SKEW CONNECTIONS IN VECTOR BUNDLES AND THEIR PROLONGATIONS

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The paper is closely related to [1] and [2]. A skew connection in a vector bundle E as defined here is a pseudo-connection (in the sense of [1]) which can be changed into a connection by transforming separately the bundle E itself and the bundle of its differentials, i.e. one-forms on the base with values in E. The properties of skew connections are thus expected to be only "algebraically" more complicated than those of connections; especially one can follow the pattern of [1], and prolong them to obtain higher order semi-holonomic and non-holonomic pseudo-connections. It is shown in this paper that under some circumstances the main theorem of [1] or [2] applies also to skew connections.

Let M be a fixed (finite-dimensional, C^∞-differentiable) manifold, E a (finite-dimensional over the reals, C^∞-differentiable) vector bundle with base M and fibre type R^n. Let the dimension of M be m. We shall always suppose that the structure group of a vector bundle is the maximal linear group (i.e. $GL(n, R)$ in the case of E), and neglect the question of its possible reducibility. Let F be another vector bundle over M, $p : E \to M$ and $p' : F \to M$ the corresponding projections. A C^∞-map $\Phi : E \to F$ (a diffeomorphism), such that $p'\Phi = p$ and Φ is linear on each fibre, is called a *bundle morphism* (isomorphism). If $T(M)$ and $T(M)^*$ are the tangent and cotangent bundles respectively to M, denote $T^1(E) = E \oplus E \otimes T(M)^*$, and by $S^1(E)$ the vector bundle over M of all one-jets of local sections of E. Denoting by $\boldsymbol{R}$ the trivial bundle $M \times R$, we have clearly $S^1(\boldsymbol{R}) = T^1(\boldsymbol{R})$. Note that the fibres of both $S^1(E)$ and $T^1(E)$ have the same dimension, and that $E \otimes T(M)^*$ can be regarded as a subbundle of both $T^1(E)$ and $S^1(E)$, identifying it with Ker π_T and Ker π_S respectively, where $\pi_T : T^1(E) \to E$ and $\pi_S : S^1(E) \to E$ are the natural bundle projections. In [1] we have defined a *pseudo-connection* in E as a bundle isomorphism $H : S^1(E) \to T^1(E)$, and we have seen that it corresponds to a usual connection iff $\pi_T H = \pi_S$ and H is the identity on $E \otimes T(M)^*$.

Let $H = H_1 + H_2$ be a pseudo-connection, where $H_1 : S^1(E) \to E$ and $H_2 : S^1(E) \to E \otimes T(M)^*$ are its natural composants. It is called a *skew connection* iff it preserves the subbundle $E \otimes T(M)^*$, i.e. iff $\pi_S(X) = 0 \Rightarrow H_1(X) = 0$. We have the evident

LEMMA 1. *A pseudo-connection H is a skew connection iff any of the two conditions is satisfied:*

(A) *There is a bundle morphism* $A : E \to E$ *such that* $H_1 = A\pi_S$;
(B) *There is a bundle morphism* $Q : E \otimes T(M)^* \to E \otimes T(M)^*$ *such that* $\pi_S(X) = 0 \Rightarrow H(X) = H_2(X) = Q(X)$.

Note that if such A or Q exists for a pseudo-connection H, then both they exist, are uniquely determined and invertible (i.e. bundle isomorphisms). Call A the *first* and Q the *second tensor* of the skew connection H. A pseudo-connection is thus a connection iff it is a skew connection with trivial (i.e. identity) first and second tensors. A skew connection is called a *relative connection with respect to a bundle isomorphism* $A : E \to E$ (or briefly an *A-connection*) if its first and second tensors are A and $A \otimes id_{(T)M^*}$ respectively.

REMARK. A pseudo-connection is a skew connection, iff its components $\Gamma^{h\alpha}_{k\beta}(h, k = 1, \cdots n; \alpha, \beta = 0, 1, \cdots m)$ in coordinate neighbourhoods (c.f. [1]) satisfy $\Gamma^{hi}_{k0} = 0$ $(h, k = 1, \cdots n;\ i = 1, \cdots m)$. In this case Γ^{h0}_{k0} are the components of the first, and Γ^{hi}_{kj} the components of the second tensors.

Both the groups Aut $S^1(E)$ or Aut $T^1(E)$, of all bundle automorphisms of $S^1(E)$ or $T^1(E)$ respectively, act freely and transitively (to the right or left respectively) on the set $PC(E)$ of all pseudo-connections in E. Each element $B \in$ Aut $T^1(E)$ is uniquely determined by a 'matrix of tensors' $(B_{ik})_{i,k=1,2}$, where $B_{11} : E \to E$, $B_{12} : E \to E \otimes T(M)^*$, $B_{21} : E \otimes T(M)^* \to E$, $B_{22} : E \otimes T(M)^* \to E \otimes T(M)^*$ are bundle morphisms subject only to the condition that the morphism $(X+Y) \mapsto (B_{11}(X)+B_{21}(Y))+(B_{12}(X)+B_{22}(Y))$ of $T^1(E)$ onto itself be invertible.

THEOREM 1. *The subset* $SC(E) \subset PC(E)$ *of skew connections in E is one of the orbits in* $PC(E)$ *with respect to the action of the subgroup* $\mathscr{B} \subset$ Aut $T^1(E)$ *characterized by the condition* $B_{21} = 0$.

The proof is evident. Note that $B_{21} = 0$ implies the invertibility of both B_{11} and B_{22}.

