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ON THE EXISTENCE AND THE CLASSIFICATION OF CRITICAL POINTS FOR NON-SMOOTH FUNCTIONALS

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ABSTRACT. We extend the min-max methods used in the critical point theory of differentiable functionals on smooth manifolds to the case of continuous functionals on a complete metric space. We study the topological properties of the min-max generated critical points in this new setting by adopting the methodology developed by Ghoussoub in the smooth case. Many old and new results are extended and unified and some applications are given.

1. **Min-max methods for continuous functionals.** While the concepts of minimum and maximum of a functional are purely topological notions, the classical Morse classification of Saddle-type critical points involves in a crucial way the differential structure of the functional and the domain. In recent years, many functionals associated to various important variational problems lacked the smoothness properties that are usually needed for the application of the classical theory. For example, it is well known that $W^{1,2}(M,N)$ is not a Banach manifold when M is a manifold of dimension larger than 2. This usually complicates the variational approach for constructing harmonic maps by finding critical points of the energy functional. Another example is the C^1 but not C^2 dual functional associated to a Hamiltonian system [5]. In order to deal with this difficulty, Hofer [14] isolated the purely topological notion of *a critical point of mountain pass type* in order to analyse the saddle points obtained in the Mountain Pass theorem of Ambrosetti and Rabinowitz [1] for functionals that fail to be in C^2. In the case of a (smooth) Morse function, these points coincide exactly with the critical points whose Morse index is equal to one. Our main goal in this paper, is to develop the non-smooth analogue of those critical points that correspond to a higher Morse index.

In order to construct and classify such critical points, we first extend to our—purely metric—setting the strong form of the min-max principle established by Ghoussoub [11]. Besides yielding the existence of critical points, this theorem provides valuable information about their location on certain *dual sets*. This information was successfully used, in the smooth case, by Ghoussoub-Preiss [13], Ghoussoub [11] and Fang [6] for the classification of min-max generated critical points [12]. The basic idea behind our results here is that the methodology of using *dual sets* for classifying critical points is metric in nature and therefore it carries over to our general setting.

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In Section 1, the strong form of the min-max principle for continuous functionals defined on a complete metric space is established. In Section 2, we study the structure of the critical set generated by the min-max principle in the case of one dimensional paths. In Section 3, we first isolate various *topological indices* that can be associated to certain critical sets and points. Then we study the structure of the critical set generated by various *homotopic, cohomotopic and homological* min-max theorems in the higher dimensional case. In Section 4, we demonstrate that the new indices coincide with the Morse indices in the classical setting.

We shall always assume in this paper that X is a complete metric space with metric d unless otherwise explicitly specified. Following [11], we first introduce the following definition:

DEFINITION 1.1. Let B be a closed subset of a complete metric space (X, d). We shall say that a class $\mathcal{F}$ of compact subsets of X is a *homotopy-stable family with boundary B* provided:

(a) every set in $\mathcal{F}$ contains B;

(b) for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X\,; X)$ satisfying $\eta(t,x) = x$ for all (t,x) in $(\{0\} \times X) \cup ([0,1] \times B)$ we have that $\eta(\{1\} \times A) \in \mathcal{F}$.

In the case B is empty, we will just say that $\mathcal{F}$ is a homotopy-stable family.

DEFINITION 1.2. Say that a closed set F is *dual* to $\mathcal{F}$ if F verifies the following:

$$F \cap B = \emptyset \text{ and } F \cap A \neq \emptyset \quad \text{for all } A \text{ in } \mathcal{F}.$$

Denote by $\mathcal{F}^*$ a family of closed sets that are dual to $\mathcal{F}$ and we say that $\mathcal{F}^*$ is a dual family to $\mathcal{F}$. Note that for such a dual family, we readily have that

$$c^* := \sup_{F \in \mathcal{F}^*} \inf_{x \in F} \varphi(x) \leq \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x) =: c.$$

Now we recall the following notion of "derivative" for a continuous function. See for instance [3] or [27].

DEFINITION 1.3. Let $\varphi: X \longrightarrow R$ be a continuous function and $u \in X$. We denote by $|d\varphi|(u)$ the supremum of the σ's in $[0, \infty)$ such that there exist $\delta > 0$ and $\mathcal{H}: B(u, \delta) \times [0, \delta] \longrightarrow X$ continuous with

$$\operatorname{dist}\big(\mathcal{H}(v,t), v\big) \leq t$$
$$\varphi\big(\mathcal{H}(v,t)\big) - \varphi(v) \leq -\sigma t.$$

The extended real number $|d\varphi|(u)$ is called the *weak slope* of φ at u. If X is a C^1 Finsler manifold and φ is a C^1 function, it turns out that $|d\varphi|(u) = \|d\varphi(u)\|$. Before considering the min-max principle, we shall study this notion in connection with Ekeland's perturbed minimization principle.

PROPOSITION 1.4. *Let φ be a bounded below continuous functional on a complete metric space (X, d). Then, for any minimizing sequence $(y_n)_n$, there exists a minimizing sequence $(x_n)_n$ such that $d(x_n, y_n) \longrightarrow 0$ and $|d\varphi|(x_n) \longrightarrow 0$.*

PROOF. For the minimizing sequence $(y_n)_n$, let

$$\epsilon_n = \begin{cases} \varphi(y_n) - \inf_X \varphi & \text{if } \varphi(y_n) - \inf_X \varphi > 0 \\ 1/n & \text{if } \varphi(y_n) - \inf_X \varphi = 0. \end{cases}$$

Then $\varphi(y_n) \leq \inf_X \varphi + \epsilon$ and $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. By Ekeland's variational principle, for each $n > 1$, there exists $x_n \in X$ such that

(a) $\varphi(x_n) \leq \varphi(y_n)$;
(b) $d(x_n, y_n) \leq \sqrt{\epsilon_n}$;
(c) $\varphi(x) > \varphi(x_n) - \sqrt{\epsilon_n} d(x_n, x)$ for all $x \in X, x \neq x_n$.

We claim that $|d\varphi|(x_n) \leq \sqrt{\epsilon_n}$ for all $n \geq 1$. If not, then there are $\delta > 0, \sigma > \sqrt{\epsilon_n}$ and $\mathcal{H}: B(x_n, \delta) \times [0, \delta] \longrightarrow X$ such that

$$d\big(\mathcal{H}(v,t), v\big) \leq t$$
$$\varphi\big(\mathcal{H}(v,t)\big) - \varphi(v) \leq \sigma t$$

for all $v \in B(x_n, \delta)$, $t \in [0, \delta]$. Put $u = \mathcal{H}(x_n, t)$. Then $\varphi(u) \leq \varphi(x_n) - \sigma t < \varphi(x_n) - \sqrt{\epsilon_n} d(u, x_n)$ which contradicts (c) and it proves the proposition. ∎

We now can state the following min-max principle for continuous functionals on X. The smooth counterpart is studied in detail in [11] including its many applications. We refer to [12] for other related topics.

THEOREM 1.5. *Let φ be a continuous functional on a complete metric space X. Consider a homotopy-stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B and a dual family $\mathcal{F}^*$ of $\mathcal{F}$. Assume that*

$$\sup_{F \in \mathcal{F}^*} \inf_{x \in F} \varphi(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x) = c$$

and is finite. Then for any sequence of sets $(A_n)_n$ in $\mathcal{F}$ and a sequence $(F_n)_n$ in $\mathcal{F}^*$ such that $\lim_n \sup_{x \in A_n} \varphi(x) = c = \lim_n \inf_{x \in F_n} \varphi(x)$ *and* $\underline{\lim}_{n \to \infty} \operatorname{dist}(F_n, B) > 0$, *there exists a sequence $(x_n)_n$ in X such that*

(i) $\lim_n \varphi(x_n) = c$;
(ii) $\lim_n |d\varphi|(x_n) = 0$;
(iii) $\lim_n \operatorname{dist}(x_n, F_n) = 0$;
(iv) $\lim_n \operatorname{dist}(x_n, A_n) = 0$.

We now recall the following definitions.

DEFINITION 1.6. A sequence $(F_n)_n$ in $\mathcal{F}^*$ is said to be a suitable *max-mining* sequence in $\mathcal{F}^*$ if $\lim_n \inf \varphi(F_n) = c^*$ and $\underline{\lim}_{n \to \infty} \operatorname{dist}(F_n, B) > 0$. A sequence $(A_n)_n$ in $\mathcal{F}$ is said to be *min-maxing* in $\mathcal{F}$ if $\lim_n \sup_{x \in A_n} \varphi(x) = c = c(\varphi, \mathcal{F})$.

DEFINITION 1.7. Say that φ verifies $(PS)_c$ (resp. $(PS)_{F,c}$) (resp. $(PS)_{F,c}$ along a min-maxing sequence $A_n \in \mathcal{F}$) (resp. $(PS)_c$ along a min-maxing sequence $A_n \in \mathcal{F}$ and a suitable max-mining sequence $F_n \in \mathcal{F}^*$) if every sequence $(x_n)_n$ that verifies (i) and (ii) (resp. (i), (ii) and (iii) with $F_n = F \in \mathcal{F}^*$) (resp. (i), (ii), (iii) with $F_n = F \in \mathcal{F}^*$ and (iv)) (resp. (i), (ii), (iii) and (iv)) above has a convergent subsequence.

Throughout this paper, we shall denote by A_∞ the set

$$A_\infty = \{x \in X\,;\, \underline{\lim}_n \operatorname{dist}(x, A_n) = 0\}$$

and by F_∞ the set

$$F_\infty = \{x \in X\,;\, \underline{\lim}_n \operatorname{dist}(x, F_n) = 0\}.$$

We shall denote by K_c the set of critical points at level c, *i.e.*,

$$K_c = \{x \in X\,;\, \varphi(x) = c, |d\varphi|(x) = 0\}.$$

For any set V, we shall denote by

$$N_\delta(V) = \{u \in X\,;\, \operatorname{dist}(u, V) < \delta\}$$

its δ-neighborhood.

COROLLARY 1.8. *Let X, φ and $\mathcal{F}$ be as in Theorem 1.5 and consider a family of sets $\mathcal{F}^*$ that is dual to $\mathcal{F}$. Assume that*

$$\sup_{F\in\mathcal{F}^*} \inf_{x\in F} \varphi(x) = \inf_{A\in\mathcal{F}} \max_{x\in A} \varphi(x) = c$$

and is finite. If φ verifies $(PS)_c$ along a min-maxing sequence $(A_n)_n$ in $\mathcal{F}$ and a suitable max-mining sequence $(F_n)_n$ in $\mathcal{F}^*$, then there exists a sequence $(x_n)_n$ in X that converges to a point in $A_\infty \cap F_\infty \cap K_c$.

To prove Theorem 1.5, we need the following lemma just as in the smooth case [12]:

LEMMA 1.9. *Let $\varphi: X \longrightarrow \mathbb{R}$ be a continuous function. Let B and C be two closed and disjoint subsets of X. Suppose that C is compact and that $|d\varphi|(x) > \epsilon > 0$ for every $x \in C$. Then there exist a positive continuous function g on X and a deformation α in $C([0,1] \times X\,;\, X)$ such that for some $t_0 > 0$, the following holds for every $t \in [0, t_0)$:*

i) $\alpha(t,x) = x$ *for every* $x \in B$;
ii) $\operatorname{dist}\big(\alpha(t,x), x\big) \le t$ *for every* $x \in X$;
iii) $\varphi\big(\alpha(t,x)\big) - \varphi(x) \le -\epsilon g(x) t$ *for every* $x \in X$;
iv) $g(x) = 1$ *for all* $x \in C$.

A version of this lemma appeared in [3] but we shall give a proof for completeness. The lemma was first formulated and established in the smooth case in [12].

PROOF. For each $x \in C$, $\exists \delta > 0$, $\sigma > \epsilon$ and $\mathcal{H}: B(x,\delta) \times [0,\delta] \longrightarrow X$ such that for each $v \in B(u,\delta)$ we have that

$$\operatorname{dist}\big(\mathcal{H}(v,t), v\big) \le t$$
$$\varphi\big(\mathcal{H}(v,t)\big) \le \varphi(v) - \sigma t.$$

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Since C is compact, there exist $x_i, \delta_i > 0$, $\mathcal{H}_i\colon B(x_i,\delta_i)\times[0,\delta_i] \longrightarrow X$ and $\sigma_i > \epsilon$ $(1 \le i \le m)$ such that $C \subseteq \bigcup_{i=1}^m B(x_i, \frac{\delta_i}{2})$ and

$$\begin{cases} \operatorname{dist}\big(\mathcal{H}_i(v,t), v\big) & \le t \\ \varphi\big(\mathcal{H}_i(v,t)\big) - \varphi(v) & \le -\sigma_i t \end{cases}$$

where $v \in B(u_i,\delta_i)$ and $1 \le i \le m$. Denote by B_i the ball $B(u_i, \frac{\delta_i}{2})$ for simplicity. Define

$$f_i(x) = \frac{\operatorname{dist}(x, X \setminus B_i)}{\sum_{i=1}^m \operatorname{dist}(x, X \setminus B_i)}$$

and

$$f = \begin{cases} 0 & x \notin \bigcup_{i=1}^m B_i \\ 1 & x \in C. \end{cases}$$

Let $\tilde{\delta} = \frac{1}{2}\min_i\{\delta_i\}$. Then we define by induction $\{\eta_i\}_{i=1}^m\colon X \times [0,\tilde{\delta}] \longrightarrow X$ such that

$$\begin{cases} \operatorname{dist}\big(\eta_i(v,t), v\big) \le tf(v)\sum_{j=1}^i f_j(v) \\ \varphi\big(\eta_i(v,t)\big) - \varphi(v) \le -\sigma t f(v)\sum_{j=1}^i f_j(v). \end{cases}$$

First, we define η_1 as follows:

$$\eta_1(v,t) = \begin{cases} \mathcal{H}_1\big(v, f(v)f_1(v)t\big) & \text{if } v \in B_1 \\ v & \text{if } v \notin B_1. \end{cases}$$

Suppose now that we have defined η_{j-1}. Since

$$\operatorname{dist}(\eta_{j-1}(v,t), v) \le f(v)\sum_{i=1}^{j-1} f_i(v)t \le \tilde{\delta} \le \delta_j,$$

we can define

$$\eta_j(v,t) = \begin{cases} \mathcal{H}_j(\eta_{j-1}(v,t), f(v)f_j(v)t) & \text{if } v \in B_j \\ \eta_{j-1}(v,t) & \text{if } v \notin B_j. \end{cases}$$

By induction, it is easy to see that $\alpha(v,t) = \eta_m$ and $g(v) = f(v)\sum_{i=1}^m f_i(v)$ verify (i), (ii), (iii) and (iv) of the lemma. ∎

Now we can prove the following theorem which is a quantitative version of Theorem 1.5.

