrm{O}(1), \quad \tilde{c}^0_{-3} = \text{const}\,.$$

Let $\varepsilon, \varepsilon_1, \varepsilon_2$ be real constants, such that $0 < \varepsilon_1 \ll |\varepsilon| \ll \varepsilon_2 \ll 1$ where the precise meaning of this inequality is given below. We shall compute the precise number of the zeros of $s\big(J_{-1}(s) + \varepsilon J_{-3}(s)\big)$ in the complex domain $\mathcal{D} \cap \{|s| < \varepsilon_2\}$, where ε_2 is fixed and $|\varepsilon|$ is sufficiently small (with respect to ε_2). Clearly this number equals the number of the zeros in the complex domain

$$\mathcal{D}'' = \mathcal{D} \cap \{|s| < \varepsilon_2\} \cap \{|s| > \varepsilon_1\}$$

shown on Figure 4.2 where ε_1 is sufficiently small (with respect to $|\varepsilon|$). To compute the zeros of $J_{-1}(s) + \varepsilon J_{-3}(s)$ in $\mathcal{D}''$ we shall use the argument principle. Namely, consider the embedding of the boundary $\partial\mathcal{D}''$ in $\mathbb{C}$ by the map

$$s \to w = J_{-1}(s) + \varepsilon J_{-3}(s).$$

Then the number of the crossings of the embedded boundary $\partial\mathcal{D}''$ with the real axis $\{w : \operatorname{Im} w = 0\}$, where the crossings are counted with signs and multiplicities, is twice as big as the number of the zeros of $J_{-1}(s) + \varepsilon J_{-3}(s)$ in $\mathcal{D}''$ (see [3]).

- Let us suppose that ε_2 is so small that the imaginary part of $J_{-1}(s)$ vanishes exactly once along the circle $\{|s| = \varepsilon_2\}$ at $s = \varepsilon_2$ (denoted $\varepsilon_2 \ll 1$). We note also that when running the circle

$$\{s = \varepsilon_2 e^{i\varphi} : -\pi \le \varphi \le \pi\}$$

in a positive direction, then $J_{-1}(s)$ crosses the real axes at the point $J_{-1}(\varepsilon_2)$ in a *negative* direction.

- Let $|\varepsilon|$ be so small with respect to ε_2 that the above remains true for the linear combination $J_{-1}(s) + \varepsilon J_{-3}(s)$ (we denote this $|\varepsilon| \ll \varepsilon_2$).
- Finally, let $\varepsilon_1 > 0$ be so small with respect to $|\varepsilon|$, that the increase of the argument $J_{-1}(s) + \varepsilon J_{-3}(s)$ along the circle $\{|s| = \varepsilon_1\}$ is close to -2π (denoted $\varepsilon_1 \ll |\varepsilon|$). This means that the imaginary part of $J_{-1}(s) + \varepsilon J_{-3}(s)$ vanishes at most three times along $\{|s| = \varepsilon_1\}$.

Therefore the embedded boundary $\mathcal{D}''$ crosses at most four times the real axes, and one of the crossings is always in a negative direction. We conclude that $J_{-1}(s) + \varepsilon J_{-3}(s)$ has at most one zero in $\mathcal{D}''$. More precisely, if the sign of $J_{-1}(s) + \varepsilon J_{-3}(s)$ at the points $s = \varepsilon_1$ and $s = \varepsilon_2$ is one and the same, then this function has no zeros in $\mathcal{D}''$. The last is equivalent to $\varepsilon c^0_{-3} c^0_{-1} < 0$. In the opposite case $\varepsilon c^0_{-3} c^0_{-1} > 0$ the function $J_{-1}(s) + \varepsilon J_{-3}(s)$ has exactly one zero in $\mathcal{D}''$, and hence in $\mathcal{D} \cap \{|s| < \varepsilon_2\}$. From geometric consideration it is clear that for small positive s the functions $J_{-1}(s)$

and $J_{-3}(s)$ have the same sign which implies $c^0_{-3}c^0_{-1} < 0$. Therefore for small positive ε the function $J_{-1}(s) + \varepsilon J_{-3}(s)$ has no zeros and for small negative ε it has exactly one zero in $\mathcal{D} \cap \{|s| < \varepsilon_2\}$.

To resume, we proved that when the parameter ε crosses the bifurcation set $\varepsilon = 0$ with a strictly negative velocity, then a simple zero of $J_{-1}(s) + \varepsilon J_{-3}(s)$ bifurcates from $s = 0$ in the complex domain $\mathcal{D}$. This implies Theorem 4.2.

The next theorem will be proved along the same lines.

Theorem 4.3 *The function* (30) *where* $(\mu_1, \mu_2, \mu_3) \neq (0, 0, 0)$, $\mu_i \in \mathbb{R}$, *is analytic in the complex domain* $\mathcal{D}$ *and vanishes at* $s = 4$. *The bifurcations set* $\mathbf{B} \subset \mathbb{RP}^2$ *of the zeros of* $G(s)$ *in* $\mathcal{D}$ *is an union of two lines and a segment (a piece of the real quadric* (36)) *joining them as on Figure 4.2. In each of the connected components of* $\mathbb{RP}^2 \setminus \mathbf{B}$ *the function* $G(s)$ *has exactly one, two, or three zeros, as it is shown on Figure 4.2.*