THEOREM 2. *If H is a skew connection in E, its first and second tensors being A and Q respectively, and* $B \in \mathscr{B}$, *then the first and second tensors of the skew connection BH are* $B_{11}A$ *and* $B_{22}Q$ *respectively.*

The proof is again evident as well as that of the

COROLLARY. *Given any pair* $A : E \to E$, $Q : E \otimes T(M)^* \to E \otimes T(M)^*$ *of bundle isomorphisms, there is a unique orbit* $C_{AQ}(E) \subset PC(E)$, *with respect to the action of the subgroup* $\mathscr{B}_0 \subset \mathscr{B} \subset$ Aut $T^1(E)$, *consisting of all the skew connections in E admitting A and Q as their first and second tensors. The subgroup* $\mathscr{B}_0$ *is characterized by the condition* $B_{21} = 0$, $B_{11} = id_E$, $B_{22} = id_{E\otimes TTM^*}$.

If H is a skew connection, A and Q its tensors as above, let B be defined by the quadrupole $B_{11} = A^{-1}$, $B_{21} = B_{12} = 0$, $B_{22} = Q^{-1}$. Then $H^0 = BH$ is a connection in E called the *associated with H connection*. Conversely, if H^0 is a connection in E, A, Q arbitrary bundle automorphisms as above, then $H = B^{-1}H^0$, where B^{-1} is the inverse of B as above, is a skew connection admitting A and Q as its first and second tensors respectively. Explicitly

$$H = j_T^1 A\pi_T H^0 + j_T^{1*} Q\pi_T^* H^0,$$

where $T^1(E)$ is represented by the direct sum diagram

$$E \underset{\pi_T}{\overset{j_T}{\rightleftarrows}} T^1(E) \underset{\pi_T{}^*}{\overset{j_T^{1*}}{\leftrightarrows}} E \otimes T(M)^*$$

(c.f. [1]). There is hence a natural one-to-one-correspondence between $SC(E)$ and all the triples consisting of connections in E and bundle automorphisms $A : E \to E$, $Q : E \otimes T(M)^* \to E \otimes T(M)^*$.

REMARK. If $B \in \operatorname{Aut} T^1(E)$, $B' \in \mathscr{B}_0 B$ then $B'_{21} = B_{21}, B'_{22} = B_{22}$; if moreover $B_{21} = 0$, then also $B'_{11} = B_{11}$. Thus the tensors B_{21}, B_{22} are invariants of the right cosets with respect to $\mathscr{B}_0$; i.e. given $H \in PC(E)$, the tensors $B_{21} = B_{21}(H)$ and $B_{22} = B_{22}(H)$ corresponding to any automorphism of $T^1(E)$ taking H into a connection are "invariants of the pseudo-connection H". It is a skew connection iff $B_{21}(H) = 0$; in that case also $B_{11} = B_{11}(H)$ is an 'invariant' and evidently $B_{11}(H)^{-1}$ and $B_{22}(H)^{-1}$ coincide with the first and second tensors of the skew connection H.

Let $\Phi : E \to E$ be a bundle morphism. We have then also bundle morphisms $S^1(\Phi) : S^1(E) \to S^1(E)$ and $T^1(\Phi) : T^1(E) \to T^1(E)$ (c.f. [1]); S^1 and T^1 are functors from the category of vector bundles over M into itself. A skew connection H in E is called *Φ-invariant* if $T^1(\Phi)H = HS^1(\Phi)$. We have again an evident

LEMMA 2. *If $H \in SC(E)$ is Φ-invariant, then so is any skew connection BH, where $B \in \mathscr{B}_0$ and B_{11} commutes with Φ, B_{22} with $\Phi \otimes id_{T(M)^*}$.*

COROLLARY. *A skew connection is Φ-invariant if the associated connection is Φ-invariant and Φ commutes with the first tensor, $\Phi \otimes id_{T(M)^*}$ with the second tensor.*

A skew connection is called *regular*, if it is A-invariant, where A is its first tensor. Thus such $H \in SC(E)$ is regular if its associated connection is A-invariant and $A \otimes id_{T(M)^*}$ commutes with the second tensor; especially an A-connection is regular if its associated connection is A-invariant.

REMARK. The Φ-invariancy of a connection H, i.e. the condition $HS^1(\Phi) = T^1(\Phi)H$, is equivalent with the condition $\nabla_X(\Phi f) = \Phi\nabla_X f$ for any local section X of $T(M)$ and any local section f of E, where $\nabla_X f = \langle X, H_2(j^1 f)\rangle$ is the covariant derivative induced by the connection H (c.f. [1], p. 144). In other words H is Φ-invariant iff the absolute differential of Φ is zero. This gives also the local conditions for the regularity of a skew connection in terms of its components Γ^h_k and Γ^h_{ki} as

$$\partial_i \Gamma^s_k + \sum_{h=1}^{n} (\Gamma^s_{hi}\Gamma^h_k - \Gamma^h_{ki}\Gamma^s_h) = 0$$

for each $i = 1, \cdots m$; $s, k = 1, \cdots n$.

