FLOWS ON HYPERMAPS

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1. Introduction. The combinatorial investigation of graphs embedded on surfaces leads one to consider a pair of permutations (σ, α) that generate a transitive group [**7**]. The permutation α is a fixed-point-free involution and the pair is called a *map*. When this condition on α is dropped the combinatorial object that arises is called a *hypermap*. Both maps and hypermaps have a topological description: for maps a classical reference is [**13**] and for hypermaps such a description can be found in [**4**] and [**6**]; a brief account of it will be given below. However, the relationship between maps and hypermaps is not simply that the latter generalize the former. Actually, with every hypermap there is associated a map, its bipartite map, and conversely every bipartite map arises in this way. We do not enter into the details of this question; we refer the reader to the work of Walsh [**16**]. In this sense hypermaps are, at the same time, a generalization and a special case of maps.

Various result are known about hypermaps, in particular result concerning their groups of automorphisms ([**6**], [**11**], [**12**]), i.e. properties of the centralizer of the group generated by the two permutations σ and α. In the present paper we take a different point of view and consider functions defined on the set B, on which σ and α act, with values in a field K. The vector space $K(B)$ thus arising will contain a subspace whose elements correspond to the flows as usually defined on graphs. Flows on oriented graphs are functions defined on the edges with the property that the sum of the values on the edges entering a vertex equals that on the edges leaving it (Kirchhoff's law). It will be seen here that the analogous concept for a hypermap (σ, α) is that of vectors orthogonal to the fixed subspaces of σ and α of $K(B)$. As in the case of a graph, the circuits of the hypermaps are special flows, in which the value of the flow is 0 or 1. Now, if a graph is embedded in a surface of genus g, the faces are circuits that generate a subspace of codimension $2g$ in the space of flows. There is an analogous concept for hypermaps, as we shall prove below. When the hypermap has only one face this gives a new interpretation and proof of a result of Brahana [**2**] and others ([**3**], [**10**]).

In the general case a basis for the space of flows is formed by considering a set of permutations on B that correspond to the complement of a spanning tree in the case of a graph.

We now turn to the topological description of a hypermap mentioned above.

Definition 1. A *topological hypermap* on an orientable surface Σ is a decomposition of Σ into the subsets of three families S, A and F such that:

(i) S and A are a union of a finite number of disjoint closed sets homeomorphic to a plane disc;

(ii) an element of S and one of A intersect in at most a finite number of points;

(iii) the complement F of $A \cup S$ in Σ is a finite union of simply connected domains (see [**4**] and [**6**]).

The choice of an orientation on Σ allows us to define a pair of permutations (σ, α) in the following way. Let B be the set of points of intersections of the elements of S and that of A. Going around a disc $D \in S$ in the sense given by the chosen orientation, the points of B on the boundary of D undergo a cyclic ordering. The various cycles thus obtained as D varies in S are the cycles of permutation σ of B. In a similar way, a permutation α is obtained from the set A. The pair (σ, α) is the combinatorial hypermap associated with the given decompositon of Σ. The domains of F being simply connected, the group generated by σ and α is transitive on B. Note that starting with a point $b \in B$ and successively applying σ and α one goes along the border of a domain of F (in the sense opposite to that given by the orientation). In this way, the domains of F are described by the cycles of the product $\sigma\alpha$.

With a hypermap on a surface Σ there is associated a graph in the following way. The vertex set V is the set B and the edge set E is the set of arcs on the border of the discs of S and A joining two points of B. Thus every vertex is of degree 4, and $|E| = 2|B|$. Let F' be the set of faces of this embedded graph. Now F' contains F and the discs of S and A; the regions of F' are simply connected, so that Euler's formula holds:

$$|V| - |E| + |F'| = 2 - 2g,$$

where g is the genus of the surface. For a given permutation γ let $z(\gamma)$ denote the number of cycles of γ. Then $|S| = z(\sigma)$, $|A| = z(\alpha)$ and $|F| = z(\sigma\alpha)$. Euler's Formula becomes:

$$2 - 2g = |V| - |E| + |F'| = |B| - 2|B| + z(\sigma) + z(\alpha) + z(\sigma\alpha)$$

so that

$$z(\sigma) + z(\alpha) + z(\sigma\alpha) = |B| + 2 - 2g \tag{1}$$

Conversely, given a pair of permutations (σ, α) that generate a transitive group on B, it can be proved that there exists a non-negative integer g such that (1) holds ([**5**], [**8**]). Geometrical considerations then show that there exists a surface of genus g and a topological hypermap on it such that the corresponding combinatorial hypermap is precisely (σ, α). Equality (1) will be referred to as the *genus formula*.