THEOREM 1.10. *Let X, φ, B, c and $\mathcal{F}$ be as in Theorem 1.5. Let F be a closed set dual to $\mathcal{F}$ and satisfying*

$$(*) \qquad \inf \varphi(F) \ge c - \delta.$$

Suppose $0 < \delta < \frac{1}{32}\operatorname{dist}^2(B,F)$, *then for any A in $\mathcal{F}$ satisfying* $\max \varphi(A) \le c + \delta$, *there exists $x_\delta \in X$ such that*

(i) $c - \delta \le \varphi(x_\delta) \le c + 9\delta$;
(ii) $|d\varphi|(x_\delta) \le 18\sqrt{\delta}$;
(iii) $\operatorname{dist}(x_\delta, F) \le 5\sqrt{\delta}$;
(iv) $\operatorname{dist}(x_\delta, A) \le 3\sqrt{\delta}$.

PROOF. Let $\delta = \varepsilon^2/8$. The hypothesis implies that

$$0 < \varepsilon < \frac{1}{2}\operatorname{dist}(B, F) \text{ and } \inf \varphi(F) \geq c - \varepsilon^2/8.$$

We shall prove the existence of $x_\varepsilon \in X$ such that

(i) $c - \varepsilon^2/8 \leq \varphi(x_\varepsilon) \leq c + 9\varepsilon^2/8$;
(ii) $\operatorname{dist}(x_\varepsilon, F) \leq 3\varepsilon/2$;
(iii) $|d\varphi|(x_\varepsilon) \leq 6\varepsilon$;
(iv) $\operatorname{dist}(x_\varepsilon, A) \leq \varepsilon/2$.

This will clearly imply the claim of the theorem. Let $F_\varepsilon = \{x \in X\,;\, \operatorname{dist}(x, F) < \varepsilon\}$ and consider the subspace $\mathcal{L}$ of $C([0,1] \times X\,;\, X)$ consisting of all deformations η such that

$$\eta(t,x) = x \quad \text{for all } (t,x) \in K_0 = (\{0\} \times X) \cup \big([0,1] \times (A \setminus F_\varepsilon) \cup B\big)$$

and $\sup\big\{\operatorname{dist}\big(\eta(t,x), x\big)\,;\, t \in [0,1], x \in X\big\} < +\infty$.

Since $(\{0\} \times X) \cup ([0,1] \times B) \subset K_0$, we get that $\eta(\{1\} \times A) \in \mathcal{F}$ for all η in $\mathcal{L}$. Clearly, the space $\mathcal{L}$ equipped with the uniform metric ρ is a complete metric space.

Set now $\psi(x) = \max\{0, \varepsilon^2 - \varepsilon \operatorname{dist}(x, F)\}$ and define a lower semi-continuous function $I\colon \mathcal{L} \longrightarrow R$ by

$$I(\eta) = \sup\big\{(\varphi + \psi)\big(\eta(1,x)\big)\,;\, x \in A\big\}.$$

Let $l = \inf\{I(\eta)\,;\, \eta \in \mathcal{L}\}$. Since $\eta(\{1\} \times A) \in \mathcal{F}$ for all $\eta \in \mathcal{L}$ and since $\psi = \varepsilon^2$ on F we get from the duality and $(*)$ that

$$I(\eta) \geq \sup\{(\varphi + \psi)(x)\,;\, x \in \eta(\{1\} \times A) \cap F\} \geq c - \varepsilon^2/8 + \varepsilon^2.$$

Hence

$$l \geq c + 7\varepsilon^2/8. \tag{1.1}$$

Consider again the identity element $\bar{\eta}$ in $\mathcal{L}$ and note that

$$l \leq I(\bar{\eta}) = \sup\{(\varphi + \psi)(x)\,;\, x \in A\} < c + \varepsilon^2/8 + \varepsilon^2 = c + 9\varepsilon^2/8. \tag{1.2}$$

Combine (1.1) and (1.2) to get that $\bar{\eta}$ verifies

$$I(\bar{\eta}) < c + 9\varepsilon^2/8 \leq l + \varepsilon^2/4 = \inf\{I(\eta)\,;\, \eta \in \mathcal{L}\} + \varepsilon^2/4. \tag{1.3}$$

Apply Ekeland's theorem to get η_0 in $\mathcal{L}$ such that

$$I(\eta_0) \leq I(\bar{\eta}), \tag{1.4}$$

$$\rho(\eta_0, \bar{\eta}) \leq \varepsilon/2, \tag{1.5}$$

$$I(\eta) \geq I(\eta_0) - (\varepsilon/2)\rho(\eta, \eta_0) \quad \text{for all } \eta \text{ in } \mathcal{L}. \tag{1.6}$$

Let $C = \{x \in \eta_0(\{1\} \times A)\,;\, (\varphi + \psi)(x) = I(\eta_0)\}$. Since $\psi = 0$ outside F_ε we get from (1.1) that

$$\sup(\varphi + \psi)(A \setminus F_\varepsilon) \leq \sup \varphi(A) < c + \varepsilon^2/8 \leq l - 3\varepsilon^2/4.$$

Hence we have that

$$C \cap (A \setminus F_\varepsilon) = \emptyset. \tag{1.7}$$

We shall now prove the following

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CLAIM. There exists $x_\varepsilon \in C$ such that $|d\varphi|(x_\varepsilon) \leq 6\varepsilon$. Before proving it, let us show how it implies Theorem 1.5. First note that since $x_\varepsilon \in C$ we have by (1.3) and (1.4) that $l \leq (\varphi + \psi)(x_\varepsilon) \leq c + 9\varepsilon^2/8$. Since $0 \leq \psi \leq \varepsilon^2$, we get from (1.1) that $c - \varepsilon^2/8 \leq \varphi(x_\varepsilon) \leq c + 9\varepsilon^2/8$ which is assertion (i). For (ii) write $x_\varepsilon = \eta_0(1, x)$ where, in view of (1.7), x is necessarily in F_ε. Hence $\text{dist}(x, F) \leq \varepsilon$. On the other hand, by (1.5) we have $d(x_\varepsilon, x) = d\big(\eta_0(1, x), x\big) \leq \rho(\eta_0, \bar{\eta}) \leq \varepsilon/2$. Hence $\text{dist}(x_\varepsilon, F) \leq 3\varepsilon/2$. Note finally that (iv) is satisfied since $x \in A$.

Back to the above claim. Suppose it is false. Apply Lemma 1.9 to the sets C and $(A \setminus F_\epsilon)$ to get $\alpha(t, x)$ satisfying the conclusion of that lemma with a suitable function g and a time $t_0 > 0$.

For $0 < \lambda < t_0$, consider the function $\eta_\lambda(t, x) = \alpha\big(t\lambda, \eta_0(t, x)\big)$. It belongs to $\mathcal{L}$ since it is clearly continuous on $[0, 1] \times X$ and since for all $(t, x) \in (\{0\} \times X) \cup \big([0, 1] \times (A \setminus F_\epsilon)\big)$, we have $\eta_\lambda(t, x) = \alpha\big(t\lambda, \eta_0(t, x)\big) = \alpha(t\lambda, x) = x$.

Since $\rho(\eta_\lambda, \eta_0) < t\lambda \leq \lambda$, we get from (1.6) that $I(\eta_\lambda) \geq I(\eta_0) - \varepsilon\lambda/2$. Since A is compact, let $x_\lambda \in A$ be such that $(\varphi + \psi)\big(\eta_\lambda(1, x_\lambda)\big) = I(\eta_\lambda)$. We have

$$(\varphi + \psi)\big(\eta_\lambda(1, x_\lambda)\big) - (\varphi + \psi)\big(\eta_0(1, x)\big) \geq -\varepsilon\lambda/2 \quad \text{for every } x \in A. \tag{1.8}$$

Since the Lipschitz constant of ψ is less than ε we get

$$\varphi\big(\eta_\lambda(1, x_\lambda)\big) - \varphi\big(\eta_0(1, x_\lambda)\big) \geq -3\varepsilon\lambda/2. \tag{1.9}$$

On the other hand, by (iii) of lemma 1.9, we have for each x_λ

$$\begin{aligned} \varphi\big(\eta_\lambda(1, x_\lambda)\big) - \varphi\big(\eta_0(1, x_\lambda)\big) &= \varphi\Big(\alpha\big(\lambda, \eta_0(1, x_\lambda)\big)\Big) - \varphi\big(\eta_0(1, x_\lambda)\big) \\ &\leq -6\varepsilon\lambda g\big(\eta_0(1, x_\lambda)\big). \end{aligned} \tag{1.10}$$

Combining (1.9) and (1.10) we get

$$-3\varepsilon/2 \leq -6\varepsilon g\big(\eta_0(1, x_\lambda)\big). \tag{1.11}$$

If now x_0 is any cluster point of (x_λ) when $\lambda \longrightarrow 0$, we have from (1.8) that $\eta_0(1, x_0) \in C$ and hence $g\big(\eta_0(1, x_0)\big) = 1$. This clearly contradicts (1.11) and therefore the initial claim was true. The proof of the theorem is complete. ∎

2. **Structure of the critical set in the 1-dimensional case.** In this section, we shall assume that the complete metric space X is contractible and locally connected. For $u, v \in X$, we denote by $\mathcal{F}_v^u$ the set of all continuous paths joining two points u, v in X *i.e.*

$$\mathcal{F}_v^u = \{g \in C([0, 1]\,; X)\,; g(0) = u \text{ and } g(1) = v\}.$$

Clearly $\mathcal{F}_v^u$ is a homotopy-stable family with boundary $\{u, v\}$. In fact by the concept introduced in the next section, $\mathcal{F}_v^u$ is a homotopy-stable family of dimension 1. We say that a closed subset F of X separates u, v if F is dual to $\mathcal{F}_v^u$. Since any connected subset of a locally connected complete metric space X is path connected, a closed subset F of X separates u and v if and only if u, v do not belong to one connected component of $X \setminus F$.

To classify the various types of critical points, we use the following notation:

$G_c = \{x \in X\,;\, \varphi(x) < c\}$,
$L_c = \{x \in X\,;\, \varphi(x) \geq c\}$,
$M_c = \{x \in K_c\,;\, x$ is a local minimum of $\varphi\}$,
$P_c = \{x \in K_c\,;\, x$ is a proper local maximum of φ, that is x is a local maximum of φ and $x \in \overline{G_c}\}$,
$S_c = \{x \in K_c\,;\, x$ is a saddle point of φ, that is in each neighborhood of x there exist two points y and z such that $\varphi(y) < \varphi(x) < \varphi(z)\}$.

Following Hofer [14], we have the following definition:

DEFINITION 2.1. Say that a point x in K_c is *of mountain-pass type* if for any neighborhood N of x, the set $\{x \in N\,;\, \varphi(x) < c\}$ is nonempty and not path connected. We denote by H_c the set of critical points of mountain pass type at the level c.

Now we state the general mountain pass principle of Ghoussoub and Preiss [13] which is a corollary of Theorem 1.5.

THEOREM 2.2 (GENERAL MOUNTAIN PASS PRINCIPLE). *Let $\varphi: X \longrightarrow \mathbb{R}$ be a continuous function on X. Take two points u and v in X and consider the number*

$$c = \inf_{g \in \mathcal{F}_v^u} \max_{0 \leq t \leq 1} \varphi\big(g(t)\big).$$

Suppose F is a closed subset of X separating u, v such that $\inf \varphi(F) \geq c$. *Then there exists a sequence $(x_n)_n$ in X verifying the following:*
(i) $\lim_n \operatorname{dist}(x_n, F) = 0$;
(ii) $\lim_n \varphi(x_n) = c$;
(iii) $\lim_n |d\varphi|(x_n) = 0$.
Moreover, if φ verifies $(PS)_{F,c}$, then $F \cap K_c \neq \emptyset$.

COROLLARY 2.3. *In Theorem 2.2, assume that P_c contains no compact set that separates u and v, then:*
(1) Either $F \cap M_c \neq \emptyset$ or $F \cap S_c \neq \emptyset$.
(2) $S_c \neq \emptyset$ if φ verifies $(PS)_{N_\epsilon(F\cup K_c),c}$ for some $\epsilon > 0$ and $u, v \neq \overline{M_c}$;

PROOF. (1) We first note that any connected subset of a locally connected complete metric space X is path connected. Since X is contractible, by a result of Whyburn (see [17] Chapter VIII, Section 57, III, Theorem 1) we can find a closed connected subset $\hat{F} \subseteq F$ that also separates u and v. Note that $\hat{F} \cap K_c = \hat{F} \cap P_c$ and the latter is relatively open in $\hat{F}$ while $\hat{F} \cap K_c$ is closed. Since $\hat{F}$ is connected, then either $\hat{F} \cap P_c = \emptyset$ or $\hat{F} \cap P_c = \hat{F}$. But the first case is impossible since by Theorem 2.2 we have $\hat{F} \cap P_c = \hat{F} \cap K_c \neq \emptyset$. Hence $\hat{F} \subset P_c$ which is impossible by assumption and this proves (1).

To prove (2), first observe that K_c is the disjoint union of S_c, M_c and P_c. By the $(PS)_{N_\epsilon(F\cup K_c),c}$ condition, we know that K_c is compact. Suppose $S_c = \emptyset$. For each $x \in M_c$, there exists a $B(x, \epsilon_x)$ such that $B(x, \epsilon_x) \subseteq L_c$. Let $N = \bigcup_{x \in M_c} B(x, \epsilon_x)$. Then $M_c \subseteq N \subseteq L_c$. Since $u, v \notin \overline{M_c}$ and $\overline{M_c}$ is compact, we may assume that $u, v \notin \bar{N}$. Now put $F_0 = (F \setminus N) \cup \partial N$. It is clear that $\inf_{x \in F_0} \varphi(x) \geq c$ and that F_0 separates u, v. Moreover,

$F_0 \cap (M_c \cup S_c) = \emptyset$. By (1), P_c must contain a compact subset that separates u, v. A contradiction that completes the proof. ■

Before we state the results about the critical set generalized by the above theorem, we introduce the following definition:

DEFINITION 2.4. For A, B two disjoint subsets of X and any nonempty subset C of X, we say that A, B are *connected* through C if there is no $F \subseteq C \cup A \cup B$ relatively both closed and open such that $A \subseteq F$ and $F \cap B = \emptyset$.

Now we are ready to state the local structure result about the critical set generated by general mountain pass principle of Ghoussoub-Preiss.

THEOREM 2.5. *In Theorem 2.2, we assume that φ verifies $(PS)_{N_\epsilon(F),c}$ for some $\epsilon > 0$. Then either $F \cap \overline{M_c} \neq \emptyset$ or $F \cap K_c$ contains a critical point of mountain-pass type.*

We also have the following.

THEOREM 2.6. *In Theorem 2.2, we further assume $u, v \notin K_c$ and that φ verifies $(PS)_{N_\epsilon(F\cup K_c),c}$. Then one of the following three assertions concerning the set K_c must be true:*

(1) P_c contains a compact subset that separates u and v;
(2) K_c contains a saddle point of mountain-pass type;
(3) There are finitely many components of G_c, say C_i $(i = 1, 2, \ldots, n)$ such that

$$S_c = \bigcup_{i=1}^{n} S_c^i, \quad S_c^i \cap S_c^j = \emptyset \quad (i \neq j\ 1 \leq i, j \leq n)$$

where $S_c^i = S_c \cap \overline{C_i}$. Moreover there are at least two of them $S_c^{i_1}, S_c^{i_2} (i_1 \neq i_2\ 1 \leq i_1, i_2 \leq n)$ such that the sets $\overline{M_c} \cap S_c^{i_1}$, $\overline{M_c} \cap S_c^{i_2}$ are nonempty and connected through M_c (see Definition 2.4).