If E and F are two vector bundles over M, H_E and H_F connections in E and F respectively, then they induce natural connections $H_E(\oplus)\, H_F$ in $E \oplus F$ and $H_E\,(\otimes)\, H_F$ in $E \otimes F$; there is also a connection H_{E^*} in the dual bundle E^* induced by the connection H_E (see e.g. again [1] including the notations). Trying to generalize this to arbitrary skew connections H_E and H_F with the first tensors A_E and A_F, the second tensors Q_E and Q_F respectively, we first pass to the associated connections H^0_E, H^0_F, form $H^0_E\,(\oplus)\, H^0_F$ or $H^0_E\,(\otimes)\, H^0_F$ or $H^0_{E^*}$ as above, and introduce $H_E\,(\oplus)\, H_F$ or $H_E\,(\otimes)\, H_F$ or H_{E^*} as the skew connections with these associated connections and the tensors 'naturally' connected with those of H_E and H_F. In the case of the direct sum this means that we put $A_{E\oplus F} = A_E \oplus A_F$, $Q_{E\oplus F} = Q_E \oplus Q_F$ for the tensors of $H_E\,(\oplus)\, H_F$, but in the case of the tensor product, to obtain the second tensor reasonably linked with Q_E and Q_F, one has to suppose that $Q_E = P_E \otimes R$, $Q_F = P_F \otimes R$, where $P_E : E \to E$, $P_F : F \to F$, $R : T(M)^* \to T(M)^*$ are some bundle automorphisms. We shall refer to this situation by saying that H_E and H_F are *R-linked*. Now if the skew connections H_E and H_F are R-linked, we define the tensors of $H_E\,(\otimes)\, H_F$ and H_E by $A_{E\otimes F} = A_E \otimes A_F$, $Q_{E\otimes F} = P_E \otimes P_F \otimes R$ and $A_{E^*} = (A_E)^*$, $Q_{E^*} = (P_E)^* \otimes R$. Note that if H_E is an A_E-connection, H_F an A_F-connection, then they are linked by the identity and $H_E\,(\otimes)\, H_F$ is an $(A_E \otimes A_F)$-connection.

An easy consequence of Lemma 3.1 and 3.2 in [1] is

LEMMA 3. *If $\Phi : E \to E$, $\Psi : F \to F$ are bundle morphisms, H_E and H_F connections in E and F respectively, then*

$$H_E S^1(\Phi)\,(\otimes)\, H_F S^1(\Psi) = (H_E\,(\otimes)\, H_F) S^1(\Phi \otimes \Psi)$$

and

$$T^1(\Phi) H_E\,(\otimes)\, T^1(\Psi) H_F = T^1(\Phi \otimes \Psi)(H_E\,(\otimes)\, H_F).$$

LEMMA 4. *Let $\Phi : E \to E$, $\Psi : F \to F$ be bundle morphisms. Let H_E be a Φ-invariant connection in E, H_F a Ψ-invariant connection in F. Then*

(a) *$H_E\,(\oplus)\, H_F$ is $(\Phi \oplus \Psi)$-invariant,*

(b) *$H_E\,(\otimes)\, H_F$ is $(\Phi \otimes \Psi)$-invariant,*

(c) *H_{E^*} is Φ^*-invariant.*

PROOF. (a) If $E \oplus F$ is represented by

$$E \underset{j_E}{\overset{\pi_E}{\rightleftarrows}} E \oplus F \underset{j_F}{\overset{\pi_F}{\rightleftarrows}} F$$

then $H_E\,(\oplus)\,H_F = T^1(j_E)H_E S^1(\pi_E) + T^1(j_F)H_F S^1(\pi_F)$ (c.f. (3.23) in [1]) and hence $H_E S^1(\Phi) = T^1(\Phi)H_E$, $H_F S^1(\Psi) = T^1(\Psi)H_F$ implies

$$\begin{aligned}(H_E\,(\oplus)\,H_F)S^1(\Phi \oplus \Psi) &= T^1(j_E)H_E S^1(\Phi\pi_E) + T^1(j_F)H_F S^1(\Psi\pi_F)\\ &= T^1(j_E\Phi)H_E S^1(\pi_E) + T^1(j_F\Psi)H_F S^1(\pi_F) = T^1(\Phi \oplus \Psi)T^1(j_E)H_E S^1(\pi_E)\\ &\quad + T^1(\Phi \oplus \Psi)T^1(j_F)H_F S^1(\pi_F).\end{aligned}$$

(b) follows directly from Lemma 3.3 in [1].

(c) Denoting by $c : E \otimes E^* \to \boldsymbol{R}$ the natural contraction, we have $c(id_E \otimes \Phi^*) = c(\Phi \otimes id_{E^*})$ and thus applying this, Lemma 3 and (b) to the relation $T^1(c)(H_E\,(\otimes)\,H_{E^*}) = S^1(c)$, (c.f. (3.49) in [1]), we get

$$\begin{aligned}T^1(c)(H_E\,(\otimes)\,[H_{E^*}S^1(\Phi^*)]) &= T^1(c)(H_E(\otimes)\,H_{E^*})(S^1(id_E) \otimes S^1(\Phi^*))\\ &= S^1(c)(S^1(\Phi) \otimes S^1(id_{E^*})) = T^1(c)([H_E S^1(\Phi)]\,(\otimes)\,H_{E^*})\\ &= T^1(c)(T^1(\Phi)H_E]\,(\otimes)\,H_{E^*}) = T^1(c)T^1(id_E \otimes \Phi^*)(H_E\,(\otimes)\,H_{E^*})\\ &= T^1(c)(H_E\,(\otimes)\,[T^1(\Phi^*)H_{E^*}]).\end{aligned}$$

Now according to the uniqueness property in Lemma 3.5 in [1], the proof is completed.

COROLLARY. *Let H_E and H_F be regular skew connections in E and F respectively, their tensors being A_E or $Q_E = P_E \otimes R$, and A_F or $Q_F = P_F \otimes R$. Let A_E commute with P_E and A_F with P_F. Then the skew connections $H_E\,(\oplus)\,H_F$, $H_E\,(\otimes)\,H_F$ and H_{E^*} are regular.*

PROOF. It is sufficient to show that $(A_E \oplus A_F) \otimes id_{T(M)^*}$ commutes with $Q_E \oplus Q_F$, and $A_E \otimes A_F \otimes id_{T(M)^*}$ with $P_E \otimes P_F \otimes R$, as well as $(A_E)^* \otimes id_{T(M)^*}$ with $(P_E)^* \otimes R$; but this is obvious from the assumptions.