DEFINITION 2. A (combinatorial) *hypermap* is a pair of permutations (σ, α) on a set B, whose elements will be called *brins*, such that the group $\langle \sigma, \alpha \rangle$ they generate is transitive on B. The *genus* of (σ, α) is the non-negative integer g appearing in (1). When $g = 0$, the hypermap is *planar*. The cycles of σ, α and $\sigma\alpha$ are the *vertices*, *edges* and *faces* of the hypermap, respectively.

The transitivity of the group $\langle \sigma, \alpha \rangle$ does not depend on the two permutations σ and α but only on the partitions of B induced by the cycles of σ and α. These two partitions constitute the *underlying hypergraph* of the hypermap; the transitivity of $\langle \sigma, \alpha \rangle$ depends on the connectedness of this hypergraph, in the sense of the following definition.

DEFINITION 3. A *hypergraph* on a set B is a pair of partitions (V, E) of B. When the classes of E all have cardinality 2, the hypergraph is called a *graph*. A hypergraph is *connected* if whenever a non-empty union of classes of V equals a union of classes of E, then this union is the whole set B. Let $B = \{1, 2, \ldots, n\}$.

Thus a hypermap is a hypergraph in which a cyclic ordering has been given to the classes of V and E. We close this section by mentioning a well-known lemma that will be used throughout the paper.

LEMMA (Serret). *Let γ be a permutation, $\tau = (i, j)$ a transposition. Then $z(\gamma\tau) = z(\gamma) + 1$ if i and j belong to one and the same cycle of γ and $z(\gamma\tau) = z(\gamma) - 1$ if i and j belong to two different cycles of γ. In the first case we shall say that τ disconnects γ, in the second that τ connects γ.*

EXAMPLE. Fig. 1 shows an embedding of the "Fano hypermap" on the torus. Each set belonging to A (drawn as a triangle) intersects exactly three sets of S (drawn as

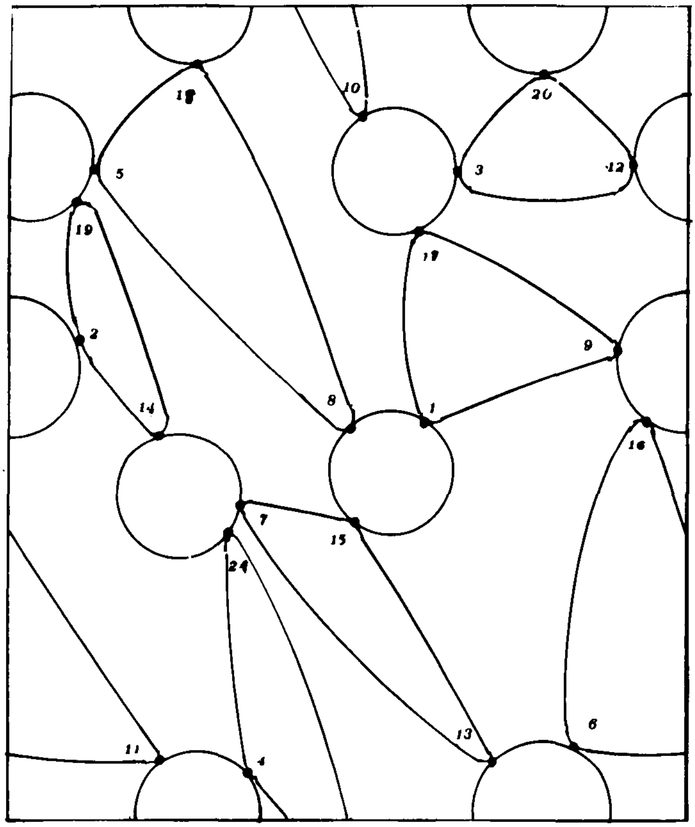

Figure 1

circles), and conversely. The two permutations σ and α are:

$$\sigma=(1, 8, 15)\quad(2, 9, 16)\quad(3, 10, 17)\quad(4, 11, 18)\quad(5, 12, 19)\quad(6, 13, 20)\quad(7, 14, 21),$$
$$\alpha=(1, 9, 17)\quad(2, 14, 19)\quad(3, 12, 20)\quad(4, 10, 21)\quad(5, 8, 18)\quad(6, 11, 16)\quad(7, 13, 15).$$

The faces are described by the product:

$$\sigma\alpha=(1, 18, 10)\quad(2, 17, 12)\quad(3, 21, 13)\quad(4, 16, 14)\quad(5, 20, 11)\quad(6, 15, 9)\quad(7, 19, 8).$$

