II

Interactions of the Standard Model

A gauge theory involves two kinds of particles, those which carry 'charge' and those which 'mediate' interactions between currents by coupling directly to charge. In the former class are the fundamental fermions and nonabelian gauge bosons, whereas the latter consists solely of gauge bosons, both abelian and nonabelian. The physical nature of charge depends on the specific theory. Three such kinds of charge, called *color*, *weak isospin*, and *weak hypercharge*, appear in the Standard Model. The values of these charges are not predicted from the gauge symmetry, but must rather be determined experimentally for each particle. The strength of coupling between a gauge boson and a particle is determined by the particle's charge, e.g., the electron–photon coupling constant is $-e$, whereas the u-quark and photon couple with strength $2e/3$. Because nonabelian gauge bosons are both charge carriers and mediators, they undergo self-interactions. These produce substantial nonlinearities and make the solution of nonabelian gauge theories a formidable mathematical problem. Gauge symmetry does not generally determine particle masses. Although gauge-boson mass would seem to be at odds with the principle of gauge symmetry, the Weinberg–Salam model contains a dynamical procedure, the *Higgs mechanism*, for generating mass for both gauge bosons and fermions alike.

II–1 Quantum Electrodynamics

Historically, the first of the gauge field theories was electrodynamics. Its modern version, Quantum Electrodynamics (*QED*), is the most thoroughly verified physical theory yet constructed. *QED* represents the best introduction to the Standard Model, which both incorporates and extends it.

U(1) gauge symmetry

Consider a spin one-half, positively charged fermion represented by field ψ. The classical lagrangian which describes its electromagnetic properties is

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$$\mathcal{L}_{\rm em} = -\frac{1}{4} F^2 + \overline{\psi} \left(i \not{D} - m \right) \psi. \tag{1.1}$$

Here, the covariant derivative is $D_\mu \psi \equiv (\partial_\mu + ieA_\mu)\psi$, m and e are, respectively, the mass and electric charge for ψ, A_μ is the gauge field for electromagnetism, $F^{\mu\nu}$ is the gauge-invariant field strength (cf. Eqs. (I–5.8), (I–5.9)), and $F^2 \equiv F^{\mu\nu} F_{\mu\nu}$. This lagrangian is invariant under the local $U(1)$ transformations

$$\psi(x) \to e^{-i\alpha(x)} \psi(x) \,, \tag{1.2}$$

$$A_\mu(x) \to A_\mu(x) + e^{-1} \partial_\mu \alpha(x). \tag{1.3}$$

The associated equations of motion are the Dirac equation

$$(i \not{\partial} - m - e \not{A} \,)\psi = 0, \tag{1.4}$$

and the Maxwell equation

$$\partial_\mu F^{\mu\nu} = e \overline{\psi} \gamma^\nu \psi. \tag{1.5}$$

It is worthwhile to consider in more detail the important subject of $U(1)$ gauge invariance, addressing both its extent and its limitations.

(i) *Universality of electric charge*: The deflection of atomic and molecular beams by electric fields establishes that the fractional difference in the magnitude of electron and proton charge is no larger than $\mathcal{O}(10^{-20})$. Likewise, there is no evidence of any difference between the electric charges of the leptons e, μ, τ. Whatever the source of this charge universality may be, it is not the $U(1)$ invariance of electrodynamics. For example assume that in addition to ψ, there exists a second charged fermion field ψ' with charge parameter βe. It is easy to see that gauge invariance alone does not imply $\beta = 1$. The electromagnetic lagrangian for the extended system is

$$\mathcal{L}_{\rm em} = -\frac{1}{4} F^2 + \overline{\psi} \left(i \not{D} - m \right) \psi + \overline{\psi}' \left(i \not{D}' - m' \right) \psi', \tag{1.6}$$

where $D'_\mu \psi' \equiv (\partial_\mu + i\beta e A_\mu(x))\psi'$. The above lagrangian is invariant under the extended set of gauge transformations

$$\begin{aligned} &\psi(x) \to e^{-i\alpha(x)} \psi(x), \qquad \psi'(x) \to e^{-i\beta\alpha(x)} \psi'(x), \\ &A_\mu(x) \to A_\mu(x) + e^{-1} \partial_\mu \alpha(x). \end{aligned} \tag{1.7}$$

This demonstration of gauge invariance is valid for arbitrary β, and thus says nothing about its value. The $U(1)$ symmetry is compatible with, but does not explain, the observed equality between the magnitudes of the electron and proton charges. We shall return to the issue of charge quantization in Sect. II–3 when we consider how weak hypercharge is assigned in the Weinberg–Salam model.