This corollary is useful for the prolongation procedure of skew connections. First let us recall briefly some basic notions and notations from [1], (c.f. also [2]).

For each integer $q \geqq 1$ denote by S^q, $\bar{S}^q$ and $\tilde{S}^q$ the covariant functors from the category of veçtor bundles over M into itself which are defined by means of the holonomic, semi-holonomic and non-holonomic jet prolongations respectively in the sense of Ch. Ehresmann. We put $E = S^0(E) = \bar{S}^0(E) = \tilde{S}^0(E)$ as well as $E = T^0(E) = \bar{T}^0(E) = \tilde{T}^0(E)$ and define for each $q \geqq 1$ recurrently

$$\begin{aligned} T^q(E) &= T^{q-1}(E) \oplus E \otimes (\overset{q}{\bigcirc}\, T(M)^*)\\ \text{(1)}\qquad \bar{T}^q(E) &= \bar{T}^{q-1}(E) \oplus E \otimes (\overset{q}{\otimes}\, T(M)^*)\\ \tilde{T}^q(E) &= \tilde{T}^{q-1}(E) \oplus \tilde{T}^{q-1}(E) \otimes T(M)^*,\end{aligned}$$

giving rise to the functors T^q, $\bar{T}^q$, $\tilde{T}^q$ from the category of vector bundles into itself. Let $\pi_S^q : S^q(E) \to S^{q-1}(E)$, $\pi_S^q : \bar{S}^q(E) \to \bar{S}^{q-1}(E)$ and $\pi_S = \tilde{\pi}_S^q : \tilde{S}^q(E) = S^1(\tilde{S}^{q-1}(E)) \to \tilde{S}^{q-1}(E)$, or correspondingly π_T^q, $\bar{\pi}_T^q$ and $\tilde{\pi}_T^q$ (c.f. (1)) be the natural surjections. Let further $\iota_S^q : S^q(E) \to \bar{S}^q(E)$, $\bar{\iota}_S^q : \bar{S}^q(E) \to \tilde{S}^q(E)$ denote the natural injections as well as ι_T^q and $\bar{\iota}_T^q$ in the other case. It is known (c.f. [1]) that $\bar{\iota}_S^q$ can be splitted into injections

$$\bar{\iota}_S^q : \bar{S}^q(E) \xrightarrow{\iota_{S^{q'}}} S^1(\bar{S}^{q-1}(E)) \xrightarrow{S^1(\iota_{S^{q-1}})} S^1(\tilde{S}^{q-1}(E)) = \tilde{S}^q(E),$$

and analogously

$$\bar{\iota}_T^q : \bar{T}^q(E) \xrightarrow{\iota_{T^{q'}}} T^1(T^{q-1}(E)) \xrightarrow{T^1(\iota_{T^{q-1}})} T^1(\tilde{T}^{q-1}(E)) = \tilde{T}^q(E).$$

Here the morphism $\bar{\iota}_T^{q'}$ is determined by

$$\bar{\iota}_T^{q'} : e \otimes \sum_{k=0}^{q} \omega_1^k \otimes \cdots \otimes \omega_k^k \mapsto e \otimes \sum_{k=0}^{q-1} \omega_1^k \otimes \cdots \otimes \omega_k^k$$
$$+ e \otimes \sum_{k=0}^{q-1} [\omega_1^{k+1} \otimes \cdots \otimes \omega_k^{k+1}] \otimes \omega_{k+1}^{k+1},$$

where $e \in E$, $\omega_i^k \in T(M)^*$ for $i = 1, \cdots k$; $k = 0, \cdots q$; $\omega_0^0 = (1, x) \in \boldsymbol{R}$ and $x \in M$ is the point 'over which' these elements are taken.

One also identifies $E \otimes (\bigcirc^q T(M)^*)$ with both the subbundles $\operatorname{Ker} \pi_S^q \subset S^q(E)$ as well as $\operatorname{Ker} \pi_T^q \subset T^q(E)$, and $E \otimes^q T(M)^*)$ with both the subbundles $\operatorname{Ker} \bar{\pi}_S^q \subset \bar{S}^q(E)$ as well as $\operatorname{Ker} \bar{\pi}_T^q \subset \bar{T}^q(E)$.

A holonomic or semi-holonomic or non-holonomic *pseudo-connection of order* $q \geqq 1$ in E is a bundle isomorphism $HH^q : S^q(E) \to T^q(E)$ or $SH^q : \bar{S}^q(E) \to \bar{T}^q(E)$ or $NH^q : \tilde{S}^q(E) \to \tilde{T}^q(E)$ respectively. Given a sequence $\{HH^q\}_{q=1}^\infty$ or $\{SH^q\}_{q=1}^\infty$ or $\{NH^q\}_{q=1}^\infty$ of pseudo-connections in E, then it is called a sequence of holonomic or semi-holonomic or non-holonomic *connections* if for each $q \geqq 1$, $\pi_T^q HH^q = HH^{q-1}\pi_S^q$; $HH^q|_{E\otimes(\bigcirc^q T)M)^*)} = id$, with $HH^0 = id_E$, or $\bar{\pi}_T^q SH^q = SH^{q-1}\bar{\pi}_S^q$; $SH^q|_{E\otimes(\otimes^q T(M)^*)} = id$, with $SH^0 = id_E$, or $\pi_T NH^q = NH^{q-1}\pi_S$; $NH^q|_{S^{q-1}(E)\otimes T(M)^*} = NH^{q-1} \otimes id_{T(M)^*}$, with $NH^0 = id_E$.