2. The space of flows. Let K be a field and let $K(B)$ be the vector space of functions from B to K. Then $K(B)$ can be thought of as the set of formal sums $u=\Sigma u_i i$, $i\in B$, and if σ is a permutation of B then σ acts on $K(B)$ by permuting the basis B:

$$u\sigma=(\Sigma u_i i)\sigma=\Sigma u_i\sigma(i)$$

Let S be the subspace of vectors fixed by σ: $S=\{u\in K(B), u\sigma=u\}$, and let $\sigma=\sigma_1\sigma_2\ldots\sigma_t$, $t=z(\sigma)$, be the cycle decomposition of σ. If $u\in S$, then the elements of σ_i all have the same coefficient in u, u_i say, and if $\boldsymbol{\sigma}_i$ denotes the sum of the elements belonging to σ_i, then the t vectors $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_t$ generate S and are independent. Thus $\dim(S)=z(\sigma)$. Similarly, $\dim(A)=z(\alpha)$ and $\dim(F)=z(\sigma\alpha)$, where A and F are the fixed spaces of α and $\sigma\alpha$, respectively.

LEMMA 1. *We have*:

$$\dim(S\cap A)=\dim(S\cap F)=\dim(A\cap F)=1$$

Proof. Let $u\in S\cap A$, $u=\Sigma u_i i$. Let $B_1=\{i\in B \mid u_i=u_1\}$. Then B_1 is both a union of orbits of σ and of α; then the transitivity of $\langle\sigma, \alpha\rangle$ implies $B_1=B$. Thus $u_i=u_1$, for all i, so that the elements $u\in S\cap A$ are of the form $u=u_1(\Sigma i)$. The elements Σi also generates the two other subspaces because $\langle\sigma, \alpha\rangle=\langle\sigma, \sigma\alpha\rangle=\langle\alpha, \sigma\alpha\rangle$. The result follows.

A scalar product is defined on $K(B)$ by $\langle i, j\rangle=\delta_{ij}$. Note that if $\theta\in S^n$, the symmetric group on $n=|B|$ elements, then $\langle\theta v, \theta u\rangle=\langle v, u\rangle$. Let $S^\perp$ and $A^\perp$ be the orthogonal subspaces of S and A, respectively.

DEFINITION 4. The *space of flows* of the hypermap (σ, α) is the subspace $S^\perp\cap A^\perp$.

LEMMA 2. *The dimension of the space of flows is*

$$\dim(S^\perp\cap A^\perp)=(n-z(\alpha))-(z(\sigma)-1)$$

Proof. This follows from Lemma 1 and from the two equalities $S^\perp\cap A^\perp=(S+A)^\perp$ and $\dim(S+A)^\perp+\dim(S+A)=n$.

REMARK. The notion of a flow only depends on the orbits of σ and α and not on the two permutations σ and α. Thus, two hypermaps having the same underlying hypergraph also have the same space of flows. When (σ, α) is a map, then the dimension of $S^\perp\cap A^\perp$ is $z(\alpha)-(z(\sigma))-1)$; this is the number of edges of the underlying graph minus the number of edges of one of its spanning trees. In this case S and A are equivalent to the

0-chain group and to the 1-chain group of the underlying graph in the terminology of [**15**]. The subspace $S^\perp \cap A^\perp$ is the *circuit subspace* extensively investigated in [**1**]. Thus, by the above definition, a flow on a hypermap is a function $f: B \to K$ such that the sum of the values $f(i)$ vanishes on the cycles of both σ and α. An example of a flow on the hypermap of Fig. 1 is the following:

$$\begin{aligned} u = {} & 1\,.\,1 + (-1)\,.\,2 + 3\,.\,3 + (-1)\,.\,4 + 0\cdot 5 + 1\cdot 6 + (-4)\,.\,7 + (-2)\,.\,8 + 1\,.\,9 \\ & + (-1)\,.\,10 + (-1)\,.\,11 + 1\,.\,12 + 3\,.\,13 + 2\,.\,14 + 1\,.\,15 + 0\,.\,16 + (-2)\,.\,17 \\ & + 2\,.\,18 + (-1)\,.\,19 + (-4)\,.\,20 + 2\,.\,21. \end{aligned}$$

It is known that in a planar graph the set of faces generates the space of flows (a basis can be obtained by removing any one of the faces). This is not the case if the graph is not planar. Theorem 1 below shows that for a hypermap of genus g the faces yield a subspace of codimension $2g$ of the space of flows. In order to associate a flow with a face we define the following function:

$$\varphi : B \to K(B)$$

given by:

$$\varphi(i) = i\theta - (i\theta)\alpha^{-1}$$

where $i\theta$ denotes the sum of the elements contained in the cycle of $\sigma\alpha$ to which i belongs. In Fig. 2, for example, 1 belongs to the face $(1, 3, 5)$, so that $1\theta = 1 + 3 + 5$ and $\varphi(1) = 1 + 3 + 5 - 6 - 2 - 4$.