We need several lemmas in order to prove the above two theorems. We begin with the following easy lemma whose proof is left to the interested reader.

LEMMA 2.7. *Let M be a subset of a metric space (X, d). Suppose $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. If M_1 is both open and closed relative to the subspace M, then there exist open sets D_1, D_2 of X such that*

$$M_1 \subseteq D_1, \quad M_2 \subseteq D_2, \quad D_1 \cap D_2 = \emptyset.$$

LEMMA 2.8. *Let S^i $(i = 1, 2, \ldots, n)$ be n mutually disjoint compact subsets of a metric space (X, d) and let M be any nonempty subset of X. If for all i, j $(i \neq j\ i, j = 1, 2, \ldots, n)$, the sets $S^i \cap \bar{M}$ and $S^j \cap \bar{M}$ are not connected through M, then there are n mutually disjoint open sets N^i $(i = 1, 2, \ldots, n)$ such that*

$$M \cup \left(\bigcup_{i=1}^{n} S^i\right) \subseteq \bigcup_{i=1}^{n} N^i \text{ and } S^i \subseteq N^i \quad \text{for all } i = 1, 2, \ldots, n. \tag{2.1}$$

PROOF. For each $i (i = 1, 2, \ldots, n)$, we denote by M^i the compact set $S^i \cap \bar{M}$. Since by assumption none of the pairs $M^i, M^j (i \neq j\ i, j = 1, 2, \ldots, n)$ are connected through M, there exist by Lemma 2.7 open sets O_{ij} and $P_{ij} (O_{ij} = P_{ji}, i \neq j\ i, j = 1, 2, \ldots, n)$ such that

$$M^i \subseteq O_{ij},\ M^j \subseteq P_{ij},\ O_{ij} \cap P_{ij} = \emptyset \quad (i \neq j\ i, j = 1, 2, \ldots, n)$$

and

$$M^i \cup M \cup M^j \subseteq O_{ij} \cup P_{ij} \quad (i \neq j\ i, j = 1, 2, \ldots, n).$$

For each $i (i = 1, 2, \ldots, n)$, let

$$O_i = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} O_{ij}, \quad P_i = \bigcup_{\substack{j=1 \\ j \neq i}}^{n} P_{ij} \tag{2.2}$$

and

$$M_s = \bigcup_{i=1}^{n} M^i, \quad \tilde{M}^i = \bigcup_{\substack{j=1 \\ j \neq i}}^{n} M^j. \tag{2.3}$$

Then

$$M^i \subseteq O_i, \quad \tilde{M}^i \subseteq P_i \tag{2.4}$$

and

$$O_i \cap P_i = \emptyset, \quad M_s \cup M \subseteq O_i \cup P_i. \tag{2.5}$$

Put for each i $(i = 1, 2, \ldots, n)$

$$O^i = O_i \bigcap \Big(\bigcap_{\substack{j=1 \\ j \neq i}}^{n} P_j \Big). \tag{2.6}$$

Then by (2.2)–(2.5), we have

$$M^i \subseteq O^i,\ O^i \cap O^j = \emptyset \quad (i \neq j\ i, j = 1, 2, \ldots, n). \tag{2.7}$$

It is not generally true that $M_s \cup M \subseteq \bigcup_{i=1}^{n} O^i$. In order to prove the lemma, we let

$$M' = (M_s \cup M) \setminus \Big(\bigcup_{i=1}^{n} O^i \Big), \quad M'' = (M_s \cup M) \cap \Big(\bigcup_{i=1}^{n} O^i \Big).$$

Then

$$M_s \cup M = M' \cup M'', \quad M' \cap M'' = \emptyset. \tag{2.8}$$

By (2.5) and (2.6), we see that M'' is both open and closed relative to $M_s \cup M$. Again by Lemma 2.7, there exist two open sets D' and D'' such that

$$M' \subseteq D', \quad M'' \subseteq D'', \quad D' \cap D'' = \emptyset. \tag{2.9}$$

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Now for each i ($i = 1,2,\dots,n$) put $O^i_D = O^i \cap D''$. By (2.7) and (2.9), then

$$(2.10)\qquad D' \cap \Big(\bigcup_{i=1}^{n} O^i_D\Big) = \emptyset,\ M^i \subseteq O^i_D,\ O^i_D \cap O^j_D = \emptyset \quad (i \neq j\ i,j = 1,2,\dots,n).$$

By the compactness of S^i and M^i, we may introduce

(2.11)

$$a_i = \operatorname{dist}(M^i, X \setminus O^i_D) > 0, \quad \delta_1 = \frac{1}{2}\min\{\operatorname{dist}(S^i, S^j)\ ;\ i \neq j\ i,j = 1,2,\dots,n\} > 0.$$

Let $\delta_2 = \frac{1}{4}\min\{a_i, \delta_1\ ;\ i = 1,2,\dots,n\}$ and

$$(2.12)\qquad Q_i = \{x \in X\ ;\ \operatorname{dist}(x, M^i) < \delta_2\}, \quad S^i_q = S^i \setminus Q_i.$$

Then

$$Q_i \subseteq O^i_D, \quad S^i_q \cap \bar{M} = \emptyset.$$

By (2.11), we see that

$$\operatorname{dist}(S^i_q, Q_j) \geq \operatorname{dist}(S^i_q, S^j_q) - \delta_2 \geq 3\delta_2.$$

By the compactness of S^i_q, we may also introduce

$$b_i = \frac{1}{4}\operatorname{dist}(S^i_q, M) > 0, \quad \delta_3 = \min\{b_i, \delta_2\ ;\ i = 1,2,\dots,n\} > 0.$$

Put

$$(2.13)\qquad P = \{x \in X\ ;\ \operatorname{dist}(x, M) < \delta_3\}$$

and

$$(2.14)\qquad N_i = Q_i \cup (O^i_D \cap P), \quad R' = D' \cap P.$$

Then

$$(2.15)\qquad M' \subseteq R', \quad M'' \subseteq \bigcup_{i=1}^{n} N_i.$$

By (2.10), we have that $R' \cap (\bigcup_{i=1}^{n} N_i) = \emptyset$ and $N_i \cap N_j = \emptyset$ ($i \neq j\ i,j = 1,2,\dots,n$). Furthermore

$$(2.16)\qquad \operatorname{dist}(S^i_q, P) \geq \operatorname{dist}(S^i_q, M) - \delta_3 \geq 3\delta_3.$$

Hence

$$(2.17)\qquad \operatorname{dist}(S^i_q, R') \geq \operatorname{dist}(S^i_q, P) \geq \operatorname{dist}(S^i_q, M) - \delta_3 \geq 3\delta_3.$$

By (2.14) and (2.16), we also have that

$$(2.18)\qquad \operatorname{dist}(S^i_q, N_j) \geq \min\{\operatorname{dist}(S^i_q, Q_j), \operatorname{dist}(S^i_q, P)\} \geq \min(3\delta_2, 3\delta_3) \geq 3\delta_3.$$

Now let

$$N^1 = N_1 \cup \{x \in X\ ;\ \operatorname{dist}(x, S^1_q) < \delta_3\} \cup R',$$
$$N^i = N_i \cup \{x \in X\ ;\ \operatorname{dist}(x, S^i_q) < \delta_3\} \quad (i \neq 1\ i = 1,2,\dots,n).$$

By (2.8), (2.12) and (2.15) it follows that

$$S^i \subseteq N^i, \quad M \subseteq \bigcup_{i=1}^{n} N^i. \tag{2.19}$$

By (2.10), (2.17) and (2.18), we see that

$$N^i \cap N^j = \emptyset \quad (i \neq j\ i,j = 1,2,\ldots,n). \tag{2.20}$$

So (2.19) and (2.20) imply that N^i satisfy (2.1) and this completes the proof of the lemma. ■

LEMMA 2.9. *Let F_0 be a closed subset of X that separates two distinct points u and v. Let Z_i $(i = 1,2,\ldots,n)$ be n mutually disjoint open subsets of X such that $u, v \notin \bigcup_{i=1}^n \overline{Z_i}$. Let G be an open subset of $X \setminus F_0$ and denote by $Y_i = Z_i \setminus G$. Then the following holds:*

(i) The set $F_1 = [F_0 \setminus (\bigcup_{i=1}^n Z_i)] \cup (\bigcup_{i=1}^n \partial Y_i)$ separates u and v;

(ii) If A_i $(i = 1,2,\ldots,n)$ are n nonempty connected components of G and for each i $(1 \leq i \leq n)$ $T_i \subseteq (Z_i \cap \partial A_i)$ is a relatively open subset of ∂Y_i such that $T_i \cap \partial L = \emptyset$ for any connected component L of G with $L \neq A_i$, then the set $F_2 = [F_0 \setminus (\bigcup_{i=1}^n Z_i)] \cup (\bigcup_{i=1}^n \partial Y_i \setminus T_i)$ also separates u and v.

PROOF. (i) Since $G \subseteq X \setminus F_0$, we have

$$F_1 = \Big[F_0 \setminus \Big(\bigcup_{i=1}^{n} Y_i\Big)\Big] \cup \Big(\bigcup_{i=1}^{n} \partial Y_i\Big). \tag{2.21}$$

Clearly F_1 is closed and $u, v \notin F_1$. We need only to show that for any $g \in \Gamma_v^u$, $g([0,1]) \cap F_1 \neq \emptyset$. If $g([0,1]) \cap (F_0 \setminus \bigcup_{i=1}^n Y_i) \neq \emptyset$, we are done. Otherwise $g([0,1]) \cap (\bigcup_{i=1}^n Y_i) \cap F_0 \neq \emptyset$ so that if $g([0,1]) \cap (\bigcup_{i=1}^n \partial Y_i) = \emptyset$, then $g([0,1]) \subseteq \bigcup_{i=1}^n Y_i \subseteq \bigcup_{i=1}^n Z_i$ which contradicts that $u, v \notin \bigcup_{i=1}^n \overline{Z_i}$.

(ii) We first prove the following claims: For $i,j = 1,2,\ldots,n$, we have:

(a) $T_i \subseteq Y_i \cap \partial Y_i$, $T_i \cap G = \emptyset$ and $A_i \cap F_2 = \emptyset$;

(b) $T_j \cap \overline{Y_i} = \emptyset$ and $T_i \cap T_j = \emptyset$ if $i \neq j$;

(c) $Z_i \cap (\partial G \setminus T_i) \subseteq \partial Y_i \setminus T_i$.

- (a) Since G is open, it is clear from the definition of T_i that $T_i \subseteq Z_i \cap \partial G$ so that $T_i \subseteq Y_i \cap \partial Y_i$ and $T_i \cap G = \emptyset$ for $i = 1,2,\ldots,n$. On the other hand, $A_i \cap \overline{Y_j} \subseteq A_i \cap \overline{(Z_j \setminus G)} \subseteq A_i \cap (\overline{Z_j} \setminus G) = \emptyset$, hence $A_i \cap F_2 = \emptyset$.
- (b) If $i,j = 1,2,\ldots,n$ and $i \neq j$, then $T_j \cap \overline{Y_i} \subseteq T_j \cap \overline{Z_i} \subseteq Z_j \cap \overline{Z_i} = \emptyset$ and $T_i \cap T_j \subseteq Z_i \cap Z_j = \emptyset$.
- (c) Since G is open, we have that for any $x \in Z_i \cap \partial G \setminus T_i$, $x \notin G$, hence $x \in Z_i \setminus G$ and $x \in Y_i$. Moreover, for any $x \in \partial G \setminus T_i$ and any $\varepsilon > 0$ there is $y \in B(x, \epsilon) \cap G$. Clearly $y \notin Y_i$ so that $x \in \partial Y_i$. Since $T_i \cap Z_i \cap (\partial G \setminus T_i) = \emptyset$, we have that $x \in \partial Y_i \setminus T_i$.

Back to the proof of the Lemma, we note first that the set F_2 is closed and is equal to

$$F_2 = \Big[F_0 \setminus \Big(\bigcup_{i=1}^{n} Y_i\Big)\Big] \cup \Big(\bigcup_{i=1}^{n} \partial Y_i \setminus T_i\Big). \tag{2.22}$$

Clearly $u, v \notin F_2$ and we need only to show that for any $g \in \Gamma_v^u$, $g([0,1]) \cap F_2 \neq \emptyset$.

Suppose not, and take $g_0 \in \Gamma_v^u$ such that $g_0([0,1]) \cap F_2 = \emptyset$. We shall work toward a contradiction.

First by (2.21), we have $g_0([0,1]) \cap (\bigcup_{i=1}^n T_i) \neq \emptyset$. Let i_1 be the first $i \in \{1, \ldots, n\}$ such that $g_0([0,1]) \cap T_i \neq \emptyset$. We shall find a $g_{i_1} \in \Gamma_v^u$ such that

$$g_{i_1}([0,1]) \cap F_2 = \emptyset,\ g_{i_1}([0,1]) \cap T_i = \emptyset \quad \text{for } 1 \leq i \leq i_1. \tag{2.23}$$

To do this, we define the following times:

$$s_1 = \inf\{t \in [0,1]\,;\, g_0(t) \in Z_{i_1}\}, \quad s_2 = \inf\{t \in [0,1]\,;\, g_0(t) \in Y_{i_1}\}, \tag{2.24}$$

$$t_1 = \sup\{t \in [0,1]\,;\, g_0(t) \in Y_{i_1}\}, \quad t_2 = \sup\{t \in [0,1]\,;\, g_0(t) \in Z_{i_1}\}. \tag{2.25}$$

We shall show the following:

(d) $0 < s_1 < s_2 < t_1 < t_2 < 1$;

(e) $g_0(t_1)$ and $g_0(s_2)$ belong to T_{i_1};

(f) $g_0(t) \in A_{i_1}$ for $t \in (s_1, s_2) \cup (t_1, t_2)$.

Indeed, it is clear that $0 \leq s_1 \leq s_2 \leq t_1 \leq t_2$. Since $u, v \notin \bigcup_{i=1}^n \bar{Z}_i$, we have $0 < s_1$ and $t_2 < 1$. On the other hand, $g_0(t_2) \notin Z_{i_1}$ since the latter is open, while $g_0(t_1) \in \partial Y_{i_1} \cap T_{i_1}$ since $g_0([0,1]) \cap F_2 = \emptyset$, hence (a) yields that $g_0(t_1) \in \partial Y_{i_1} \cap T_{i_1} = T_{i_1} \subset Z_{i_1}$. Modulo a similar reasoning for s_1, s_2, (d) and (e) are therefore verified.

To prove (f), we note first that $g_0(t) \in G$ for $t \in (s_1, s_2) \cup (t_1, t_2)$, since otherwise $g_0(t) \in Y_{i_1}$ which contradicts (2.24) and (2.25). So, for any $t \in (t_1, t_2)$, $g_0(t) \in U$ for some connected component U of G. If $U \neq A_{i_1}$, we have that $T_{i_1} \cap \partial U = \emptyset$ and since $g_0(t_1) \in T_{i_1}$, we see that $g_0(t_1) \notin \partial U$. Hence there must be $t_3 \in (t_1, t)$ such that $g_0(t_3) \in \partial U \subseteq \partial G \setminus T_{i_1}$. By (c) we see that $g_0(t_3) \in F_2$ which is a contradiction. So $U = A_{i_1}$ and consequently, $g_0(t) \in A_{i_1}$ for all $t \in (t_1, t_2)$, and (f) is proved.