(ii) *A candidate quantum lagrangian*: The quantum version of $\mathcal{L}_{\rm em}$ is in fact the most general Lorentz-invariant, hermitian, and renormalizable lagrangian which is $U(1)$ invariant. Consider the seemingly more general structure

$$\mathcal{L}_{\rm gen} = -\frac{1}{4} Z F^2 + i Z_R \overline{\psi}_R \not{D} \psi_R + i Z_L \overline{\psi}_L \not{D} \psi_L - M \overline{\psi}_R \psi_L - M^* \overline{\psi}_L \psi_R, \quad (1.8)$$

where $Z,\ Z_{R,L}$ are constants, $\not{D}$ is the covariant derivative of Eq. (1.1), and M can be complex-valued. This lagrangian not only apparently differs from $\mathcal{L}_{\rm em}$, but seemingly is CP-violating due to the complex mass term. However, under the rescalings

$$A'_\mu = Z^{1/2} A_\mu, \qquad e' = Z^{-1/2} e, \qquad \psi'_{R,L} = Z_{R,L}^{1/2} \psi_{R,L}, \quad (1.9)$$

we obtain

$$\mathcal{L}'_{\rm gen} = -\frac{1}{4} F'^2 + i \overline{\psi}' \not{D}' \psi' - M' \overline{\psi}'_R \psi'_L - M'^* \overline{\psi}'_L \psi'_R, \quad (1.10)$$

where $M' = (Z_R Z_L)^{-1/2} M$. A subsequent global chiral change of variable

$$\psi''_{L,R} = e^{-i\alpha\gamma_5} \psi'_{L,R} \qquad (\alpha = \text{constant}) \quad (1.11)$$

does not affect the covariant derivative term but modifies the mass terms,

$$\mathcal{L}''_{\rm gen} = -\frac{1}{4} F'^2 + i \overline{\psi}'' \not{D}' \psi'' - M' e^{2i\alpha} \overline{\psi}''_R \psi''_L - (M' e^{2i\alpha})^* \overline{\psi}''_L \psi''_R. \quad (1.12)$$

Choosing the parameter α so that $\text{Im}\,(M' e^{2i\alpha}) = 0$ and defining $m \equiv \text{Re}(M' e^{2i\alpha})$, we see that $\mathcal{L}''_{\rm gen}$ reduces to $\mathcal{L}_{\rm em}$ which appears in Eq. (1.1).

(iii) *Renormalizability and $U(1)$*: Renormalizability plays a role in the preceding discussion because $U(1)$ symmetry by itself would admit a larger set of interaction terms. In principle, $U(1)$ invariant terms like $\overline{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$, $\overline{\psi}\psi F^{\mu\nu} F_{\mu\nu}$, $\overline{\psi}\gamma^\mu\gamma^\nu \gamma^\alpha\gamma^\beta\psi F_{\mu\nu} F_{\alpha\beta}$, etc. could appear in the *QED* lagrangian. However, they do not because the condition of renormalizability admits only those contributions which have dimension $d \le 4$. As discussed in App. C–3, the canonical dimension of boson and fermion fields is $d = 1,\ 3/2$ respectively, and each derivative adds a unit of dimension. Accordingly, the above candidate operators have $d = 5,\ 7,\ 7$ and thus are ruled out. There remains an operator, $F_{\mu\nu}\tilde{F}^{\mu\nu}$, which *is* gauge-invariant and has dimension 4.[1] A noteworthy aspect of this quantity is that, unlike the other operators encountered thus far, it is odd under CP. This follows from writing it as $-4\mathbf{E}\cdot\mathbf{B}$ and realizing that under CP, $\mathbf{E} \to \mathbf{E}$ and $\mathbf{B} \to -\mathbf{B}$. However, a simple exercise shows that we can identify this operator as a four-divergence $F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu$, where $K^\mu \equiv 2\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta$. Thus, a contribution proportional to $F_{\mu\nu}\tilde{F}^{\mu\nu}$

[1] We define the tensor $\tilde{F}^{\mu\nu}$ which is dual to $F^{\mu\nu}$ as $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2$.