LEMMA 3. $\operatorname{Im} \varphi \subseteq S^\perp \cap A^\perp$.

Proof. Let $i \in B$, $s \in S$. Then

$$\langle i\theta\alpha^{-1}, s\rangle = \langle i\theta\alpha^{-1}, s\sigma\rangle = \langle i\theta, s\sigma\alpha\rangle = \langle i\theta(\sigma\alpha)^{-1}, s\rangle = \langle i\theta, s\rangle.$$

Now,

$$\langle \varphi(i), s\rangle = \langle i\theta, s\rangle - \langle i\theta\alpha^{-1}, s\rangle = \langle i\theta, s\rangle - \langle i\theta, s\rangle = 0,$$

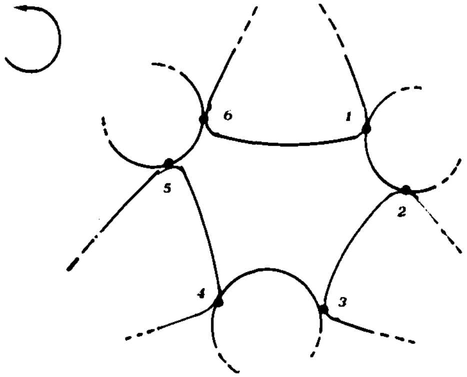

Figure 2

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so that $\varphi(i) \in S^{\perp}$. If $a \in A$, then

$$\langle \varphi(i), a \rangle = \langle i\theta, a \rangle - \langle i\theta\alpha^{-1}, a \rangle = \langle i\theta, a \rangle - \langle i\theta, a\alpha \rangle = \langle i\theta, a \rangle - \langle i\theta, a \rangle = 0.$$

This proves $\varphi(i) \in A^{\perp}$ and the lemma.

LEMMA 4. $\dim(\operatorname{Im} \varphi) = z(\sigma\alpha) - 1$.

Proof. Let $T = \{b_1, b_2, \ldots, b_t\}$, $t = z(\sigma\alpha)$, be a transversal of the orbits of $\sigma\alpha$. Since two elements belonging to one and the same face have the same image in φ, the elements $\varphi(b_l)$, $i = 1, 2, \ldots, t$ generate $\operatorname{Im} \varphi$. Suppose we have a relation:

$$\sum_{l \in I} k_l \varphi(b_l) = 0$$

where the k_l are not all zero and $I \subseteq T$. The vector

$$u = \sum k_l(b_l\theta) = \sum k_l(b_l\theta\alpha^{-1})$$

is fixed by both $\sigma\alpha$ and α. By Lemma 1, the k_l are all equal, $k_l = k$, say, and $u = k(\sum_{i \in B} i)$. All the i's must then appear in the above relation, so that $I = T$. Thus $\sum_{l \in B} \varphi(b_l) = 0$ is the only relation among the $\varphi(b_l)$'s, and the lemma is proved.

As a consequence of the genus formula and of Lemmas 2 and 4 we have the following theorem.

THEOREM 1. *Let (σ, α) be a hypermap of genus g. Then*

$$\dim((S^{\perp} \cap A^{\perp})/\operatorname{Im} \varphi) = 2g.$$

3. One-faced hypermaps. In this section we consider the special case of a hypermap (σ, α) in which $\sigma\alpha$ is a cyclic permutation, and give a method that provides a generating system for the space of flows of (σ, α). For each $i \in B$, let τ be the transposition $(i, \alpha(i))$, and let

$$U(i) = i\theta - (i\theta)\alpha^{-1}$$

where $i\theta$ denotes the sum of the elements contained in the cycle of $\sigma\alpha\tau$ to which i belongs. Let $A_i = \{u \in K(B) \mid (u)\alpha\tau = u\}$. Then $A_i \supset A$, $A_i^{\perp} \subset A^{\perp}$ and $S^{\perp} \cap A_i^{\perp} \subseteq S^{\perp}$. By Lemma 3, the image of $U(i)$ is contained in $S^{\perp} \cap A_i^{\perp}$, so that $U(i) \subseteq S^{\perp} \cap A^{\perp}$. The $U(i)$ thus obtained generate $S^{\perp} \cap A^{\perp}$; this is shown in Theorem 2 below. We first need a lemma.