REMARK. These definitions are in accordance with the definitions of higher order connections in vector bundles in [3], [4] or [5]. On the other hand a higher order connection as introduced by C. Ehresmann corresponds in the case of vector bundles to a surconnection (and not connection) of P. Libermann (c.f. [3]). See also [6] for the relation of these two definitions.

As in [1], we restrict our interest to the semi-holonomic and non-holonomic cases. The following sequences of first order pseudo-connections have been also introduced in [1]:

$$\{\tilde{H}_S^q\}, \text{ with } \tilde{H}_S^q : S^1(\tilde{S}^{q-1}(E)) \to T^1(\tilde{S}^{q-1}(E));$$
$$\{\tilde{H}_T^q\}, \text{ with } \tilde{H}_T^q : S^1(\tilde{T}^{q-1}(E)) \to T^1(\tilde{T}^{q-1}(E));$$

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$$\{\bar{H}^q_S\}, \text{ with } \bar{H}^q_S : S^1(\bar{S}^{q-1}(E)) \to T^1(\bar{S}^{q-1}(E));$$
$$\{\bar{H}^q_T\}, \text{ with } \bar{H}^q_T : S^1(\bar{T}^{q-1}(E)) \to T^1(\bar{T}^{q-1}(E)).$$

Such a sequence $\{\tilde{H}^q_S\}$ (or $\{\tilde{H}^q_T\}$) is called *reducible* to a sequence $\{\bar{H}^q_S\}$ (or $\{\bar{H}^q_T\}$) if for each $q \geqq 1$ the relation $\tilde{H}^q_S S^1(\bar{\imath}^{q-1}_S) = T^1(\bar{\imath}^{q-1}_S)\bar{H}^q_S$ (or $\tilde{H}^q_T S^1(\bar{\imath}^{q-1}_T) = T^1(\bar{\imath}^{q-1}_T)\bar{H}^q_T$) holds. A sequence

$$\text{(a) } \{SH^q\}; \quad \text{(b) } \{\bar{H}^q_S\}; \quad \text{(c) } \{\bar{H}^q_T\}$$

of pseudo-connections is called *regular* if for each $q \geqq 1$ the following condition is satisfied:

(a) $\bar{\pi}^q_T SH^q = SH^{q-1}A^{q-1}\bar{\pi}^q_S$ for some sequence $\{A^{q-1}\}$ of automorphisms $A^{q-1} : \bar{S}^{q-1}(E) \to \bar{S}^{q-1}(E)$ or, equivalently, $\bar{\pi}^q_S(SH^q)^{-1} = (SH^{q-1})^{-1}(B^{q-1})^{-1}$ $\bar{\pi}^q_T$ for some sequence $\{B^{q-1}\}$ of automorphisms $B^{q-1} : \bar{T}^{q-1}(E) \to \bar{T}^{q-1}(E)$;

(b) $\pi_T \bar{H}^q_S \iota^{q'}_S = A^{q-1}\bar{\pi}^q_S$ and $T^1(A^{q-1}\bar{\pi}^q_S)\bar{H}^{q+1}_S \iota^{q+1'}_S = \bar{H}^q_S \iota^{q'}_S A^q \bar{\pi}^{q+1}_S$ for some sequence $\{A^{q-1}\}$ of automorphisms as sub (a);

(c) $\pi_S(\bar{H}^q_T)^{-1}\bar{\imath}^{q'}_T = (B^{q-1})^{-1}\bar{\pi}^q_T$ and $S^1((B^{q-1})^{-1}\bar{\pi}^q_T)(\bar{H}^{q+1}_T)^{-1}\bar{\imath}^{q+1'}_T = (\bar{H}^q_T)^{-1}\bar{\imath}^{q'}_T(B^q)^{-1}\bar{\pi}^{q+1}_T$ for some sequence $\{B^{q-1}\}$ of automorphisms as sub (a).

The relations

$$NH^q = T^1(NH^{q-1})\tilde{H}^q_S \langle = \rangle \tilde{H}^q_S = T^1(NH^{q-1})^{-1}NH^q \tag{2}$$

and

$$NH^q = \tilde{H}^q_T S^1(NH^{q-1}) \langle = \rangle \tilde{H}^q_T = NH^q S^1(NH^{q-1})^{-1} \tag{3}$$

define a 'one-to-one-to-one' correspondence $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$ between the three sequences dealt with in the non-holonomic case. The main theorem in [1] states that if there is a triple of sequences in such a correspondence, then the following conditions are equivalent:

(I) $\{NH^q\}$ is reducible to a regular sequence $\{SH^q\}$ with the automorphisms $\{A^{q-1}\}$ (or $\{B^{q-1} = SH^{q-1}A^{q-1}(SH^{q-1})^{-1}\}$);

(II) $\{\tilde{H}^q_S\}$ is reducible to a regular sequence $\{\bar{H}^q_S\}$ with the automorphisms $\{A^{q-1}\}$;

(III) $\{\tilde{H}^q_T\}$ is reducible to a regular sequence $\{\tilde{H}^q_T\}$ with the automorphisms $\{B^{q-1}\}$.

In particular it has been shown there that if H is a (first order) connection in E, h a (first order) connection in the tangent bindle $T(M)$, then one can get 'by prolongation' sequences which satisfy (III) and hence all the above conditions. This can be generalized with some restrictions to the case where H is a skew connection in E, h a skew connection in $T(M)$.