Since now A_{i_1} is path connected, then for $s_1 < s^{i_1} < s_2$, $t_1 < t^{i_1} < t_2$, we can use a path in A_{i_1} to join $g_0(s^{i_1})$ and $g_0(t^{i_1})$. In this way, we get a path $g_{i_1} \in \Gamma_v^u$ such that $g_{i_1}([0,1]) \cap T_{i_1} = \emptyset$ and $g_{i_1}([0,1]) \cap T_i = \emptyset$ for $1 \leq i \leq i_1$, since by (a), $A_{i_1} \cap T_i = \emptyset$ for all $i = 1, 2, \ldots, n$. On the other hand, since $A_{i_1} \cap F_2 = \emptyset$, we get that $g_{i_1}([0,1]) \cap F_2 = \emptyset$ and (2.23) is established.

Next, let i_2 be the first $i \in \{1, \ldots, n\}$ such that $g_{i_1}([0,1]) \cap T_i \neq \emptyset$. Clearly $i_1 < i_2 \leq n$. In the same way, we can construct $g_{i_2} \in \Gamma_v^u$ such that for $1 \leq i \leq i_2$,

$$g_{i_2}([0,1]) \cap F_2 = \emptyset \quad \text{and} \quad g_{i_2}([0,1]) \cap T_i = \emptyset.$$

By iterating a finite number of times, we will get a $g_n \in \Gamma_v^u$ such that for $1 \leq i \leq n$,

$$g_n([0,1]) \cap F_2 = \emptyset \quad \text{and} \quad g_n([0,1]) \cap T_i = \emptyset.$$

But this contradicts assertion (i) and the lemma is proved. ∎

PROOF OF THEOREM 2.5. We shall prove it by contradiction. Suppose $F \cap K_c$ contains no critical point of mountain-pass type and $F \cap \overline{M_c} = \emptyset$. Let $\tilde{F} = F \cap L_c$. Then we claim that:

There exist finitely many components of G_c, say $C_1, \ldots, C_n$ and $\mu_1 > 0$ such that

$$G_c \cap \{x\,;\, \mathrm{dist}(x, \tilde{F} \cap K_c) < \mu_1\} \subseteq C_1 \cup C_2 \cup \cdots \cup C_n. \tag{2.26}$$

Indeed, if not, we could find a sequence x_i in S_c and a sequence $(C_i)_i$ of different components of G_c such that $\mathrm{dist}(x_i, C_i) \longrightarrow 0$. But then any limit point of the sequence x_i would be a saddle point for φ of mountain-pass type, thus contradicting our assumption. The claim is hence proved. We clearly may assume that $C_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$.

Clearly for all i,j ($i,j = 1, 2, \ldots, n$ $i \neq j$), we have

$$(\tilde{F} \cap K_c \cap \overline{C_i}) \cap (\tilde{F} \cap K_c \cap \overline{C_j}) = (\tilde{F} \cap K_c) \cap (\overline{C_i} \cap \overline{C_j}) = \emptyset. \tag{2.27}$$

Indeed, otherwise $\tilde{F}$ and hence F will contain a critical point of mountain-pass type. Put

$$\tilde{S}_c^i = \tilde{F} \cap K_c \cap \overline{C_i} = F \cap L_c \cap K_c \cap \overline{C_i}.$$

By the compactness of $\tilde{S}_c^i$ and (2.27), we may find for each i ($i = 1, 2, \ldots, n$) an open set N^i such that

$$\tilde{S}_c^i \subseteq N^i, \quad \overline{N^i} \cap \overline{N^j} = \emptyset \quad \text{for all } i,j = 1, 2, \ldots, n \quad i \neq j. \tag{2.28}$$

Since $F \cap \overline{M_c} = \emptyset$ and $u, v \notin F$, we may assume

$$\overline{M_c} \cap \Big(\bigcup_{i=1}^{n} \overline{N^i}\Big) = \emptyset \quad u, v \notin \bigcup_{i=1}^{n} \overline{N^i}. \tag{2.29}$$

Next for each i ($1 \leq i \leq n$), for any $x \in \tilde{S}_c^i$ there must be $B(x, \epsilon_x)$ such that $B(x, \epsilon_x) \cap U = \emptyset$ for any component U of G_c with $U \neq C_i$. Put

$$\tilde{T}_i^c = \bigcup_{x \in \tilde{S}_c^i} B(x, \epsilon_x/2) \cap \partial C_i \cap N^i.$$

Then let

$$Y_i^c = N^i \setminus G_c \quad \text{and} \quad \hat{F} = \Big[\tilde{F} \setminus \Big(\bigcup_{i=1}^{n} N^i\Big)\Big] \cup \Big[\bigcup_{i=1}^{n} \Big(\partial Y_i^c \setminus T_i^c\Big)\Big]. \tag{2.30}$$

Clearly, $\inf_{x \in \hat{F}} \varphi(x) \geq c$. Since T_i^c is open relative to $N^i \cap \partial C_i$ and $\tilde{S}_c^i \subseteq T_i^c$ by (2.29), we see that we can apply Lemma 2.9 to conclude that $\hat{F}$ separates u, v. By (2.28) and (2.30), we may assume that $\bigcup_{i=1}^{n} \overline{Y_i^c} \subseteq N_\epsilon(F)$. Hence by Theorem 2.2, we have $\hat{F} \cap K_c \neq \emptyset$. On the other hand by (2.26), (2.29) and the assumption that $F \cap \overline{M_c} = \emptyset$, we have $\hat{F} \cap K_c = \emptyset$. This is a contradiction. ∎

PROOF OF THEOREM 2.6. Suppose assertions (2) and (3) are not true. In order to prove the theorem, we need to show that assertion (1) holds true.

As in the proof of Corollary 2.3, we know that K_c is the disjoint union of S_c, M_c and P_c. Also by the $(PS)_{N_\epsilon(F \cup K_c), c}$ condition, K_c is compact. It is also clear that S_c is closed and compact. We will assume that $S_c \neq \emptyset$ since otherwise we conclude by Corollary 2.3. We start with the following:

CLAIM 1. There exist finitely many components of G_c, say C_i $(i = 1, 2, \ldots, n)$ and $\eta_1 > 0$ such that

$$G_c \cap \{x\,;\, \mathrm{dist}(x, S_c) < \eta_1\} \subseteq \bigcup_{i=1}^{n} C_i. \tag{2.31}$$

Indeed, if not, we could find a sequence x_i in S_c and a sequence $(C_i)_i$ of different components of G_c such that $\mathrm{dist}(x_i, C_i) \longrightarrow 0$. But then any limit point of the sequence x_i would be a saddle point for φ of mountain-pass type, thus contradicting our assumption that assertion (2) is false. Claim 1 is hence proved. We clearly may assume that $C_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$.

Next for each $i = 1, 2, \ldots, n$, let $S_c^i = S_c \cap \overline{C_i}$. Clearly they all are compact and mutually disjoint. Also we have that

$$S_c = \bigcup_{i=1}^{n} S_c^i. \tag{2.32}$$

CLAIM 2. There are n mutually disjoint open sets N^i $(i = 1, 2, \ldots, n)$ such that $u, v \notin \bigcup_{i=1}^{n} \overline{N^i}$ and

$$S_c \cup M_c \subseteq \bigcup_{i=1}^{n} N^i \text{ and } S_c^i \subseteq N^i \quad \text{for all } i = 1, 2, \ldots, n. \tag{2.33}$$

Indeed, we have two cases to consider.

CASE 1: $M_c = \emptyset$. This is a trivial case. By the initial assumption that $u, v \notin K_c$, for each i $(i = 1, 2, \ldots, n)$ there exists an open neighborhood N^i of S_c^i such that $u, v \notin \ni$. Since the S_c^i's are mutually disjoint compact sets, we may take the N^i's in such a way that they are also mutually disjoint. This proves Claim 2 in Case 1.

CASE 2: $M_c \neq \emptyset$. In this case we are in a situation where we have n mutually disjoint compact sets S_c^i $(i = 1, 2, \ldots, n)$ and a nonempty set M_c. Moreover all the pairs $S_c^i \cap \overline{M_c}, S_c^j \cap \overline{M_c}$ $(i \neq j\; i, j = 1, 2, \ldots, n)$ are not connected through M_c since assertion (3) is assumed false. Applying Lemma 2.8, we can then find n mutually disjoint open sets N^i such that (2.33) is verified. Since $u, v \notin K_c$, we may clearly assume that $u, v \notin \bigcup_{i=1}^{n} \ni$. Claim 2 is proved in both cases.

In order to finish the proof of Theorem 2.6, we still need the following

CLAIM 3. There exists a closed set $\hat{F}$ such that $\hat{F}$ separates u, v while

$$\inf_{x \in \hat{F}} \varphi(x) \geq c \quad \text{and} \quad \hat{F} \cap (S_c \cup M_c) = \emptyset. \tag{2.34}$$

To prove Claim 3, we first let for each i $(i = 1, 2, \ldots, n)$

$$Y_i^c = N^i \setminus G_c. \tag{2.35}$$

Then for each i ($1 \leq i \leq n$), for any $x \in S_c^i$, there must by $B(x, \epsilon_x)(\epsilon_x > 0)$ such that for any connected component U of G_c with $C_i \neq U$, $B(x, \epsilon_x) \cap U = \emptyset$. Otherwise x is a saddle point of mountain-pass type and this contradicts that assertion (2) is assumed false. Put

$$T_i^c = \bigcup_{x \in S_c^i} B(x, \epsilon_x/2) \cap \partial C_i \cap N^i \cap \{x \in X\, ; \, \mathrm{dist}(x, S_c^i) \leq \eta_1\}. \tag{2.36}$$

Clearly

$$S_c^i \subseteq T_i^c, \quad T_i^c \subseteq N^i \cap \partial C_i \tag{2.37}$$

and T_i^c is open relative to $N^i \cap \partial C_i$. Also $T_i^c \cap \partial U = \emptyset$ for any component U of G_c with $U \neq C_i$. Now let

$$\hat{F} = \left[(F \cap L_c) \setminus \left(\bigcup_{i=1}^n N^i\right)\right] \cup \left(\bigcup_{i=1}^n \partial Y_i^c \setminus T_i^c\right).$$

Then clearly, $\inf_{x \in \hat{F}} \varphi(x) \geq c$. Since $F \cap L_c$ separates u, v and in view of Claim 1, Claim 2, (2.35) and (2.37), we see that we can apply Lemma 2.9 with $A_i = C_i$, $G = G_c$, $Z_i = N^i$, $Y_i = Y_i^c$, $T_i = T_i^c$ for all $i = 1, 2, \ldots, n$ to conclude that $\hat{F}$ separates u, v. On the other hand, since $M_c \cap (\overline{G_c} \setminus G_c) = \emptyset$, we have by (2.33) and (2.35), that $\partial Y_i^c \cap M_c = \emptyset$. Therefore by (2.32) and (2.36), we have $\bigcup_{i=1}^n (\partial Y_i^c \setminus T_i^c) \cap (S_c \cup M_c) = \emptyset$. Hence $\hat{F} \cap (M_c \cup S_c) = \emptyset$ and Claim 3 is thus proved.

Finally by Corollary 2.3, we see that $\hat{F} \cap P_c$ and hence P_c must contain a compact subset that separates u, v which implies assertion (1). This clearly finishes the proof of the theorem. ∎

It is important to know the number of critical points. Rather surprisingly, we have the following corollary concerning the cardinality of the critical set K_c generated by Theorem 2.2. In the following corollary we let bind(X) to be the least cardinality of all the subset U of X such that $X \setminus U$ is not connected.

COROLLARY 2.10. *Under the hypothesis of Theorem 2.6, one of the following three assertions must be true:*

(1) K_c has a saddle point of mountain-pass type;

(2) The cardinality of P_c is at least the same as bind(X) *(see above);*

(3) The cardinality of M_c is at least the same as the continuum.

PROOF. If K_c does not contain a saddle point of mountain-pass type, then either assertion (1) or assertion (3) in Theorem 2.6 is true. Let us first assume that assertion (3) is true. Then there exist two disjoint nonempty closed subsets of K_c, say, M_c^1 and M_c^2 which are connected through M_c. Clearly $\mathrm{dist}(M_c^1, M_c^2) = d > 0$. For any $0 < \sigma < d$, let

$$M_\sigma = \{x \in X\, ; \, \mathrm{dist}(x, M_c^1) < \sigma\}.$$

Then $\overline{M_\sigma} \cap M_c^2 = \emptyset$, $M_c^1 \subseteq M_\sigma$. We claim that $\partial M_\sigma \cap M_c \neq \emptyset$. Otherwise, there will be two disjoint open sets M_σ and $X \setminus \overline{M_\sigma}$ such that

$$M_c^1 \subseteq M_\sigma, \quad M_\sigma \cap M_c^2 = \emptyset, \quad M_c \cup M_c^1 \cup M_c^2 \subseteq M_\sigma \cup (X \setminus \overline{M_\sigma}).$$

This contradicts that M_c^1, M_c^2 are connected through M_c. Now let $m_\sigma \in \partial M_\sigma \cap M_c$. Then we have a map f from $(0, d)$ to M_c defined as:

$$f: \sigma \in (0, d) \longrightarrow m_\sigma \in M_c.$$

Clearly f is injective. Hence assertion (3) in Corollary 2.10 is true. If instead, assertion (1) in Theorem 2.6 is true, then since P_c separates u and v we have that $X \setminus P_c$ is not connected. Hence by the definition of bind(X), we see the assertion (2) is true and this completes the proof the corollary. ■

As an interesting application of the above corollary, we have the following.

COROLLARY 2.11. *Suppose φ has a local maximum and a local minimum on a Banach space X. If φ satisfies (PS) and if* $\dim(X) \geq 2$, *then necessarily φ has a third critical point.*

We need the following lemma.

LEMMA 2.12. *Let φ be continuous functional on a Banach space X.*

(i) If φ is bounded below and verifies $(PS)_c$ with $c = \inf_X \varphi$, then every minimizing sequence for φ is relatively compact. In particular, φ achieves its minimum at a point in K_c;

(ii) If $d = \liminf_{\|u\| \to \infty} \varphi(u)$ is finite, then φ does not verify $(PS)_d$.

PROOF. (i) It is an immediate application of Proposition 1.4.