LEMMA 5. *Let (σ, α) be a hypermap having only one face $\sigma\alpha = \zeta = (1, 2, \ldots, n)$ and assume that* $i < j < \alpha(i)$ implies $i < \alpha(j) < \alpha(i)$. *Then (σ, α) is planar.*

Proof. The hypotheses of the lemma imply the existence of $i \in B$ such that $\alpha(i) = \zeta(i) = i + 1$. Let τ be the transposition $\tau = (i, i + 1)$; then the restriction of $(\sigma, \alpha\tau)$ to $B \setminus (i)$ is a hypermap satisfying the hypotheses of the lemma. By induction on $|B|$, this hypermap is planar; it follows that (σ, α) is also planar. (See Fig. 3.)

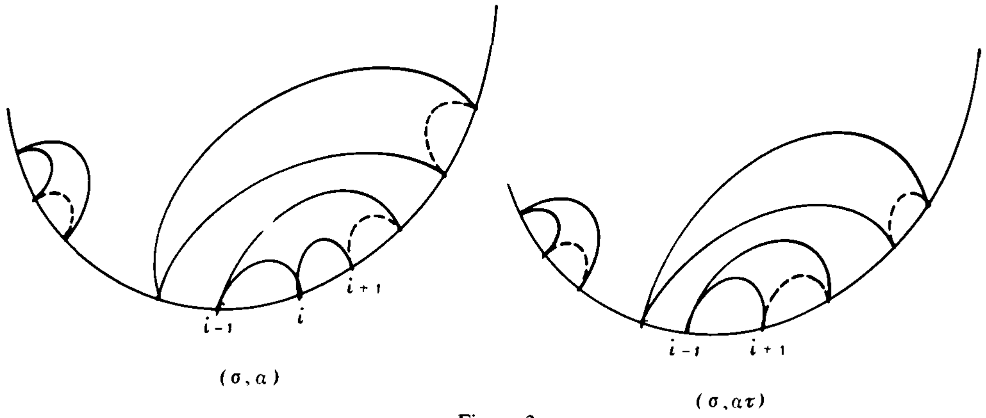

Figure 3

REMARK. The converse of this lemma is also true [**4**, L.IV2].

THEOREM 2. *Let* (σ, α) *be a hypermap having only one face. Then the space of flows of* (σ, α) *is generated by the* $U(i)$, $i \in B$.

Proof. The proof is by induction on the genus g of (σ, α). If $g = 0$ then $S^\perp \cap A^\perp = \{0\}$, and we are done. Let $\sigma\alpha = \zeta = (1, 2, \ldots, n)$ and let $g > 0$; then Lemma 5 implies the existence of $a < c < b < d$ such that $\alpha(a) = b$ and $\alpha(c) = d$. We consider the hypermap $(\sigma, \tilde{\alpha})$ where $\tilde{\alpha} = \alpha(a, b)(c, d)$. Then $(\sigma, \tilde{\alpha})$ still has only one face and $\tilde{\alpha}(a) = a$, $\tilde{\alpha}(c) = c$; clearly $g(\sigma, \tilde{\alpha}) = g - 1$. Let U' be the mapping associated with the hypermap $(\sigma, \tilde{\alpha})$; then, by induction, the $U'(i)$, $i \in B$, generate the space of flows of $(\sigma, \tilde{\alpha})$, so that

$$\dim(\langle U'(i), i \in B \rangle) = 2(g-1).$$

We now show that:

(i) the subspace generated by the $U'(i)$ is contained in that generated by the $U(i)$.

(ii) $U(a)$ and $U(c)$ are linearly independent and no linear combination of them belongs to the subspace generated by the $U'(i)$.

Then $\dim(\langle U(i), i \in B \rangle) \geq 2(g-1) + 2 = 2g$; this space being contained in $S^\perp \cap A^\perp$, which has dimension $2g$, we have the result.

Proof of (i). Let x be the set of the vectors corresponding to the brins lying between a and c, y between c and b, z between b and d, and t between d and a (Fig. 4). The proof consists of a detailed analysis of the various cases arising according to the possible values of $\alpha(b)$ and $\alpha(d)$. We shall only consider a few of them, the remaining ones being similarly handled. Assume

(a) $\alpha(b) \in t$, and let t_1 be the set of brins between d and $\alpha(b)$

(b) $\alpha(d) \in x$, and let x_1 be the set of brins between a and $\alpha(d)$

(c) $i \in x$, $\alpha(i) \in t$; x_2 and t_2 defined as above.