Thus suppose $H \in SC(E)$ with the tensors A and Q, $h \in SC(T(M))$ with the tensors a and q are R-linked skew connections, i.e. $Q = P \otimes R$, $q = p \otimes R$ for some fixed bundle automorphism $R : T(M)^* \to T(M)^*$. We have already seen that one can construct then two canonical sequences $\{\bar{H}^q_T\}$ and $\{\tilde{H}^q_T\}$, where each $\bar{H}^q_T$ $(q \geqq 1)$ is a skew connection in $\bar{T}^{q-1}(E)$ with the first tensor $\bar{A}^q_T = A \otimes \sum_{k=0}^q \otimes^k a^*$, the second tensor $\bar{Q}^q_T = \bar{P}^q_T \otimes R$, with $\bar{P}^q_T = P \otimes \sum_{k=0}^q \otimes p^*$, and each $\tilde{H}^q_T$ $(q \geqq 1)$ is a skew connection in $\tilde{T}^{q-1}(E)$ with the first tensor $\tilde{A}^q_T = A \otimes (\otimes^{q-1}(id_{\mathbf{R}} \otimes a^*))$, the second tensor $\tilde{Q}^q_T = \tilde{P}^q_T \otimes R$, with $\tilde{P}^q_T = P \otimes (\otimes^{q-1}(id_{\mathbf{R}} \otimes p^*))$. Denote by $\{(\bar{H}^q_T)^0\}$ and $\{(\tilde{H}^q_T)^0\}$ the sequences of the corresponding associated connections – they are constructed from the associated to H and h connections H^0 and h^0 respectively as in [1].

Lemma 5. *The sequence $\{\tilde{H}^q_T\}$ is reducible to the sequence $\{\bar{H}^q_T\}$, i.e. for each $q \geqq 1$,*

$$H^q_T S^1(i^{q-1}_T) = T^1(i^{q-1}_T)\bar{H}^q_T.$$

Proof. Such a relation certainly holds for the sequences $\{(\tilde{H}^q_T)^0\}$ and $\{(\bar{H}^q_T)^0\}$ (c.f. [1] or [2]). On the other hand the relation between skew connections and associated connections gives in this case

$$\begin{aligned} \tilde{H}^q_T &= j^1_T \tilde{A}^q_T \pi_T (\tilde{H}^q_T)^0 + j^{1*}_T (\tilde{P}^q_T \otimes R)\pi^*_T (\tilde{H}^q_T)^0, \\ \bar{H}^q_T &= j^1_T \bar{A}^q_T \pi_T (\bar{H}^q_T)^0 + j^{1*}_T (\bar{P}^q_T \otimes R)\pi^*_T (\bar{H}^q_T)^0, \end{aligned} \tag{4}$$

and thus by (2.14-15) and (2.67-68) of [1] we get subsequently

$$\begin{aligned} \tilde{H}^q_T S^1(i^{q-1}_T) &= j^1_T \tilde{A}^q_T \pi_T T^1(i^{q-1}_T)(\bar{H}^q_T)^0 + j^{1*}_T(\tilde{P}^q_T \otimes R)\pi^*_T(T^1(i^{q-1}_T)(\bar{H}^q_T)^0 \\ &= j^1_T \tilde{A}^q_T i^{q-1}_T \pi_T (\bar{H}^q_T)^0 + j^{1*}_T(\tilde{P}^q_T \otimes R)(i^{q-1}_T \otimes id_{T(M)^*})\pi^*_T(\bar{H}^q_T)^0 \\ &= j^1_T i^{q-1}_T \bar{A}^q_T \pi_T (\bar{H}^q_T)^0 + j^{1*}_T (i^{q-1}_T \otimes id_{T(M)^*})(\bar{P}^q_T \otimes R)\pi^*_T(\bar{H}^q_T)^0 \\ &= j^1_T i^{q-1}_T \pi_T j^1_T \bar{A}^q_T \pi_T (\bar{H}^q_T)^0 + j^{1*}_T (i^{q-1}_T \otimes id_{T(M)^*})\pi^*_T j^{1*}_T (\bar{P}^q_T \otimes R)\pi^*_T(\bar{H}^q_T)^0 \\ &= T^1(i^{q-1}_T)\bar{H}^q_T. \end{aligned}$$

Here we have used the obvious relations

$$A^q_T i^{q-1}_T = i^{q-1}_T \bar{A}^q_T \text{ and } \tilde{P}^q_T i^{q-1}_T = i^{q-1}_T \bar{P}^q_T.$$

Theorem 3. *Let H be a skew connection in E with the tensors A and $Q = P \otimes R$ which is regular and such that A commutes with P. Let h be a skew connection in $T(M)$ with a trivial first tensor (i.e. $a = id_{T(M)}$) and the second tensor $q = p \otimes R$ (especially let h be a connection in $T(M)$). Then the canonical sequence $\{\tilde{H}^q_T\}$ of skew connections is reducible to the canonical sequence $\{\bar{H}^q_T\}$, which is regular.*

Proof. According to Lemma 5, all we have to prove is that $\{\bar{H}^q_T\}$ is regular. By the corollary of Lemma 4 we easily conclude, that each skew connection $\bar{H}^q_T$ is regular, i.e. $\bar{A}^q_T$-invariant, i.e. $T^1(\bar{A}^{q+1}_T)\bar{H}^{q+1}_T = \bar{H}^{q+1}_T S^1(\bar{A}^{q+1}_T) \Rightarrow S^1(\bar{\pi}^q_T)$ $S^1(\bar{A}^{q+1}_T)^{-1}(\bar{H}^{q+1}_T)^{-1} i^{q+1'}_T = S^1(\bar{\pi}^q_T)(\bar{H}^{q+1}_T)^{-1} T^1(\bar{A}^{q+1}_T)^{-1} i^{q+1'}_T$. Now we have evidently $S^1(\bar{\pi}^q_T)S^1(\bar{A}^{q+1}_T)^{-1} = S^1(\bar{A}^q_T)^{-1}S^1(\bar{\pi}^q_T)$, and from (4) we also derive $T^1(\bar{\pi}^q_T)\bar{H}^{q+1}_T = \bar{H}^q_T S^1(\bar{\pi}^q_T)$, i.e. $S^1(\bar{\pi}^q_T)(\bar{H}^{q+1}_T)^{-1} = (\bar{H}^q_T)^{-1} T^1(\bar{\pi}^q_T)$. Finally by