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can be of no physical consequence. Upon integration over spacetime, it becomes a surface term evaluated at infinity. There is nothing in the structure of *QED* which would cause such a surface term to be anything but zero.

QED to one loop

The perturbative expansion of *QED* is carried out about the free field limit, and is interpreted in terms of Feynman diagrams. Two distinct phenomena are involved, scattering and renormalization. The latter encompasses both an additive mass shift for the fermion (but not for the photon) and rescalings of the charge parameter and of the quantum fields. To carry out the calculational program requires a quantum lagrangian $\mathcal{L}_{QED}$ to establish the Feynman rules, a regularization procedure to interpret divergent loop integrals, and a renormalization scheme.

One can develop *QED* using either canonical or path-integral methods. In either case a proper treatment necessitates modification of the classical lagrangian. As we have seen, the $U(1)$ gauge symmetry implies a certain freedom in defining the $A^\mu(x)$ field. Regardless of the quantization procedure adopted, this freedom can cause problems. For canonical quantization, the procedure of selecting a complete set of coordinates and their conjugate momenta is upset by the freedom to gauge transform away a coordinate at any given time. For path integration, the integration over gauge copies of specific field configurations gives rise to specious divergences (cf. App. A–6). In either case, superfluous gauge degrees of freedom can be eliminated by introducing an auxiliary condition which constrains the gauge freedom. There are a variety of ways to accomplish this. The one adopted here is to employ the following *gauge-fixed* lagrangian,

$$\mathcal{L}_{QED} = -\frac{1}{4}F^2 - \frac{1}{2\xi_0}(\partial \cdot A)^2 + \overline{\psi}\,(i\partial\!\!\!/ - e_0 A\!\!\!/ - m_0)\psi, \tag{1.13}$$

where e_0 and m_0 are, respectively, the fermion charge and mass parameters. The quantity ξ_0 is a real-valued, arbitrary constant appearing in the gauge-fixing term. This term is Lorentz-invariant but not $U(1)$-invariant. One of its effects is to make the photon propagator explicitly dependent on ξ_0. The value $\xi_0 = 1$ corresponds to *Feynman* gauge, whereas the limit $\xi_0 \to 0$ defines the *Landau* gauge.

The zero subscripts on the mass, charge, and gauge-fixing parameters denote that these *bare* quantities will be subject to renormalizations, as will the quantum fields. This process is characterized in terms of quantities Z_i and δm,

$$\begin{aligned} &\psi = Z_2^{1/2}\psi^r, && A_\mu = Z_3^{1/2}A_\mu^r\,, \\ &e_0 = Z_1 Z_2^{-1} Z_3^{-1/2} e, && m_0 = m - \delta m, \\ &\xi_0 = Z_3 \xi, \end{aligned} \tag{1.14}$$

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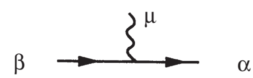
where the superscript 'r' labels renormalized fields. The renormalization constants Z_1, Z_2, and Z_3 (associated respectively with the fermion–photon vertex, the fermion wavefunction, and the photon wavefunction) and the fermion mass shift δm are chosen order by order to cancel the divergences occurring in loop integrals. For vanishing bare charge $e_0 = 0$, they reduce to $Z_{1,2,3} = 1,\ \delta m = 0$.

The Feynman rules for *QED* are:

fermion–photon vertex:

$$-i\, e_0\, (\gamma_\mu)_{\alpha\beta} \tag{1.15}$$

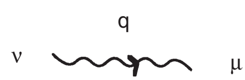


fermion propagator $i\, S_{\alpha\beta}(p)$:

$$\frac{i\, (\not{p} + m_0)_{\alpha\beta}}{p^2 - m_0^2 + i\epsilon} \tag{1.16}$$

photon propagator $i\, D^{\mu\nu}(q)$:

$$\frac{i}{q^2 + i\epsilon}\left(-g^{\mu\nu} + (1 - \xi_0)\frac{q^\mu q^\nu}{q^2 + i\epsilon}\right) \tag{1.17}$$



In the above ϵ is an infinitesimal positive number.