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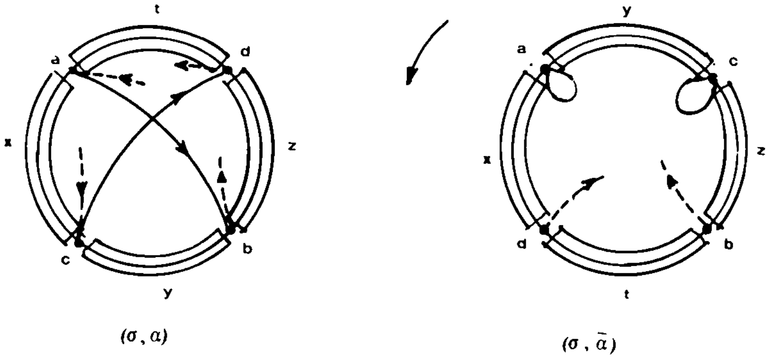

Figure 4

Then we have†

$$U(a) = x + y + c - x\alpha^{-1} - y\alpha^{-1} - c\alpha^{-1}$$
$$U(b) = z + t + d - z\alpha^{-1} - t\alpha^{-1} - d\alpha^{-1}$$
$$U(c) = y + z + b - y\alpha^{-1} - z\alpha^{-1} - b\alpha^{-1}$$
$$U(d) = t + x_1 + a - t\alpha^{-1} - x_1\alpha^{-1} - a\alpha^{-1}$$

(the image in α^{-1} of the various elements appearing above is the same as that in $\alpha(a, b)$, $\alpha(b, \alpha(b))$, etc.). Now

$$U(i) = x_2 + y + z + t_2 + c + b + d - x_2\alpha^{-1} - y\alpha^{-1} - z\alpha^{-1} - t_2\alpha^{-1} - c\alpha^{-1} - b\alpha^{-1} - d\alpha^{-1}$$

and

$$U'(i) = x_2 + t_2 + d - x_1\tilde{\alpha}^{-1} - t_1\tilde{\alpha}^{-1} - d\tilde{\alpha}^{-1}$$

(here too $x\tilde{\alpha}^{-1} = x_1\alpha^{-1}$, etc.). It is easily seen that $U'(i) = U(i) - U(c)$. In case $\alpha(b) = c$ one obtains, for instance,

$$U'(b) = U(b) + U(a) + U(c).$$

Proof of (ii). Since $\tilde{\alpha}(a) = a$ and $\tilde{\alpha}(c) = c$, we have $a, c \in \tilde{\alpha}$. If $u \in \alpha^{-1}$, then $\langle u, a\rangle = \langle u, c\rangle = 0$; therefore,

$$\langle U'(i), a\rangle = \langle U'(i), c\rangle = 0$$

because $U'(i) \subseteq \tilde{A}^{\perp}$. Now, $\langle U(a), c\rangle = 1$ and $\langle U(c), a\rangle = -1$, so that neither $U(a)$ nor $U(c)$ belongs to the subspace generated by the $U'(i)$. Moreover, $U(a)$ and $U(c)$ are linearly independent (in the case considered above, $b \notin U(a)$ whereas $b \in U(c)$), and no linear combination of them belongs to the subspace generated by the $U'(i)$. The proof is complete.

The special case in which (σ, α) is a map is worth mentioning. $U(i)$ is then equal to

† For the sake of simplicity, we denote by $x, y, \ldots$ the sum of the vectors belonging to the sets $x, y, \ldots$

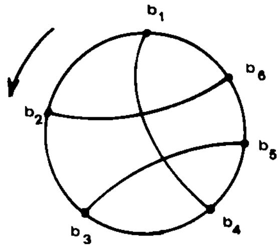

Figure 5

the sum of the brins between b and $\alpha(b)$ minus the image in α of this sum. Equal brins with opposite signs cancel; the brins that are left belong to the edges that cross $(b, \alpha(b))$. In the example of Fig. 5 we have

$$U(b_2) = b_3 + b_4 + b_5 - b_5 - b_1 - b_3 = b_4 - b_1$$

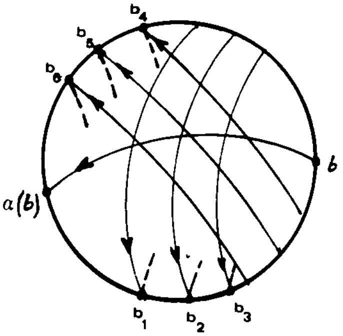

Figure 6

When (σ, α) is a hypermap we have (Fig 6) $U(b) = b_1 + b_2 + b_3 - b_4 - b_5 - b_6$, that is the sum of the brins that are at the end of the arcs going "inside" minus those of the arcs going "outside". Each edge contains the same number of in- and outgoing arcs; when (σ, α) is a map, each edge is made up of a pair "ingoing–outgoing" arcs. The matrix representing the linear transformation U for the example of Fig. 5 is

	1	2	3	4	5	6
1	0	1	1	0	−1	−1
2	−1	0	0	1	0	0
3	−1	0	0	1	0	0
4	0	−1	−1	0	1	1
5	1	0	0	−1	0	0
6	1	0	0	−1	0	0

When i and j are on the same edge, line i can be obtained by multiplying line j by -1.