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(2.64) of [1] we get $T^1(\bar{\pi}^q_T)T^1(\bar{A}^{q+1}_T)^{-1}\bar{i}^{p+1'}_T = T^1(\bar{A}^q_T)^{-1}\bar{i}^{q'}_T\bar{\pi}^{q+1}_T$ and this completes the proof, since we have

$$(5) \qquad T^1(\bar{A}^q_T)^{-1}\bar{i}^{q'}_T = \bar{i}^{q'}_T(\bar{A}^{q+1}_T)^{-1}$$

because of $a = id_{T(M)}$.

The results just obtained can be summarized in the following way: If the assumptions of Theorem 3 are satisfied – especially if H is a regular relative connection in E and h a connection in $T(M)$ – then the prolongation procedure described in [1] and [2] 'works' in essentially the same manner as for connections. That is, we get a canonical sequence $\{NH^q\}$ of non-holonomic pseudo-connections in E reducible to a regular sequence $\{SH^q\}$ of semi-holonomic pseudo-connections in E, and they are uniquely connected also with a sequence $\{\tilde{H}^q_S\}$ of first order pseudo-connections in the nonholonomic jet prolongations of E reducible to a regular sequence $\{\bar{H}^q_S\}$ of pseudo-connections in the semi-holonic jet prolongations of E. Since $\{SH^q\}$ is regular, $\bar{\pi}^q_S(X) = 0 \Rightarrow \bar{\pi}^q_T SH^q(X) = 0$, and we have also

THEOREM 4. *Under the assumptions of Theorem* 3, *all the $\tilde{H}^q_S$ and $\bar{H}^q_S$ are skew connections.*

PROOF. By (4.8-9) of [1], $\tilde{H}^q_S = T^1(NH^{q-1})^{-1}NH^q = T_1(NH^{q-1})^{-1}\tilde{H}^q_T S^1(NH^{q-1})$, i.e. $\pi_T\tilde{H}^q_S = (NH^{q-1})^{-1}\tilde{A}^q_T NH^{q-1}\pi_S$, which proves that $\tilde{H}^q_S$ is a skew connection. Similarly $\pi_S(X) = 0 \Rightarrow \pi_S S^1(SH^{q-1})(X) = SH^{q-1}\pi_S(X) = 0$ and thus also $\pi_T\bar{H}^q_T S^1(SH^{q-1}) = 0$ which means by (4.44) of [1] that $SH^{q-1}\pi_T\bar{H}^q_S = 0$, i.e. $\bar{H}^q_S$ is a skew connection.

One can define, in an evident manner, the functors $\bar{T}^q$, $\tilde{T}^q$, $\bar{S}^q$, $\tilde{S}^q$ from the category of vector bundles over M into itself. Note that for $A : E \to E$ we have by our notations now $\bar{T}^q(A) = \bar{A}^{q+1}_T$, $\tilde{T}^q(A) = \tilde{A}^{q+1}_T$, and $\tilde{S}^q(A) = S^1(\tilde{S}^{q-1}(A))$ recurrently also satisfies

$$(6) \qquad \bar{i}^q_S\bar{S}^q(A) = \tilde{S}^q(A)\bar{i}^q_S.$$

THEOREM 5. *If H is a regular A-connection in E and h a connection in $T(M)$ then the canonical prolongations are such that each $\tilde{H}^q_T$ is a $\tilde{T}^{q-1}(A)$-connection, each $\bar{H}^q_T$ is a $\bar{T}^{q-1}(A)$-connection, each $\tilde{H}^q_S$ is a $\tilde{S}^{q-1}(A)$-connection, and each $\bar{H}^q_S$ is a $\bar{S}^{q-1}(A)$-connection.*

PROOF. The statement is evident for the $\tilde{H}^q_T$ and $\bar{H}^q_T$ from their construction. We shall first show that for $q \geqq 1$

$$(7) \qquad NH^q\tilde{S}^q(A) = \tilde{T}^q(A)NH^q.$$

This being evident for $q = 1$, we proceed by induction using (2) and get $NH^q\tilde{S}^q(A) = \tilde{H}^q_T S^1(NH^{q-1}\tilde{S}^{q-1}(A)) = \tilde{H}^q_T S^1(\tilde{T}^{q-1}(A))S^1(NH^{q-1}) = \tilde{T}^q(A)\tilde{H}^q_T S^1(NH^{q-1}) = \tilde{T}^q(A)NH^q$, because by the Corollary of Lemma 4 the skew connection $\tilde{H}^q_T$ is regular. Using this relation we have as in the proof of the preceding theorem