(ii) For $r \geq 0$, let $m(r) = \inf_{\|u\| \geq r} \varphi(u)$ and $D_r = \{x \in X ; \|x\| \geq r\}$. Clearly $m(r)$ is nondecreasing and $|d\varphi|_X(x) = |d\varphi|_{D_r}(x)$ for each $x \in \operatorname{Int} D_r$ the interior of D_r. We shall prove that for any $\frac{1}{2} > \epsilon > 0$ and $\hat{r} > 0$, there exists $y_\epsilon \in \operatorname{Int} D_{\hat{r}}$ such that $|d\varphi|_X(y_\epsilon) \leq \epsilon$ and $|\varphi(y_\epsilon) - d| \leq \epsilon^2$. This will clearly prove the lemma. To see this, choose $r > \max\{1, \hat{r}\}$ such that $m(r) > d - \epsilon^2$. Then choose $u \in D_{2r}$ such that $\varphi(u) \leq m(2r) + \epsilon^2 \leq d + \epsilon^2$. By Ekeland's variational principle we have a $v \in D_r$ such that

$$(**) \qquad \varphi(v) \leq \varphi(x) - \epsilon\|x - v\| \quad \text{for all } x \in D_r.$$

Hence $d^2 - \epsilon^2 \leq m(r) \leq \varphi(v) \leq \varphi(u) - \epsilon\|u - v\|$. From this we have $\|u - v\| \leq 2\epsilon < 1$ which means that $v \in \operatorname{Int}(D_r)$. On the other hand, by $(**)$ we see as in the proof of Proposition 1.4 that $|d\varphi|_{D_r}(v) \leq \epsilon$. So $|d\varphi|_X(v) \leq \epsilon$. Clearly $|\varphi(v) - d| \leq \epsilon^2$ and this proves the lemma. ■

PROOF OF COROLLARY 2.11. Suppose u_1 is a local maximum and u_2 is a local minimum. If φ is not bounded below, then we have a mountain pass situation with u_2 as an initial point and Corollary 2.10 applies to give either an infinite number of critical points or a saddle point of mountain pass type which is necessarily distinct from u_1 and u_2.

If, on the other hand, φ is bounded below then, since it satisfies $(PS)_c$, Lemma 2.12 yields that φ cannot be bounded above. Hence we have a mountain pass situation for $-\varphi$ with u_1 as an initial point. Again Corollary 2.10 applies to yield our claim. ■

3. Structure of the critical set in general case. As in the last section, we shall continue to study the structure of the critical set of continuous functionals, generated by min-max principles. But here we shall deal with the case of "n-dimensional" homotopy-stable families when $n \geq 2$. In order to do this, we first introduce the concepts of *(weak) saddle-type point* and *co-saddle point of order k* which can be seen as the higher dimensional analogue of Hofer's points of *mountain pass type*. We shall see in the next section that these notions are closely related to the classical Morse indices whenever these indices can be defined; that is when φ is a C^2-functional and when the critical points are non-degenerate.

3.1. *Preliminary.* We shall always assume in this chapter that S^k is a standard k-sphere in $\mathbb{R}^{k+1}$. We shall adopt the following definitions from [12].

DEFINITION 3.1. A family $\mathcal{F}$ of subsets of X is said to be *homotopic of dimension n with boundary B* if there exists a compact subset D of $\mathbb{R}^n$ containing a closed subset D_0 and a continuous function σ from D_0 onto B such that

$$\mathcal{F} = \{A \subset X \, ; A = f(D) \text{ for some } f \in C(D \, ; X) \text{ with } f = \sigma \text{ on } D_0\}.$$

Dually, we can introduce the *cohomotopic classes*. For that, fix a continuous map $\sigma^*: B \longrightarrow S^k$ and for any closed subset A of X containing B, set

$$\gamma(A \, ; B, \sigma^*) = \inf\{n \, ; \exists f \in C(A \, ; S^n) \text{ with } f = \sigma^* \text{ on } B\}.$$

DEFINITION 3.2. A family $\mathcal{F}$ of subsets of X is said to be *cohomotopic of dimension n with boundary B* if there exists a continuous $\sigma^*: B \longrightarrow S^n$ such that

$$\mathcal{F} = \{A \, ; A \text{ compact subset of } X, A \supset B \text{ and } \gamma(A \, ; B, \sigma^*) \geq n\}.$$

DEFINITION 3.3. A family $\mathcal{F}$ of subsets of X is said to be *a homological family of dimension n with boundary B* if for some non-trivial class α in the n-dimensional relative homology group $H_n(X, B)$ we have that

$$\mathcal{F} =: \mathcal{F}(\alpha) = \{A \, ; \; A \text{ compact subset of } X, A \supset B \text{ and } \alpha \in \operatorname{Im}(i_*^A)\}$$

where i_*^A is the homomorphism $i_*^A: H_n(A, B) \longrightarrow H_n(X, B)$ induced by the *immersion* $i: A \longrightarrow X$.

Suppose now that F is a closed subset of X that is disjoint from B. It is readily seen that F is dual to $\mathcal{F}(\alpha)$ if and only if $\alpha \notin \operatorname{Im}(i_*)$ where $i_*: H_n(X \setminus F, B) \longrightarrow H_n(X, B)$. We shall only use singular homology with rational or real coefficients.

For convenience, we also introduce the following notation.

DEFINITION 3.4. A compact subset L of K_c is said to be *an isolated critical set for* φ *in* K_c if it has a neighborhood in which φ has no critical points at the level c other than the ones that are already in L.

We shall need the following results from dimension theory which can be found in the book of Nagata [20].

DEFINITION 3.5. The *topological dimension* (or *covering dimension*) of a metric space D (in short, $\operatorname{topdim} D$) is the least integer m such that the following property holds: for any finite open covering $\mathcal{O}$ of D, there is an open covering $\mathcal{O}_1$ refining $\mathcal{O}$ such that any $p \in D$ belongs to at most $m+1$ elements of $\mathcal{O}_1$.

The following theorem summarizes the properties of topological dimension that will be needed in the sequel.

THEOREM 3.6. *Let X be a metric space. Then the following holds:*
i) $\operatorname{topdim} X_1 \leq \operatorname{topdim} X$ *for any subspace X_1 of X;*
ii) If X has a finite covering consisting of closed sets $\{X_i\,;\, i \in \mathbb{N}\}$ with $\operatorname{topdim} X_i \leq m$, *then* $\operatorname{topdim} X \leq m$;
iii) $\operatorname{topdim} \mathbb{R}^m = m$.

The following basic theorem is well known. It relates the topological dimension of a space to certain extension properties for non-linear mappings into euclidian spheres.

THEOREM 3.7. *A metric space X has a topological dimension at most m if and only if for every closed subset $X_1 \subseteq X$ and every continuous mapping f of X_1 into S^m (the standard m-sphere in $\mathbb{R}^{m+1}$) there is a continuous extension $\tilde{f}$ of f to all of X.*

We shall show in the next few sections that certain topological properties of a critical point or critical set generated by a min-max procedure are related to the topological dimensions (defined above) of homotopy-stable families (homotopic, cohomotopic and homological) under consideration.

3.2. *The homotopic case.* Recall that

$$K_c = \{x \in X\,;\, \varphi(x) = c, |d\varphi|(x) = 0\} \quad L_c = \{x \in X\,;\, \varphi(x) \geq c\} \quad G_c = X \setminus L_c$$

and that $\sup \varphi(\emptyset) = -\infty$ by convention. To avoid some complications, we shall assume that X is a Banach space throughout this subsection.

DEFINITION 3.8. Let φ be a continuous functional on X and let K be a subset of K_c. We say that K is a *weak saddle-type set of order k* if k is the least integer such that there is a neighborhood N of K verifying that for any sub-neighborhood $M \subseteq N$ of K and any $\epsilon_0 > 0$, $M \cap G_{c-\epsilon}$ is not $(k-1)$-connected for some $0 \leq \epsilon \leq \epsilon_0$. We shall then write w-sad(K) = k.

If the above holds for $\epsilon_0 = 0$, we then say that K is a *saddle-type set of order k* and we write sad(K) = k.

If K is a singleton $\{x\}$ we shall then say that x is a *weak saddle-type* (resp. a *saddle-type*) point of order k.

From the definition we clearly have that sad(K) $\geq$ w-sad(K).

REMARK 3.9. By convention we say that a set is -1-connected if it is nonempty. Hence a critical point x of mountain-pass type is a critical point with $\text{sad}(x) = 1$. x is a minimum if and only if x has $\text{sad}(x) = 0$ which holds if and only if w-sad(x) = 0.

In the case where regular Morse indices are defined, we shall see in the next section that a critical point x has Morse index k if and only if $\text{sad}(x) = w\text{-sad}(x) = k$.

We shall prove the following result which roughly speaking, implies that a homotopic family $\mathcal{F}$ of dimension n will necessarily generate a weak saddle-type critical point of order at most n.

THEOREM 3.10. *Let φ be a continuous functional on X and consider a homotopic family $\mathcal{F}$ of dimension n with closed boundary B. Let $\mathcal{F}^*$ be a family dual to $\mathcal{F}$ such that*

$$c := \sup_{F\in\mathcal{F}^*} \inf_{x\in F} \varphi(x) = \inf_{A\in\mathcal{F}} \max_{x\in A} \varphi(x)$$

and is finite. Assume that φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ in $\mathcal{F}$ and a suitable max-mining sequence $(F_k)_k$ in $\mathcal{F}^*$. Suppose $\tilde{K}_c := K_c \cap F_\infty \cap A_\infty$ is isolated in K_c. Then, for any neighborhood N of $\tilde{K}_c$, there is a connected component M of N such that $M \cap \tilde{K}_c \neq \emptyset$ and w-sad$(M \cap \tilde{K}_c) \leq n$.

Moreover, if we assume that $\tilde{K}_c$ consists of isolated critical points, then there is $x \in \tilde{K}_c$ with w-sad$(x) \leq n$.

If we assume that $\tilde{K}_c$ consists of isolated critical points and $F_k = F$ for all $k \geq 1$, then we have the following corollary.

COROLLARY 3.11. *Let φ be a continuous functional on X and consider a homotopic family $\mathcal{F}$ of dimension n with closed boundary B. Suppose that $c := c(\varphi, \mathcal{F})$ is finite and that F is dual to $\mathcal{F}$ with* $\inf \varphi(F) \geq c$. *If φ verifies $(PS)_{F,c}$ along a min-maxing sequence $(A_k)_k$ and if the set $K_c \cap A_\infty \cap F$ consists of isolated critical points, then there exists x in $K_c \cap F \cap A_\infty$ with* $\text{sad}(x) \leq n$.

If we suppose that $\sup \varphi(B) < c$, then the above applies to the dual set $F = \{\varphi \geq c\}$ and we get the following

COROLLARY 3.12. *Let φ be a continuous functional on X and consider a homotopic family $\mathcal{F}$ of dimension n with closed boundary B. Suppose that $c := c(\varphi, \mathcal{F})$ is finite and that* $\sup \varphi(B) < c$. *If φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ and if the set $K_c \cap A_\infty$ consists of isolated critical points, then there exists x in $K_c \cap A_\infty$ with* $\text{sad}(x) \leq n$.

The following corollary of Theorem 1.5 will be crucial in the proof of the main results of this chapter.

COROLLARY 3.13. *Under the hypothesis of Theorem 1.5, assume φ verifies $(PS)_c$ along a min-maxing sequence $(A_n)_n$ in $\mathcal{F}$ and a suitable max-mining sequence $(F_n)_n$ in $\mathcal{F}^*$. Suppose $\tilde{K}_c := K_c \cap F_\infty \cap A_\infty$ is isolated in K_c and let $\epsilon > 0$ be such that*

$N_\epsilon(\tilde{K}_c) \cap K_c = \tilde{K}_c$. *Put* $F'_k = F_k \cup \big(L_{c_k} \cap N_\epsilon(\tilde{K}_c)\big)$ *where* $c_k = \min \varphi(F_k)$. *Then for any* $\delta > 0$ *and any* $k_0 > 0$, *there exist* $A \in \mathcal{F}$ *and a* F'_k *with* $k > k_0$ *such that*

$$A \subseteq (X \setminus F'_k) \cup N_\delta(F_\infty \cap A_\infty \cap K_c).$$

PROOF. If not, then for some $\delta > 0$ there is an increasing sequence n_i such that the set $F''_{n_i} = F'_{n_i} \setminus N_\delta(F_\infty \cap A_\infty \cap K_c)$ are dual to $\mathcal{F}$ for all i. Since $\lim_{i\to\infty} \inf \varphi(F''_{n_i}) = c$, we have by Theorem 1.5, that $F''_\infty \cap A_\infty \cap K_c \neq \emptyset$ which is absurd. ∎

REMARK 3.14. (a) If we apply the above corollary to $F_n = L_c$ for all n, we get the existence of an $A \in \mathcal{F}$ such that

$$A \subseteq G_c \cup N_\delta(K_c);$$

(b) Under the classical condition: $\sup \varphi(B) < c$, we obtain the well known result about the existence of $A \in \mathcal{F}$ with $A \subseteq G_c \cup N_\delta(K_c)$.

The proof of Theorem 3.10 needs some algebraic topological tools. We shall first recall and prove some of the needed results. As in general, for a simplicial complex K, We denote by $|K|$ its underlying topological space and for simplexes s and t we write $t \leq s$ ($t < s$) if t is a (proper) face of s. For a simplex s, we denote s° to be the open simplex of s. Here is a lemma from [15] (pp. 108–125).

LEMMA 3.15. *Let* $D \subseteq \mathbb{R}^n$ *be a compact subset. Then for any* $\delta > 0$, *there is a finite simplicial complex* K *of* $\mathbb{R}^n$ *such that*

$$D \subseteq |K| \subseteq N_\delta(D).$$

We shall also need the following lemma. Since we can not find a reference for it, we give a proof for completeness.

LEMMA 3.16. *Let* K *be a finite simplicial complex of* $\mathbb{R}^n$. *Then there is a simplicial subcomplex* L *of* K *such that* $|L| = \partial|K|$.

PROOF. We assert that for any $a \in K$ with $|a^\circ| \cap \partial|K| \neq \emptyset$ then $|a| \subseteq \partial|K|$. Note first that $m = \dim a \leq n-1$ if $a^\circ \cap \partial|K| \neq \emptyset$. We prove the assertion by induction on m downward. It is clear that $|a| \subseteq \partial|K|$ if $|a^\circ| \cap \partial|K| \neq \emptyset$ and $m = n-1$. Suppose that it is true for all m with $k \leq m \leq n-1$, we need to show that it is true for $m = k-1$. For each $x \in |a^\circ| \cap \partial|K|$, since K is a finite simplicial complex there is an n-dimensional ball $B(x, \epsilon_x)$ with $B(x,\epsilon_x) \cap |a| \subseteq |a^\circ|$ such that for any $b \in K$ if $B(x,\epsilon_x) \cap |b| \neq \emptyset$ then either $b = a$ or $a < b$.

If there is a $b \in K$ with $a < b$ and $|b^\circ| \cap \partial|K| \neq \emptyset$, then by the induction assumption, $|b| \subseteq \partial|K|$, hence $|a| \subseteq |b| \subseteq \partial|K|$. If not, we will have $|a^\circ| \subseteq \partial|K|$ *i.e.* $|a| \subseteq \partial|K|$ as well. To see this, we note that $B(x,\epsilon_x) \setminus |a^\circ|$ is connected since $\dim a \leq n-2$ and that for any path joining y, z with $y \in \operatorname{Int}|K|$ and $z \in \mathbb{R}^n \setminus |K|$, then the path must intersect $\partial|K|$. So $B(x,\epsilon_x) \cap \operatorname{Int}|K| = \emptyset$ *i.e.* $B(x,\epsilon_x) \cap |a^\circ| \subseteq B(x,\epsilon_x) \cap |K| \subseteq \partial|K|$. This shows that $|a^\circ| \cap \partial |K|$ is open in $|a^\circ|$, also closed since $\partial |K|$ is closed. But $|a^\circ|$ is connected, therefore $|a^\circ| \cap \partial |K| = |a^\circ|$ *i.e.* $|a^\circ| = |K|$.