Consider now the incidence matrix of the edges (over GF (2)):

	(1, 4)	(2, 6)	(3, 5)
(1, 4)	0	1	1
(2, 6)	1	0	0
(3, 5)	1	0	0

This is the upper left minor of the previous matrix. The ranks of the two matrices are equal. This is easily seen to hold in general. The latter matrix is the incidence matrix of the cross graph of the chord diagram. This graph has been investigated by many authors ([**2**], [**3**], [**10**], [**14**]) who have proved that the matrix in question has rank $2g$.

4. A basis for the space of flows. In this section a sequence of subspaces of the space of flows is constructed. Each subspace has codimension 1 in the preceding 1, and the sequence corresponds to a sequence of transposition $\tau_1, \tau_2, \ldots, \tau_m$, $m = \dim(S^\perp \cap A^\perp)$, such that τ disconnects $\alpha_{i-1} = \alpha\tau_1\tau_2 \ldots \tau_{i-1}$, the pair (σ, α_i) still being a hypermap. A sequence of flows $\{u_{\tau_i}\}$ can then be defined, the i-th of which belongs to the i-th subspace but not to the $(i+1)$-th, yielding a basis of $S^\perp \cap A^\perp$.

LEMMA 6. *Let $\langle\beta, \gamma\rangle$ be a transitive group, $z(\beta) > 1$. Then there exist i and j belonging to one and the same cycle of γ and to two different cycles of β.*

Proof. If all pairs of elements belonging to the same cycle of γ also belong to the same cycle of β, then the cycles of γ are contained in those of β so that the group cannot be transitive.

LEMMA 7. *Let (σ, α) be a hypermap of genus g, $z(\sigma, \alpha) > 1$. Then there exist $p = z(\sigma\alpha) - 1$ transpositions $\tau_1, \tau_2, \ldots, \tau_p$, such that*

(i) *τ_i disconnects $\alpha_{i-1} = \alpha\tau_1\tau_2 \ldots \tau_{i-1}$ and connects $\sigma\alpha\tau_1\tau_2 \ldots \tau_{i-1}$*

(ii) *the pair (σ, α'), $\alpha' = \alpha\tau_1\tau_2 \ldots \tau_p$, is a hypermap with only one face and of genus g.*

Proof. By the previous lemma, there exists τ disconnecting α and connecting $\sigma\alpha$. Thus $(\sigma, \alpha\tau)$ is a hypermap, and its genus is g. Repeated application of the lemma yields the result.

The spaces of flows of the hypermaps (σ, α_i) give the sequence of subspaces mentioned at the beginning of this section. This is the content of the following theorem.

THEOREM 3. *With the set of p transpositions of the previous lemma there is associated a sequence of subspaces*

$$S^\perp \cap A^\perp \supset S^\perp \cap A_1^\perp \supset \ldots \supset S^\perp \cap A_{p-1}^\perp \supset S^\perp \cap A_p^\perp$$

each having codimension 1 in the preceding one. Moreover,

$$\dim(S^\perp \cap A_p^\perp) = 2g.$$

Proof. The fixed space A_i of α_i is contained in that of α_{i+1} since one of the orbits of α_i splits into two orbits of α_{i+1}: the sum of the brins in this orbit of α_i is obtained by summing the brins in the two corresponding orbits of α_{i+1}. This also shows that $\dim(A_{i+1}/A_i) = 1$. As to the dimension of $S^\perp \cap A_p^\perp$ simply observe that

$$\dim(S^\perp \cap A_p^\perp) = \dim(S^\perp \cap A^\perp) - p = n - z(\alpha) - z(\sigma) + 1 - z(\sigma\alpha) + 1 = 2g.$$

The hypermap (σ, α_p) only has one face and is of genus g. If $g = 0$ the construction ends here. Otherwise we proceed as follows.