$\pi_T \tilde{H}^q_S = (NH^{q-1})^{-1}\tilde{T}^{q-1}(A)NH^{q-1}\pi_S = \tilde{S}^{q-1}(A)\pi_S$. Also if $X \in \operatorname{Ker}\pi_S = \tilde{T}^{q-1}(E) \otimes T(M)^* \subset S^1(\tilde{T}^{q-1}(E))$ then $\tilde{H}^q_T(X) = (T^{q-1}(A) \otimes id_{T(M)^*})(X)$ and thus by (2), $\tilde{T}^{q-1}(A) \otimes id_{T(M)^*} = NH^q|_{\operatorname{Ker}\pi_S \subset S^1(\tilde{S}^{q-1}(E))}[(NH^{q-1})^{-1} \otimes id_{T(M)^*}]$, i.e. by (7), $NH^q|_{\operatorname{Ker}\pi_S} = NH^{q-1}\tilde{S}^{q-1}(A) \otimes id_{T(M)^*}$. But then for $X \in \operatorname{Ker}\pi_S \subset S^1(\tilde{S}^{q-1}(E))$ we have again by (2), $\tilde{H}^q_S(X) = T^1(NH^{q-1})^{-1}NH^q(X) = [(NH^{q-1})^{-1} \otimes id_{T(M)^*}][NH^{q-1}\tilde{S}^{q-1}(A) \otimes id_{T(M)^*}](X)$, from where we conclude that $\tilde{H}^q_S$ is a $\tilde{S}^{q-1}(A)$-connection, As for the $\bar{H}^q_S$, consider (6) together with the reducibility condition of $\{\tilde{H}^q_S\}$ to $\{\bar{H}^q_S\}$. From the just proved result about $\tilde{H}^q_S$ we get $\bar{i}^{q-1}_S\pi_T\bar{H}^q_S = \pi_T T^1(\bar{i}^{q-1}_S)\bar{H}^q_S = \tilde{S}^{q-1}(A)\pi_S S^1(\bar{i}^{q-1}_S) = \tilde{S}^{q-1}(A)\bar{i}^{q-1}_S\pi_S = \bar{i}^{q-1}_S\bar{S}^{q-1}(A)\pi_S$, and hence $\pi_T\bar{H}^q_S = \bar{S}^{q-1}(A)\pi_S$, because $\bar{i}^{q-1}_S$ is injective. Also if $X \in \operatorname{Ker}\pi_S \subset S^1(\bar{S}^{q-1}(E))$, then $S^1(\bar{i}^{q-1}_S)(X) \in \operatorname{Ker}\pi_S \subset S^1(\tilde{S}^{q-1}(E))$ and thus by the already proved result about $\tilde{H}^q_S$ we have $T^1(\bar{i}^{q-1}_S)\bar{H}^q_S(X) = \tilde{H}^q_S S^1(\bar{i}^{q-1}_S)(X) = (\tilde{S}^{q-1}(A) \otimes id_{T(M)^*})S^1(\bar{i}^{q-1}_S)(X) = (\tilde{S}^{q-1}(A)\bar{i}^{q-1}_S \otimes id_{T(M)^*})(X) = (\bar{i}^{q-1}_S \otimes id_{T(M)^*})(\bar{S}^{q-1}(A) \otimes id_{T(M)^*})(X) = T^1(\bar{i}^{q-1}_S)(\bar{S}^{q-1}(A) \otimes id_{T(M)^*})(X)$, and this proves the last relation because of the injectivity of $T^1(\bar{i}^{q-1}_S)$.

THEOREM 6. *Under the assumptions of Theorem 5, all the relative connections $\tilde{H}^q_T$, $\bar{H}^q_T$, $\tilde{H}^q_S$, $\bar{H}^q_S$ are regular.*

PROOF. It is again evident from the Corollary of Lemma 4 that this is true for $\tilde{H}^q_T$ and $\bar{H}^q_T$. Thus we have only to prove $T^1(\tilde{S}^{q-1}(A))\tilde{H}^q_S = \tilde{H}^q_S\tilde{S}^q(A)$, and $T^1(\bar{S}^{q-1}(A))\bar{H}^q_S = \bar{H}^q_S S^1(\bar{S}^{q-1}(A))$. The first relation follows by (2) and (3) from (7) as $\tilde{H}^q_S\tilde{S}_q(A) = T^1(NH^{q-1})^{-1}NH^q\tilde{S}^q(A) = T^1((NH^{q-1})^{-1}\tilde{T}^{q-1}(A))NH^q = T^1(\tilde{S}^{q-1}(A))T^1(NH^{q-1})^{-1}NH^q = T^1(\tilde{S}^{q-1}(A))\tilde{H}^q_S$. The second relation is obtained from this, the reducibility of $\{\tilde{H}^q_S\}$ to $\{\bar{H}^q_S\}$ and (6) as $T^1(\bar{i}^{q-1}_S)\bar{H}^q_S S^1(\bar{S}^{q-1}(A)) = \tilde{H}^q_S S^1(\bar{i}^{q-1}_S\bar{S}^{q-1}(A)) = \tilde{H}^q_S S^1(\tilde{S}^{q-1}(A))S^1(\bar{i}^{q-1}_S) = T^1(\tilde{S}^{q-1}(A))\tilde{H}^q_S S^1(\bar{i}^{q-1}_S) = T^1(\tilde{S}^{q-1}(A))T^1(\bar{i}^{q-1}_S)\bar{H}^q_S = T^1(\bar{i}^{q-1}_S)T^1(\bar{S}^{q-1}(A))\bar{H}^q_S$, Q.E.D., since $T^1(\bar{i}^{q-1}_S)$ is injective.

REMARK. Restricting ourselves to the most important case of a skew connection, namely to that of a relative connection, we have seen here that 'the prolongation procedure works' only if the initial (regular) relative connection in E is 'pushed' by a (strict) connection in $T(M)$. This is due to our definition of the regularity of the sequence $\{\bar{H}^q_T\}$. If H and h were both arbitrary regular relative connections, we would still get the prolonged sequence $\{\tilde{H}^q_T\}$ reducible to $\{\bar{H}^q_T\}$, however not necessarily regular, the obstacle being essentially only with the relation (5). It seems likely that one could generalize the notion of a relative connection (most probably by developing the formalism in the category of 'all' vector bundles rather than only of those over a fixed M), and get a deeper condition for the 'initial' correlation (of the relative connection in $T(M)$ to the relative connection in E) in order to 'let the prolongation procedure work'.

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