Finally we put

$$L = \{a\,;\, a \in K, |a| \subseteq \partial |K|\}.$$

Clearly L is a simplicial subcomplex of K, by the assertion established above, we have that $|L| = \partial |K|$. This proves the lemma. ∎

Next we recall an elementary lemma from obstruction theory in algebraic topology. Let K be a CW complex and L be a CW subcomplex of K. Let K^m be m-dimensional skeleton of K and $\bar{K}^m = L \cup K^m$.

LEMMA 3.17 ([15] PP. 174–179). *Let $K, L, K^m, \bar{K}^m$ be as above and let $g\colon L \longrightarrow Y$ be continuous. If Y is an m-connected topological space for some $m \geq 0$, then g has a continuous extension over $\bar{K}^{m+1}$.*

It is well known that there is a natural way to identify any simplicial complex as a CW complex.

COROLLARY 3.18. *Let $K \subseteq \mathbb{R}^n$ be a finite simplicial subcomplex and $f\colon \partial |K| \longrightarrow Y$ be continuous. If Y is path connected for $n = 1$ and each path connected component is $(n-1)$-connected for $n > 1$, then f has a continuous extension over $|K|$.*

PROOF. For $n = 1$, the corollary follows directly from Lemma 3.17. For $n > 1$, we observe that $|K|$ has only finite path connected components and f maps each path connected component into a path connected component of Y. Then applying Lemma 3.17 on each path connected component of $|K|$, we see that the corollary is proved. ∎

For any $x \in X, \epsilon > 0$, we let $B(x, \epsilon) = \{y \in X\,;\, \|x - y\| < \epsilon\}$.

LEMMA 3.19. *Let G, B, M be subsets of X with B compact and G open. Let D_0, D be compact subsets of $\mathbb{R}^n$ with $D_0 \subseteq D$. Assume $\bar{M} \cap B = \emptyset$ and choose $0 < \nu < 1/2\,\mathrm{dist}(\bar{M}, B)$. Let $f\colon D \longrightarrow G \cup B \cup M$ be continuous such that $f(D_0) = B$ and suppose there is a subset G' of G with $G' \cap N_\nu(M) = G \cap N_\nu(M)$ such that each of its path connected component is $(n-1)$-connected, then there is $g\colon D \longrightarrow X$ such that*

$$g(D) \subseteq G \cup B \text{ and } g(x) = f(x) \quad \text{for all } x \in D_0.$$

PROOF. Let $U = f(D) \cap \overline{N_{\frac{\nu}{2}}(M)}$. If U is empty, then the lemma is true. Otherwise let $V = f^{-1}(\bar{U})$. We have then an extension $\hat{f}\colon \mathbb{R}^n \longrightarrow X$ of f. Clearly there is an open neighborhood D_1 of D such that $\hat{f}(D_1) \subseteq N_\nu\big(f(D)\big)$. Since $\hat{f}(D_0) = f(D_0) = B$, V is compact and $B \cap \overline{N_\nu(M)} = \emptyset$, there is $\delta > 0$ such that

$$N_\delta(V) \subseteq D_1 \setminus D_0, \quad \hat{f}\big(N_\delta(V)\big) \subseteq N_\nu(M) \cap G.$$

By Lemma 3.15 and Lemma 3.16, there is a finite simplicial complex K of $\mathbb{R}^n$ and a simplicial subcomplex L of K such that

$$|L| = \partial |K|, \quad V \subseteq \bar{N}_{\delta/2}(V) \subseteq |K| \subseteq N_\delta(V).$$

Clearly

$$\hat{f}(|L|) \subseteq \big(N_\nu(M)\big) \cap (G \cup B) \subseteq G' \cap \big(N_\nu(M)\big) \subseteq G'.$$

By Corollary 3.18, we have $\bar{f}\colon |K| \longrightarrow G'$. Now define

$$g(x) = \begin{cases} f(x) & \text{if } x \in D \setminus |K| \\ \bar{f}(x) & \text{if } x \in |K|. \end{cases}$$

Then $g(x)\colon D \longrightarrow G \cup B$ and $g(x)$ is continuous with $g(x) = f(x)$ on $x \in D_0$. ∎

PROOF OF THEOREM 3.10. Since $\tilde{K}_c$ is compact and N is a neighborhood of $\tilde{K}_c$, we have a finite number of connected component $(M^i)_{i=1}^m$ of N such that $\tilde{K}_c \subseteq \bigcup_{i=1}^m M^i$ and $\tilde{K}_c \cap M^i \neq \emptyset$ for all $1 \leq i \leq m$. Let $M_c^i = \tilde{K}_c \cap M^i$. Clearly M is a neighborhood of the compact set M_c^i for $1 \leq i \leq m$. Hence there is $\tau > 0$ such that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap B = \emptyset \text{ and } N_{4\tau}(M_c^i) \subseteq M^i \quad \text{for all } 1 \leq i \leq m. \tag{3.2.1}$$

Since we suppose that $\tilde{K}_c$ is isolated in K_c, we may assume that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap K_c = \tilde{K}_c. \tag{3.2.2}$$

Let $\delta_k = c - \inf \varphi(F_k)$ and $F_k' = F_k \cup \bigcup_{i=1}^m \big(L_{c-\delta_k} \cap \overline{N_{4\tau}(M_c^i)}\big)$. Clearly F_k' is dual to $\mathcal{F}$ and $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$.

Suppose the theorem is not true. Then for each M_c^i, there exist $\epsilon_i > 0$ and a subneighborhood $\hat{M}^i \subseteq N_{4\tau}(M_c^i)$ of M_c^i such that each path connected component of $\hat{M}^i \cap G_{c-\epsilon}$ is $(n-1)$-connected for all $0 \leq \epsilon \leq \epsilon_i$. Take $\epsilon = \min_{1 \leq i \leq m} \epsilon_i$ and $0 < \alpha < \tau$ small such that $N_{4\alpha}(M_c^i) \subseteq \hat{M}^i$ for all $1 \leq i \leq m$. Let $k_0 > 0$ such that $\delta_k \leq \epsilon$ for all $k \geq k_0$. Now we may assume that $\mathcal{F}$ is given explicitly as in Definition 3.1 with D, D_0 and σ. Note that $B \subseteq X \setminus F_k'$ for all $k \geq 1$. Then by (3.2.1), (3.2.2) and Corollary 3.13, there exist $f\colon D \longrightarrow X$ continuous with $f(x) = \sigma(x)$ on D_0 and a F_k' with $k > k_0$ such that

$$f(D) \subseteq (X \setminus F_k') \cup N_\alpha(\tilde{K}_c). \tag{3.2.3}$$

Note that

$$(X \setminus F_k') \cap N_{4\tau}(M_c^i) = G_{c-\delta_k} \cap N_{4\tau}(M_c^i).$$

Now we shall prove that there is $g\colon D \longrightarrow X$ with $g(x) = \sigma(x)$ on D_0 such that

$$g(D) \subseteq (X \setminus F_k') \tag{3.2.4}$$

which is clearly a contradiction since F_k' is dual to $\mathcal{F}$. By induction and starting with $g^0 = f$, we shall construct $(g^i)_{i=1}^m\colon D \longrightarrow X$ continuous with $g^i(x) = \sigma(x)$ on D_0 such that

$$g^i(D) \subseteq G^i \tag{3.2.5}$$

where the sets $(G^i)_{i=1}^m$ are defined as:

$$G^m = X \setminus F_k', \; G^i = (X \setminus F_k') \cup \bigcup_{j=i+1}^m N_\alpha(M_c^j) \quad \text{for all } 1 \leq i \leq m-1. \tag{3.2.6}$$

For $i = 1$, by (3.2.3) we have that

$$g^0(D) \subseteq G^1 \cup N_\alpha(M_c^1). \tag{3.2.7}$$

Put $G' = (X \setminus F_k') \cap \hat{M}^1 \subseteq G^1$. Note that $G' = \hat{M}^1 \cap G_{c-\delta_k}$ since $\hat{M}^1 \subseteq N_{4r}(M_c^i)$. Note also that $\operatorname{dist}\big(N_\alpha(M_c^i), B\big) \geq 3\alpha$. Hence we have that

$$G' \cap N_{2\alpha}(M_c^1) = G^1 \cap N_{2\alpha}(M_c^1).$$

On the other hand, each path connected component of G' is $(n-1)$-connected by assumption and $\delta_k < \epsilon$. Hence we can apply Lemma 3.19 with this G' and $G = G^1$ to have $g^1 : D \longrightarrow X$ continuous with $g^1(x) = \sigma(x)$ on D_0 such that

$$g^1(D) \subseteq G^1$$

which is asserting (3.2.5) for $i = 1$. Next, suppose we have constructed $(g^i)_{i=1}^I$ for $1 \leq i \leq I\,(1 \leq I < m)$ so that (3.2.5) is verified. Note that

$$g^I(D) \subseteq G^I \subseteq G^{I+1} \cup N_\alpha(M_c^{I+1})$$

and $\operatorname{dist}\big(N_\alpha(M_c^{I+1}), B\big) \geq 3\alpha$. Put $G'' = (X \setminus F_k') \cap \hat{M}^{I+1} \subseteq G^{I+1}$. Then we have

$$G'' \cap N_{2\alpha}(M_c^{I+1}) = G^I \cap N_{2\alpha}(M_c^{I+1}).$$

Again by Lemma 3.19 with $G' = G''$ here and $G = G^{I+1}$ we have g^{I+1} such that

$$g^{I+1}(D) \subseteq G^{I+1}$$

which verifies (3.2.5) for $i = I + 1$. This finishes the inductive construction of $(g^i)_{i=1}^m$. Finally, $g = g^m$ gives the required map and Theorem 3.10 is proved. ∎

REMARK 3.20. The above proof actually shows that for $n \geq 2$ there exist an M such that for any $\epsilon_0 > 0$ and any open sub-neighborhood $\hat{M} \subseteq M$ of $M \cap \tilde{K}_c$, one of the path connected components of $\hat{M} \cap G_{c-\epsilon}$ is not $k-1$-connected for some $2 \leq k \leq n$ and $0 \leq \epsilon \leq \epsilon_0$.

3.3. *The cohomotopic case.* In this section we study the topological properties of the critical points generated by the min-max procedure in the cohomotopic case. For convenience, we introduce the following notation. For any subset D of X and a functional φ on X, we let

$$L_\varphi(D) = \{f \in C(X,X)\ ;\ \varphi \circ f \leq \varphi, f(D) \subseteq D \text{ and } f(x) = x \text{ on } X \setminus D\}.$$

We shall drop the subscript φ when no confusion arises in the sequel.

DEFINITION 3.21. Let φ be a continuous functional on X and let K be a subset of K_c, the critical set of φ at level c. We say that K is a *co-saddle type set of order k* if k is the least integer such that for any neighborhood N of K, there exist a sub-neighborhood $M \subseteq N$ of K and f in $L(N)$ such that $\operatorname{topdim} f(M) \leq k$. We then write $\operatorname{sad}^*(K) = k$.

If K is a singleton $\{x\}$ we shall then say that x is a *co-saddle type point of order k*.

Here is the theorem which basically says that a cohomotopic family $\mathcal{F}$ of dimension n will necessarily generate a co-saddle type critical point of order at least n.

THEOREM 3.22. *Let φ be a continuous functional on X and consider a cohomotopic family $\mathcal{F}$ of dimension n with closed boundary B. Let $\mathcal{F}^*$ be a family dual to $\mathcal{F}$ such that*

$$c := \sup_{F\in\mathcal{F}^*} \inf_{x\in F} \varphi(x) = \inf_{A\in\mathcal{F}} \max_{x\in A} \varphi(x)$$

and is finite. Assume that φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ in $\mathcal{F}$, and a suitable max-mining sequence $(F_k)_k$ in $\mathcal{F}^*$. Suppose that $\tilde{K}_c := K_c \cap F_\infty \cap A_\infty$ is isolated in K_c. Then, for any neighborhood N of $\tilde{K}_c$, there is a connected component M of N such that $M \cap \tilde{K}_c$ is not empty and $\mathrm{sad}^*(K_c \cap M) \geq n$.

Moreover if $\tilde{K}_c$ consists of isolated critical points, then there exists $x \in \tilde{K}_c$ with $\mathrm{sad}^*(x) \geq n$.

If we suppose that $\sup \varphi(B) < c$, then the above applies to the dual set $F = \{\varphi \geq c\}$ and we get the following:

COROLLARY 3.23. *Let φ be a continuous functional on X and consider a cohomotopic family $\mathcal{F}$ of dimension n with closed boundary B. Suppose that $c := c(\varphi, \mathcal{F})$ is finite and that* $\sup \varphi(B) < c$. *If φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ and if the set $K_c \cap A_\infty$ consists of isolated critical points, then there exists $x \in K_c \cap A_\infty$ with* $\mathrm{sad}^*(x) \geq n$.

The proof of Theorem 3.22 needs the following easy lemma which singles out an important stability property enjoyed by cohomotopic families.

LEMMA 3.24. *Let $\mathcal{F}$ be a cohomotopic family of dimension n with boundary B in a metric space X. Then, for any $A \in \mathcal{F}$, any continuous function $f: A \longrightarrow X$ with $f(x) = x$ on B and any open set U such that $\bar{U} \cap B = \emptyset$ and* $\mathrm{topdim} f(\bar{U}) \leq n-1$, *we have that $f(A \setminus U) \in \mathcal{F}$.*

PROOF. Suppose that $f(A \setminus U)$ does not belong to $\mathcal{F}$. Then there exists a continuous map $h: f(A \setminus U) \longrightarrow S^{n-1}$ such that $h = \sigma$ (the boundary data) on B. Let h' be the restriction of such a map to $f(A \cap \partial U)$. Since $\mathrm{topdim} f(\bar{U}) \leq n-1$, Theorem 3.7 applies to yield an extension h'' of h' from $f(A \cap U)$ into S^{n-1}. It is now clear that the map

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in f(A \setminus U) \\ h''(x) & \text{if } x \in f(A \cap U) \end{cases}$$

is a continuous map from $f(A)$ into S^{n-1} that is equal to σ on B. In other words, $\gamma\big(f(A)\,;B,\sigma\big) \leq n-1$, which is a contradiction since $f(A) \in \mathcal{F}$. ∎

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PROOF OF THEOREM 3.22. Since $\tilde{K}_c$ is compact and N is a neighborhood of $\tilde{K}_c$, we have a finite number of connected component $(M^i)_{i=1}^m$ of N such that $\tilde{K}_c \subseteq \bigcup_{i=1}^m M^i$ and $\tilde{K}_c \cap M^i \neq \emptyset$ for all $1 \leq i \leq m$. Let $M_c^i = \tilde{K}_c \cap M^i$. Clearly M^i is a neighborhood of the compact set M_c^i for $1 \leq i \leq m$. Hence there is $\tau > 0$ such that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap B = \emptyset \text{ and } N_{4\tau}(M_c^i) \subseteq M^i \quad \text{for all } 1 \leq i \leq m. \tag{3.3.1}$$

Since we assume that $\tilde{K}_c$ is isolated in K_c, we may assume that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap K_c = \tilde{K}_c. \tag{3.3.2}$$

Let $\delta_k = c - \inf \varphi(F_k)$ and $F_k' = F_k \cup \bigcup_{i=1}^m \left(L_{c-\delta_k} \cap \overline{N_{4\tau}(M_c^i)}\right)$. Clearly F_k' is dual to $\mathcal{F}$ and $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$.