Proof A vector $(\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$ represents a point $[\mu_1 : \mu_2 : \mu_3]$ on the projective plane $\mathbb{RP}^2$. The proof of Theorem 4.2 shows that if a zero bifurcates from $s = 0$ in the complex domain $\mathcal{D}$, then the point $[\mu_1 : \mu_2 : \mu_3]$ belongs to the line $\mu_3 = 0$, and if a zero bifurcates from $s = \infty$, then the point $[\mu_1 : \mu_2 : \mu_3]$ belongs to the line $\mu_2 - 2\mu_1 = 0$. Finally if a zero bifurcates from a point $s \in (0, -\infty)$ then we have

$$(5\mu_1 + 5\mu_2 - 4\mu_3)s + 36\mu_3 = 0, \quad (\mu_2 - 2\mu_1)s + 12\mu_1 + 4\mu_3 = 0.$$

For a fixed s denote

$$\Delta(s) = \{[\mu_1 : \mu_2 : \mu_3] \in \mathbb{RP}^2 : (5\mu_1 + 5\mu_2 - 4\mu_3)s + 36\mu_3 = 0, (\mu_2 - 2\mu_1)s + 12\mu_1 + 4\mu_3 = 0\}.$$

The set $\Delta(s)$ is a point provided that $s \neq 4$ and hence the bifurcation set

$$\Delta = \bigcup_{s \in (0,-\infty)} \Delta(s)$$

is a connected curve. Let $Q \subset \mathbb{RP}^2$ be the real quadric

$$Q = \{[\mu_1 : \mu_2 : \mu_3] \in \mathbb{RP}^2 : (5\mu_1 + 5\mu_2 - 4\mu_3)(12\mu_1 + 4\mu_3) - 36\mu_3(\mu_2 - 2\mu_1) = 0\}.$$

An elementary computation (which we omit) shows that Δ is the piece of the real quadric Q

$$\Delta = \{[\mu_1 : \mu_2 : \mu_3] \in Q : \mu_3(5\mu_1 + 5\mu_2 - 4\mu_3) > 0\} \tag{36}$$

shown on Figure 4.2. We conclude that $\mathbf{B} = \{\mu_3 = 0\} \cup \{\mu_2 - 2\mu_1 = 0\} \cup \Delta$. If $[\mu_1 : \mu_2 : \mu_3] \in Q$ then

$$G(s) = (as + b)\big(cJ_{-1}(s) + dJ_{-3}(s)\big)$$

where a, b, c, d are real constants depending on μ_1, μ_2, μ_3 which are easily computed. This combined with Theorem 4.2 gives the exact number of the zeros of $G(s)$ when $[\mu_1 : \mu_2 : \mu_3] \in Q$. Of course this gives also the exact number of the zeros of $G(s)$ in each connected component of $\mathbb{RP}^2 \setminus \mathbf{B}$ except in the connected component which has no common points with Q. Finally we note that the type of the bifurcation of zeros, when $[\mu_1 : \mu_2 : \mu_3]$ crosses transversally the line $\mu_3 = 0$ follows from Step 3 in the proof of Theorem 4.2. This gives the exact number of the zeros of $G(s)$ in the last connected component of $\mathbb{RP}^2 \setminus \mathbf{B}$. The result is resumed in Figure 4.2.

Corollary 3.1 *Any small quadratic perturbation of the isochronous center* $\mathcal{S}_3$ *produces around a focus* ζ *satisfying* (H1) *at most two limit cycles that escape uniformly to infinity.*

4.3 The Lotka-Volterra and the Hamiltonian Cases

We comment in brief these two systems. Case (I) has been reduced to the quadratic-like-linear case (i), where the periodic orbits around the origin are circles. Therefore the limit cycles produced by the period annulus after any quadratic perturbation are determined by the zeros of the elementary function (see [8])

$$\begin{aligned} I(h) &= \iint_{H<h} (\mu_1 + \mu_2 x^{-1} + \mu_3 x^{-2})\, dx\, dy \\ &= 2\int_{1-\sqrt{1+2h}}^{1+\sqrt{1+2h}} (\mu_1 + \mu_2 x^{-1} + \mu_3 x^{-2})\sqrt{2h + 2x - x^2}\, dx, \quad h \in (-1/2, 0). \end{aligned}$$

One can check that $I(h)$ has up to two zeros.

In the Hamiltonian case (II), $C_0 \neq -1$, in order to estimate how many limit cycles around ζ tend uniformly to infinity, we need an upper bound for the number of limit cycles produced simultaneously from the center and the period annulus under a small quadratic perturbation of the Hamiltonian vector field in (ii). Recall the reader that in this case the period annulus is surrounded by a loop containing a non-Morsean singularity and the needed upper bound is known be two, see [18]. To summarize, we state the following result

Theorem 4.4 *Any small quadratic perturbation of the Lotka-Volterra center* (I) *and the Hamiltonian system with a center and two saddles* (II), $C_0 \neq -1$ *produces around a focus* ζ *satisfying* (H1) *at most two limit cycles tending uniformly to infinity.*

Remark 4.2 In view of the cases considered above, the reader might conclude that two is always the maximal number of limit cycles around the focus coming from infinity, which is not the case. For example, in case (III), if one chooses $A_0 = \frac{3}{2}$, then (12) becomes a small perturbation of the system

$$\dot{Z} = -iZ + 4Z^2 + 2Z\bar{Z} + 2\bar{Z}^2$$

belonging to the intersection of two strata of the quadratic center manifold, Q_3^R and Q_4 (see [20]). It is known for this case [8] that up to three limit cycles can appear in a neighborhood of the origin under a small quadratic perturbation.

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