LEMMA 8. *Let (σ, α) be a hypermap such that for all transpositions τ the group $\langle \sigma, \alpha\tau \rangle$ is not transitive. Then (σ, α) has only one face and is planar.*

Proof. If $z(\sigma\alpha) > 1$ then for any transposition τ connecting $\sigma\alpha$ the group $\langle \sigma, \sigma\alpha\tau \rangle = \langle \sigma, \alpha\tau \rangle$ is transitive. This proves the first statement. Let $\sigma\alpha = (1, 2, \ldots, n)$ and suppose there exist i and j such that $i < j < \alpha(i) < \alpha(j)$. Let $\tau = (i, \alpha(i))$; then $\sigma\alpha\tau$ has two cycles one containing i and j and the other one $\alpha(i)$ and $\alpha(j)$. But j has the same image in α and $\alpha\tau$; this implies that $\langle \sigma\alpha\tau, \alpha\tau \rangle$ is transitive, contrary to the assumption. Thus $i < j < \alpha(i)$ implies $i < \alpha(j) < \alpha(i)$, and by Lemma 5 (σ, α) is planar.

LEMMA 9. *Let (σ, α) be a hypermap of genus $g > 0$. Then there exist $2g$ transpositions $\tau_1, \tau_2, \ldots, \tau_{2g}$* such that

(i) *τ_i disconnects $\alpha_{i-1} = \alpha\tau_1\tau_2 \ldots \tau_{i-1}$, $(\alpha_0 = \alpha)$;*

(ii) *τ_i disconnects $\sigma\alpha_{i-1}$ for i odd and connects $\sigma\alpha_i$ for i even.*

Proof. The existence of τ_i for i odd is assured by Lemma 8; that of τ_i for i even by Lemma 6.

REMARK. With the notation of the previous lemma, the hypermaps (σ, α_k) and (σ, α_{k+1}), $k = 1, 3, \ldots, 2g - 1$, are of genus $g - (k+1)/2$, that is, the genus decreases by 1 at step k when the hypermap has two faces and remains unchanged at step $k + 1$ when the hypermap has one face. The hypermap (σ, α_{2g}) we end with is planar.

The following theorem is now clear.

THEOREM 4. *With the set of $2g$ transposition of Lemma 9 there is associated a sequence of subspaces*

$$S^\perp \cap A_p^\perp \supset S^\perp \cap A_p'^\perp \supset \ldots \supset S^\perp \cap A_{2g}'^\perp = (0)$$

each one having codimension 1 in the preceding one.

We now construct a set of flows giving a basis of $S^\perp \cap A^\perp$. The following lemma is a consequence of transitivity.

LEMMA 10. *Let $\langle \beta, \gamma \rangle$ be a transitive group, $z(\gamma) > 1$, and let γ_1 and γ_2 be two different cycles of γ. Then there exists a sequence*

$$b_1, b_2, \ldots, b_{2k}$$

such that

(i) $b_1 \in \gamma_1$, $b_{2k} \in \gamma_2$;
(ii) b_{2i} *and* b_{2i+1} *are in the same cycle of* γ;
(iii) b_{2i-1} *and* b_{2i} *are in the same cycle of* β;
(iv) *no* b_i, $i > 1$, *belongs to* γ_1.

Consider now a transposition τ disconnecting α and such that $(\sigma, \alpha\tau)$ is still a hypermap; τ exists by Lemma 8. A cycle of α splits into two cycles, α_1 and α_2, say. The vector

$$u_\tau = \sum_{i=1}^{2u} (-1)^i b_i$$

where $b_1, b_2, \ldots, b_{2k}$ is the sequence of brins given by the lemma is a flow of (σ, α) but not of $(\sigma, \alpha\tau)$ because α_1 contains b_1 but no other brin of the sequence. Thus $u_\tau \in (S^\perp \cap A^\perp) \setminus (S^\perp \cap A_\tau^\perp)$, where A_τ is the fixed space of $\alpha\tau$.

REMARK. If τ connects $\sigma\alpha$, then the sequence $b_1, b_2, \ldots, b_{2k}$ can be obtained as $b_2 = (b_1)\sigma$, $b_3 = (b_2)\alpha, \ldots, b_{2k} = (b_{2k-1})\sigma$ and the flow u_τ is equal to $\varphi(b_1)$ (where φ is defined as in the previous section).

We state the above result in the form of a theorem.

THEOREM 5. *Let* (σ, α) *be a hypermap of genus* g, $\tau_1, \tau_2, \ldots, \tau_p$, $p = z(\sigma\alpha) - 1$ *be the transpositions defined in Lemma* 7, $\tau_{p+1}, \tau_{p+2}, \ldots, \tau_m$, $m = \dim(S^\perp \cap A^\perp)$, *those defined in Lemma* 9. *Then the above constructed flows* u_{τ_i}, $i = 1, 2, \ldots, m$, *are a basis for the space of flows of* (σ, α).

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