Suppose the theorem is not true. Then for each M_c^i, neighborhood $N_{4\tau}(M_c^i)$, there exist a sub-neighborhood $\hat{M}^i \subseteq N_{4\tau}(M_c^i)$ of M_c^i and $f_i \in L\big(N_{4\tau}(M_c^i)\big)$ such that $\operatorname{topdim} f_i(\hat{M}^i) \leq n-1$. By taking sub-neighborhood of M_c^i inside of $\hat{M}^i$ if necessary, we may assume that $\hat{M}^i$ is closed. Note that $B \subseteq X \setminus F_k'$ for all $k \geq 1$. By (3.3.1), (3.3.2) and Corollary 3.13, there is $A \in \mathcal{F}$ and F_k' such that

$$A \subseteq (X \setminus F_k') \cup \bigcup_{i=1}^m \hat{M}^i.$$

Note $(X \setminus F_k') \cap N_{4\tau}(M_c^i) = G_{c-\delta_k} \cap N_{4\tau}(M_c^i)$. Let $f = f_m \circ f_{m-1} \circ \cdots \circ f_1$ and $\tilde{A} = f(A \setminus \bigcup_{i=1}^m \hat{M}^i)$. Clearly $A \setminus \bigcup_{i=1}^m \hat{M}^i \subseteq X \setminus F_k'$. Since $\varphi \circ f \leq \varphi$, $f(x) = x$ on $X \setminus \bigcup_{i=1}^m N_{4\tau}(M_c^i)$ and $(X \setminus F_k') \cap N_{4\tau}(M_c^i) = G_{c-\delta_k} \cap N_{4\tau}(M_c^i)$ we have that $\tilde{A} \subseteq X \setminus F_k'$. On the other hand, we have that $\tilde{A} \in \mathcal{F}$ by Lemma 3.24. But this is a contradiction since F_k' is dual to $\mathcal{F}$. ∎

Now we can combine the previous results to get some two-sided information about the critical points generated by min-max principles.

THEOREM 3.25. *Let φ be a continuous functional on X and consider a homotopic family $\mathcal{F}$ (resp. a cohomotopic family $\bar{\mathcal{F}}$) of dimension n with closed boundary B. Let $\mathcal{F}^*$ (resp. $\bar{\mathcal{F}}^*$) be a family dual to $\mathcal{F}$ (resp. $\bar{\mathcal{F}}$) such that*

$$c := \sup_{F \in \mathcal{F}^*} \inf_{x \in F} \varphi(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x)$$

(resp.

$$\bar{c} := \sup_{\bar{F} \in \bar{\mathcal{F}}^*} \inf_{x \in \bar{F}} \varphi(x) = \inf_{\bar{A} \in \bar{\mathcal{F}}} \max_{x \in \bar{A}} \varphi(x))$$

and is finite. Assume that φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ in $\mathcal{F}$ and a suitable max-mining sequence $(F_k)_k$ in $\mathcal{F}^*$. Suppose that $\tilde{K}_c := K_c \cap F_\infty \cap A_\infty$ is isolated in K_c. If $\mathcal{F} \subset \bar{\mathcal{F}}$, $c = \bar{c}$ and F_k is dual to $\bar{\mathcal{F}}$ for all $k \geq 1$, then for any neighborhood N of $\tilde{K}_c$, there exists a connected component M of N such that w-sad$(M \cap \tilde{K}_c) \leq n \leq$ sad$^*(M \cap \tilde{K}_c)$.

Moreover if we assume that $\tilde{K}_c$ consists of isolated critical points, then there exists $x \in \tilde{K}_c$ such that w-sad$(x) \leq n \leq$ sad$^*(x)$.

If we assume that $F_k = F$ for all $k \geq 1$, we then have the following

COROLLARY 3.26. *Let φ be a continuous functional on X and consider a homotopic family $\mathcal{F}$ (resp. a cohomotopic family $\bar{\mathcal{F}}$) of dimension n with closed boundary B. Assume that $c := c(\varphi, \mathcal{F})$ (resp. $\bar{c} := c(\varphi, \bar{\mathcal{F}})$) is finite and that F is dual to $\mathcal{F}$ with* $\inf \varphi(F) \geq c$. *Suppose that φ verifies $(PS)_{F,c}$ along a min-maxing sequence $(A_k)_k$ and that the set $K_c \cap A_\infty \cap F$ consists of isolated critical points. If $\mathcal{F} \subset \bar{\mathcal{F}}$, $c = \bar{c}$ and F is dual to $\bar{\mathcal{F}}$, then there exists $x \in K_c \cap F \cap A_\infty$ such that* $\mathrm{sad}(x) \leq n \leq \mathrm{sad}^*(x)$.

If we suppose that $\sup \varphi(B) < c$, then again the above applies to the dual set $F = \{\varphi \geq c\}$ and we get the following

COROLLARY 3.27. *Let φ be a continuous functional on X and consider a homotopic family $\mathcal{F}$ (resp. a cohomotopic family $\bar{\mathcal{F}}$) of dimension n with closed boundary B. Suppose that $c := c(\varphi, \mathcal{F})$ (resp. $\bar{c} := c(\varphi, \bar{\mathcal{F}})$) is finite and that* $\sup \varphi(B) < c$. *Assume that φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ and that the set $K_c \cap A_\infty$ consists of isolated critical points. If $\mathcal{F} \subseteq \bar{\mathcal{F}}$ and $c = \bar{c}$, then there exists x in $K_c \cap A_\infty$ such that* $\mathrm{sad}(x) \leq n \leq \mathrm{sad}^*(x)$.

PROOF OF THEOREM 3.25. Since $\tilde{K}_c$ is compact and N is a neighborhood of $\tilde{K}_c$, we have a finite number of connected component $(M^i)_{i=1}^m$ of N such that $\tilde{K}_c \subseteq \bigcup_{i=1}^m M^i$ and $\tilde{K}_c \cap M^i \neq \emptyset$ for all $1 \leq i \leq m$. Let $M_c^i = \tilde{K}_c \cap M^i$. Clearly M is a neighborhood of the compact set M_c^i for $1 \leq i \leq m$. Hence there is $\tau > 0$ such that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap B = \emptyset \text{ and } N_{4\tau}(M_c^i) \subseteq M^i \quad \text{for all } 1 \leq i \leq m. \tag{3.3.3}$$

Since we assume that $\tilde{K}_c$ is isolated in K_c, we may assume that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap K_c = \tilde{K}_c. \tag{3.3.4}$$

Let $\delta_k = c - \inf \varphi(F_k)$ and $F_k' = F_k \cup \bigcup_{i=1}^m \left(L_{c-\delta_k} \cap \overline{N_{4\tau}(M_c^i)}\right)$. Clearly F_k' is dual to $\mathcal{F}$ and $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$.

Suppose the theorem is not true. Then without loss of generality, we may assume that for $1 \leq i \leq I < m$ and each M_c^i, there exist $\epsilon_i > 0$ and a neighborhood $\hat{M}^i \subseteq N_{4\tau}(M_c^i)$ of M_c^i such that $\hat{M}^i \cap G_{c-\epsilon}$ is $(n-1)$-connected for all $1 \leq \epsilon \leq \epsilon_i$. Also for all $I+1 \leq i \leq m$, each M_c^i and neighborhood $N_{4\tau}(M_c^i)$, there exist sub-neighborhood $\hat{M}^i \subseteq N_{4\tau}(M_c^i)$ of M_c^i and $f_i \in L\left(N_{4\tau}(M_c^i)\right)$ such that $\mathrm{topdim} f(\hat{M}^i) \leq n-1$. Take $\epsilon = \min_{1 \leq i \leq I} \epsilon_i$ and $0 < \alpha < \tau$ small such that $N_{4\alpha}(M_c^i) \subseteq \hat{M}^i$ for all $1 \leq i \leq m$. Next we may assume that $\mathcal{F}$ is given explicitly as in Definition 3.1 with D, D_0 and σ. Note that $B \subseteq X \setminus F_k$ for all $k \geq 1$. Let $k_0 > 0$ such that $\delta_k < \epsilon$ for all $k \geq k_0$. Then by (3.3.3), (3.3.4) and Corollary 3.13, there exist $f : D \longrightarrow X$ continuous with $f(x) = \sigma(x)$ on D_0 and a F_k' with $k > k_0$ such that

$$f(D) \subseteq (X \setminus F_k') \cup N_\alpha(\tilde{K}_c).$$

Note that

$$(X \setminus F_k') \cap N_{4\tau}(M_c^i) = G_{c-\delta_k} \cap N_{4\tau}(M_c^i).$$

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Now just as in the proof of Theorem 3.10, we will have a continuous map $g: D \longrightarrow X$ with $g(x) = \sigma(x)$ on D_0 such that that

$$g(D) \subseteq (X \setminus F_k') \cup \bigcup_{i=I+1}^{m} N_\alpha(M_c^i).$$

Put $A = g(D)$ and note that $g(D) \in \bar{\mathcal{F}}$ since $\mathcal{F} \subseteq \bar{\mathcal{F}}$. Let $\tilde{f} = f_m \circ \cdots \circ f_{I+1}$ and $\tilde{A} = \tilde{f}\big(A \setminus \bigcup_{i=I+1}^{m} N_\alpha(M_c^i)\big)$. Since by assumption $\operatorname{topdim} f_i(\hat{M}^i) \leq n-1$ for all $I+1 \leq i \leq m$, we have also that $\operatorname{topdim} \tilde{f}(\bigcup_{i=I+1}^{m} \hat{M}^i) \leq n-1$. So $\operatorname{topdim} \tilde{f}\big(\bigcup_{i=I+1}^{m} \overline{N_\alpha(M_c^i)}\big) \leq n-1$. Then as in the proof of Theorem 3.22, we have that $\tilde{A} \in \bar{\mathcal{F}}$ by Lemma 3.24. Next we have $\tilde{A} \subseteq X \setminus F_k'$ since that $\varphi\big(\tilde{f}(x)\big) \leq \varphi(x)$ and $\tilde{f}(x) = x$ on $X \setminus \bigcup_{i=I+1}^{m} N_{4\tau}(M_c^i)$. This is a contradiction since by assumption that F_k' is dual to $\bar{\mathcal{F}}$. ∎

3.4. *The homological case.* Like the homotopic and cohomotopic cases, a homological family $\mathcal{F}$ of dimension n will also necessarily generate a critical point with some topological properties. To describe these properties, we introduce the following concept.

DEFINITION 3.28. Let φ be a continuous functional on X and K be a subset of K_c, the critical set at level c. We define $\operatorname{Ord}_w(K)$ to be the set of all integers $k \geq 1$ verifying that there a neighborhood N of K such that for any $\epsilon_0 > 0$ and any open sub-neighborhood $M \subseteq N$ of K with $H_k(M) = 0$, we have that $H_{k-1}(G_{c-\epsilon} \cap M) \neq 0$ for some $0 \leq \epsilon \leq \epsilon_0$.

We also write $\operatorname{Ord}(K)$ for the set of all integers $k \geq 1$ verifying the above with $\epsilon_0 = \epsilon = 0$.

We shall show in the next section that a critical point x has regular Morse index of n if and only if $\operatorname{Ord}_w(x) = \operatorname{Ord}(x) = \{n\}$. Here is the main result of the section.

THEOREM 3.29. *Let φ be a continuous functional on X. Consider a homological family $\mathcal{F}$ of dimension n with boundary B. Let $\mathcal{F}^*$ be a family dual to $\mathcal{F}$ such that*

$$c := \sup_{F \in \mathcal{F}^*} \inf_{x \in F} \varphi(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x)$$

and is finite. Assume that φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ in $\mathcal{F}$, and a suitable max-mining sequence $(F_k)_k$ in $\mathcal{F}^*$. Suppose $\tilde{K}_c := K_c \cap F_\infty \cap A_\infty$ is isolated in K_c. Then for any neighborhood N of $\tilde{K}_c$, there exists a connected component M of N with $M \cap \tilde{K}_c \neq \emptyset$ such that $n \in \operatorname{Ord}_w(M \cap \tilde{K}_c)$.

Moreover if $\tilde{K}_c$ consists of isolated critical points, then there is an $x \in \tilde{K}_c$ with $n \in \operatorname{Ord}_w(x)$.

If we assume that $\tilde{K}_c$ consists of isolated critical points and $F_k = F$ for all $k \geq 1$, then we have the following corollary.

COROLLARY 3.30. *Let φ be a continuous functional on X and consider a homological family $\mathcal{F}$ of dimension n with closed boundary B. Assume that $c := c(\varphi, \mathcal{F})$ is finite and that F is dual to $\mathcal{F}$ with $\inf \varphi(F) \geq c$. If φ verifies $(PS)_{F,c}$ along a min-maxing sequence $(A_k)_k$ and if the set $K_c \cap F \cap A_\infty$ consists of isolated critical points, then there exists x in $K_c \cap F \cap A_\infty$ with $n \in \operatorname{Ord}(x)$.*

If we suppose that $\sup \varphi(B) < c$, then the above applies to the dual set $F = \{\varphi \geq c\}$ and we get the following corollary.

COROLLARY 3.31. *Let φ be a continuous functional on X and consider a homological family $\mathcal{F}$ of dimension n with closed boundary B. Set $c = c(\varphi, \mathcal{F})$ and assume that $\sup(B) < c$. If φ verifies $(PS)_c$ along a min-maxing sequence $(A_k)_k$ and if the set $K_c \cap A_\infty$ consists of isolated critical points, then there exists x in $K_c \cap A_\infty$ with $n \in \mathrm{Ord}(x)$.*

PROOF OF THEOREM 3.29. Since $\tilde{K}_c$ is compact and N is a neighborhood of $\tilde{K}_c$, we have a finite number of connected component $(M^i)_{i=1}^m$ of N such that $\tilde{K}_c \subseteq \bigcup_{i=1}^m M^i$ and $\tilde{K}_c \cap M^i \neq \emptyset$ for all $1 \leq i \leq m$. Let $M_c^i = \tilde{K}_c \cap M^i$. Clearly M^i is a neighborhood of the compact set M_c^i for $1 \leq i \leq m$. Hence there is $\tau > 0$ such that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap B = \emptyset \text{ and } N_{4\tau}(M_c^i) \subseteq M^i \quad \text{for all } 1 \leq i \leq m. \tag{3.4.1}$$

Since we assume that $\tilde{K}_c$ is isolated in K_c, we may assume that

$$\overline{N_{4\tau}(\tilde{K}_c)} \cap K_c = \tilde{K}_c. \tag{3.4.2}$$

Let $\delta_k = c - \inf \varphi(F_k)$ and $F_k' = F_k \cup \bigcup_{i=1}^m \big(L_{c-\delta_k} \cap \overline{N_{4\tau}(M_c^i)}\big)$. Clearly F_k' is dual to $\mathcal{F}$ and $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$.

Suppose now that the theorem is not true. Then for the neighborhood M^i of M_c^i and the sub-neighborhood $N_{4\tau}(M_c^i)$ of M_c^i, there exist $\epsilon_0 > 0$ and an open sub-neighborhood $\hat{M}^i \subseteq N_{4\tau}(M_c^i)$ of M_c^i such that $H_n(\hat{M}^i) = H_{n-1}(G_{c-\epsilon} \cap \hat{M}^i) = 0$ for all $0 < \epsilon < \epsilon_0$. Since $\hat{M}^i$ is open, we have the following Mayer-Vietoris exact sequence

$$H_n(X \setminus F_k', B) \oplus H_n(\hat{M}^i) \longrightarrow H_n\big((X \setminus F_k') \cup \hat{M}^i, B\big) \longrightarrow H_{n-1}\big((X \setminus F_k') \cap \hat{M}^i\big).$$

Since $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$, we have that there is $k_0 \geq 1$ such that $0 < \delta_k < \epsilon$ for all $k \geq k_0$. Since $\hat{M}^i \subseteq N_{4\tau}(M_c^i)$, we have that $(X \setminus F_k') \cap \hat{M}^i = G_{c-\delta_k} \cap \hat{M}^i$. By assumption we have for all $k \geq k_0$, that $H_{n-1}\big((X \setminus F_k') \cap \hat{M}^i\big) = 0$. So for $k \geq k_0$, we have that

$$j_*: H_n(X \setminus F_k', B) \longrightarrow H_n\big((X \setminus F_k') \cup \hat{M}^i, B\big)$$

is onto where j_* is induced by the inclusion $j: (X \setminus F_k', B) \longrightarrow \big((X \setminus F_k') \cup \hat{M}^i, B\big)$. Hence we have that the set $(F_k' \setminus \bigcup_{i=1}^m \hat{M}^i)$ is dual to $\mathcal{F}$ for all $k \geq k_0$. Since $\lim_{k\to\infty} \inf \varphi(F_k' \setminus \bigcup_{i=1}^m \hat{M}^i) = c$, we have by Theorem 1.5 that $(F_\infty \setminus \bigcup_{i=1}^m \hat{M}^i) \cap A_\infty \cap K_c \neq \emptyset$. This is a contradiction. ∎

3.5. *Application to standard variational settings.* Let $E = Y \oplus Z$ with $\dim(Y) = n$ and consider the following class

$$\mathcal{F} = \{A\ ;\ \exists h: B_Y \longrightarrow E \text{ continuous}, h(x) = x \text{ on } S_Y \text{ and } A = h(B_Y)\}.$$

It is clear that $\mathcal{F}$ is a homotopic class of dimension n with boundary S_Y. Let now

$$\bar{\mathcal{F}} = \{A\ ;\ A \text{ compact}, A \supset S_Y \text{ and } 0 \in f(A) \text{ whenever } f \in C(A\ ;\ Y) \text{ and } f(x) = x \text{ on } S_Y\}.$$

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$\bar{\mathcal{F}}$ is clearly a cohomotopic class of dimension n and with boundary S_Y. Note also that $\mathcal{F} \subset \bar{\mathcal{F}}$.

Regard now $\sigma = [S_Y]$ as the generator of the homology $H_{n-1}(S_Y, \emptyset)$ and let $\beta \in H_n(E, S_Y)$ be such that $\partial_* \beta = \sigma$ where ∂_* is the map in the exact sequence

$$\longrightarrow H_n(S^Y) \longrightarrow H_n(E) \longrightarrow H_n(E, S_Y) \xrightarrow{\partial_*} H_{n-1}(S^Y) \longrightarrow .$$

Consider $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\beta)$ to be the corresponding homological family. Since $\sigma \neq 0$ in $H_{n-1}(E \setminus Z)$, it follows that Z is dual to the class $\tilde{\mathcal{F}}$.

COROLLARY 3.32. *Let φ be a continuous functional on the Hilbert space E such that*

$$\alpha := \inf \varphi(Z) \geq 0 \geq \sup \varphi(S_Y).$$

Let $c = c(\varphi, \mathcal{F})$, $\bar{c} = c(\varphi, \bar{\mathcal{F}})$ and $\tilde{c} = c(\varphi, \tilde{\mathcal{F}})$. Assume that φ verifies (PS) and that the critical points are non-degenerate. Then the following holds:

If $0 < \bar{c}$, then

1) there exists x_1 in K_c with $\operatorname{sad}(x_1) \leq n$;

2) there exists x_2 in $K_{\bar{c}}$ with $\operatorname{sad}^*(x_2) \geq n$;

3) there exists x_3 in $K_{\tilde{c}}$ with $n \in$ $\operatorname{Ord}(x_3)$;

4) if $c = \bar{c}$, there exists x_4 in K_c with $\operatorname{sad}(x_4) \leq n \leq \operatorname{sad}^*(x_4)$.

4. Morse indices of min-max critical points. In this section, we assume that φ is a C^2-functional on a Hilbert space E and we use the results of the last section to relate the *topological* properties of the homotopy-stable class $\mathcal{F}$ to the *Morse indices* of those critical points obtained by min-maxing over $\mathcal{F}$ and which are located on an—*a priori*—given dual set. We shall be able to find one-sided relations between the *Morse index* and the *homotopic* (resp. *cohomotopic*) dimension of the class, while for *homological* families, two-sided estimates are available. We do that in the non-degenerate case by simply finding relations between the topological indices of critical points introduced in previous sections (saddle-type point, *etc.*) and the standard Morse indices associated to such points.

In this section, we will always assume E, a Hilbert space with inner product $\langle , \rangle$ and norm $\| \ \|$, $\varphi \in C^2(E, \mathbb{R})$. For any $u \in E$, we let $D^2\varphi(u)$ denote the unique bounded self-adjoint linear operator $T: E \longrightarrow E$ such that $\varphi''(u)(v)(w) = \langle Tw, v \rangle$ for all $u, v, w \in E$. We shall write $m(v)$ for the Morse index of the nondegenerate critical point v.

We shall first recall some basic concepts of Morse theory. The following lemma is standard.

LEMMA 4.1. *Assume φ is a C^2-functional on a Hilbert space E. If v_0 is a non-degenerate critical point for φ (i.e. if $d^2\varphi(v_0)$ is invertible), then there exists a Lipschitz homeomorphism H from a neighborhood W of 0 in E onto a neighborhood M of v_0 with* $H(0) = v_0$ *in such a way that*

$$\varphi\big(H(z)\big) = \varphi(v_0) + \|z_+\|^2 - \|z_-\|^2$$

where $z \longrightarrow (z_-, z_+)$ *corresponds to the decomposition of* E *into the positive and negative spaces* E_+ *and* E_- *associated to the operator* $d^2\varphi(v_0)$. *The Morse index of* v_0 *will be the dimension of* E_-.

The proof of the above standard lemma can be found in many books and papers. See for instance [12].

COROLLARY 4.2. *Let* φ *be a* C^2*-functional on a Hilbert space* E *and* v_0 *be a non-degenerate critical point for* φ *with* $m(v_0) = k$. *Then for any* $r > 0$, *there exist a neighborhood* N *of* v_0 *with* $N \subseteq B(v_0, r)$ *and* $\epsilon_0 > 0$ *such that for all* $0 \leq \epsilon \leq \epsilon_0$

$$(\dagger) \qquad N \cap G_{\varphi(v_0)-\epsilon} \cong B_+^\circ \times S^{k-1} \times (0,1).$$

where $B_+ = \{u_+ \,;\, u_+ \in E_+ \text{ and } \|u_+\| \leq 1\}$.

PROOF. Let E, E_+, E_-, H, M and W be given as in the above lemma associated to v_0. Let r_1 be small such that $B(0, r_1) \subseteq W$ and put $\psi(z) = \varphi\big(H(z)\big) - \varphi(v_0) = \|z_+\|^2 - \|z_-\|^2$. We claim that for any r_2 and ϵ_1 with $0 < \epsilon_1 < r_2 < r_1$ we have for all $0 \leq \epsilon \leq \epsilon_1$ that

$$B(0, r_2) \cap \{\psi(z) \,;\, \psi(z) < -\epsilon\} \cong B_+ \times S^{k-1} \times (0,1).$$

Indeed, for any $r_2 > r_3 > \epsilon > 0$ and $z_- \in E_-$ with $\|z_-\| = r_3$ we have that

$$\{z_+ \,;\, \|z_+\| < r_3 - \epsilon\} \subseteq B(0, r_3) \cap \{z \,;\, \psi(z) < -\epsilon\}.$$

Let t be small enough such that $H\big(B(0,t)\big) \subseteq B(v_0, r) \cap B(v_0, r_1)$. Then $N = H\big(B(0,t)\big)$ together with $\epsilon_0 = t/2$ will verify $(\dagger)$ and the corollary is proved. ∎

We also need the following lemma which is due basically to Lazer-Solimini [18].

LEMMA 4.3. *Let* φ *be a* C^2*-functional on a Hilbert space* E *and* v *be a non-degenerate critical point with Morse index* n. *Then for any* $r > 0$, *there are* $0 < r', r'' < r$ *and a continuous map* f *on* E *such that the following holds:*

(i) $f(x) = x$ *on* $E \setminus B(v, r')$;

(ii) $\varphi\big(tf(x) + x(1-t)\big) \leq \varphi(x)$ *on* E *for all* $0 \leq t \leq 1$;

(iii) $f\big(B(v, r'')\big)$ *is homeomorphic to a subset of* $\mathbb{R}^n$.

PROOF. Since v is a non-degenerate critical point for φ on E, let H be the *change of variables map* associated to v_0 by the Lemma 4.1 and write $E = E_- \oplus E_+$. Choose $r_1 > 0$ and $r_2 > 0$ small enough so that if B_- (resp. B_+) denotes the closed ball in E_- (resp. E_+) of radius r_1 (resp. r_2) centered at 0, then $2B_- + 2B_+$ is contained in the domain of H. We may also assume $4r_1^2 + 4r_2^2 < r^2$. Let α be a Lipschitz function from $\mathbb{R}$ to [0,1] so that $\alpha = 0$ on $(-\infty, 0]$ and $\alpha = 1$ on $[1, +\infty)$. Let $\eta \colon E \longrightarrow E$ be defined by

$$\eta(z_- + z_+) = z_- + z_+\left[\alpha\left(\frac{\|z_-\|}{r_1} - 1\right)\left[1 - \alpha\left(\frac{\|z_+\|}{r_2} - 1\right)\right] + \alpha\left(\frac{\|z_+\|}{r_2} - 1\right)\right]$$

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and consider the following transformation $f: E \longrightarrow E$

$$f(x) = \begin{cases} x & \text{on } E \setminus (2B_- + 2B_+) \\ H \circ \eta \circ H^{-1}(x) & \text{on } 2B_- + 2B_+. \end{cases}$$

Clearly f is a continuous map on E. Now take $r' = 2\sqrt{r_1^2 + r_2^2}$ and r'' small such that $B(v, r'') \subseteq B_- + B_+$. Then (i) is obvious. For (ii), we first note that it is true when $t = 0$. Then we need to note that for any $z = z_- + z_+$ we have that $tf(z) + (1 - t)z = z_- + z_+ g(z, t)$ where

$$g(z,t) = t\left[\alpha\left(\frac{\|z_-\|}{r_1} - 1\right)\left[1 - \alpha\left(\frac{\|z_+\|}{r_2} - 1\right)\right] + \alpha\left(\frac{\|z_+\|}{r_2} - 1\right)\right] + (1 - t)$$

and that $g(z, t) \leq 1$ for all z and $0 \leq t \leq 1$. (iii) follows obviously from the definition of f. ∎

We shall need the following basic result from algebraic topology.

LEMMA 4.4. *For all $n \geq 1$, we have that $\pi_r(S^n) = 0$ if $0 \leq r < n$ and $H_r(S^n) = 0$ if $r < n$ and $r \neq 0$. Hence S^n is $(n - 1)$-connected. Moreover we have that $H_r(S^n) = 0$ if $r > n$.*

Now we can prove the following.

THEOREM 4.5. *Let φ be a C^2-functional on a Hilbert space E and let v_0 be a non-degenerate critical point of φ with $m(v_0) = k$ $(k \geq 1)$. Then the following holds:*

(1) w-sad(v_0) = sad(v_0) = m(v_0) = sad$^*(v_0)$.

(2) Ord$_w(v_0)$ = Ord(v_0) = $\{k\}$.

PROOF. (1) We first prove that sad$(v_0) \geq w$-sad$(v_0) \geq$ m(v_0) and sad$^*(v_0) \leq m(v_0)$. By Corollary 4.2, we see that for any neighborhood N of v_0, there exist a sub-neighborhood $M \subseteq N$ of v_0 and an $\epsilon_0 > 0$ such that for all $0 \leq \epsilon \leq \epsilon_0$ we have that

$$M \cap G_{\varphi(v_0)-\epsilon} \cong B_+^{\circ} \times S^{k-1} \times (0, 1).$$

By Lemma 4.4, we see that $M \cap G_{\varphi(v_0)-\epsilon}$ is $k - 2$-connected. Hence we have that sad$(v_0) \geq w$-sad$(v_0) \geq$ k. As for sad$^*(v_0)$, we first note that by Lemma 4.3, for any neighborhood N of v_0, there exist $r_1, r_2 > 0$ and f verifying the conclusion of that lemma. Clearly from (iii) of that lemma, we have that sad$^*(v_0) \leq k$. Next we show that $m(v_0) \geq$ sad$(v_0) \geq w$-sad(v_0) and sad$^*(v_0) \geq m(v_0)$. To do this we consider $\varphi(z) = \|z_+\|^2 - \|z_-\|^2$ on Hilbert space $E = E_+ \oplus E_-$ where $E_- \cong \mathbb{R}^k$ and $z \longrightarrow (z_-, z_+)$ corresponds to the decomposition of E into the positive and negative spaces E_+ and E_-. Note that $\dim(E_-) = k = m(v_0)$ and that φ verifies (PS) condition with 0 being the only critical point. Next set up a canonical min-max process as in Section 3.5 for both homotopic and cohomotopic cases. Then by Corollary 3.32, we see that $m(v_0) \geq \mathrm{sad}(v_0) \geq w\text{-sad}(v_0)$ and $\mathrm{sad}^*(v_0) \geq m(v_0)$.

(2) As above by Corollary 4.2, we see that for any neighborhood N of v_0, there exist a sub-neighborhood $M \subseteq N$ of v_0 and an $\epsilon_0 > 0$ such that for all $0 \leq \epsilon < \epsilon_0$ we have that

$$M \cap G_{\varphi(v_0)-\epsilon} \cong B_+^\circ \times S^{k-1} \times (0, 1).$$

Now by Lemma 4.4 we see that $\mathrm{Ord}_w(v_0) = \mathrm{Ord}(v_0) = \{k\}$. ∎

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