The remainder of this section is devoted to a discussion of the one-loop radiative correction experienced by the photon propagator.[2] Throughout, we shall work in Feynman gauge.

Let us define a *proper* or *one-particle irreducible* ($1PI$) Feynman graph such that there is no point at which only a single internal line separates one part of the diagram from another part. The proper contributions to photon and to fermion propagators are called *self-energies*. The point of finding the photon self-energy is that the full propagator $i\,D'_{\mu\nu}$ can be constructed via iteration as in Fig. II–1. Performing a summation over self-energies, we obtain

Fig. II–1 The full photon propagator as an iteration.

[2] We shall leave calculation of the fermion self-energy to Prob. II–3 and analysis of the photon-fermion vertex to Sect. V–1.

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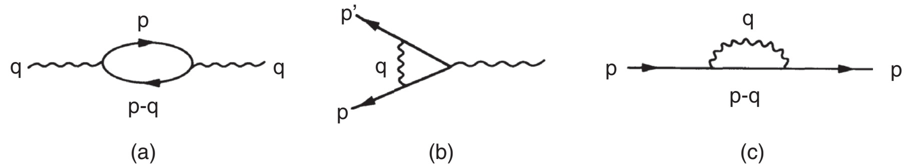

Fig. II–2 One-loop corrections to (a) photon propagator, (b) fermion-photon vertex, and (c) fermion propagator.

$$\begin{aligned} iD'_{\mu\nu} &= iD_{\mu\nu} + iD_{\mu\alpha}(i\Pi^{\alpha\beta})\,iD_{\beta\nu} + \cdots \\ &= \frac{-i}{q^2}\left[\frac{1}{1+\Pi(q)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \xi_0 \frac{q_\mu q_\nu}{q^2}\right], \end{aligned} \tag{1.18}$$

where the proper contribution

$$i\Pi^{\alpha\beta}(q) = (q^\alpha q^\beta - q^2 g^{\alpha\beta})i\Pi(q) \tag{1.19}$$

is called the *vacuum polarization tensor*. It is depicted in Fig. II–2(a) (along with corrections to the photon-fermion vertex and fermion propagator in Figs. II–2(b)–(c)), and is given to lowest order by

$$i\Pi^{\alpha\beta}(q) = -(-ie_0)^2 \int \frac{d^4p}{(2\pi)^4}\ \mathrm{Tr}\left[\gamma^\alpha \frac{i}{\not p - m + i\epsilon}\gamma^\beta \frac{i}{\not p - \not q - m + i\epsilon}\right]. \tag{1.20}$$

This integral is quadratically divergent due to singular high-momentum behavior. To interpret it and other divergent integrals, we shall employ the method of *dimensional regularization* [BoG 72, 'tHV 72, Le 75].

Accordingly, we consider $\Pi^{\alpha\beta}(q)$ as the four-dimensional limit of a function defined in d spacetime dimensions. Various mathematical operations, such as summing over Lorentz indices or evaluating loop integrals, are carried out in d dimensions and the results are continued back to $d = 4$, generally expressed as an expansion in the variable[3]

$$\epsilon \equiv \frac{4-d}{2}. \tag{1.21a}$$

Formulae relevant to this procedure are collected in App. C–5. For all theories described in this book, we shall define the process of dimensional regularization such that all parameters of the theory (such as e^2) retain the dimensionality they

[3] We shall follow standard convention is using the symbol ϵ for both the infinitesimal employed in Feynman integrals and the variable for continuation away from the dimension of physical spacetime.

have for $d = 4$. In order to maintain correct units while dimensionally regularizing Feynman integrals, we modify the integration measure over momentum to

$$\int \frac{d^4 p}{(2\pi)^4} \rightarrow \mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d}. \tag{1.21b}$$

The parameter μ is an arbitrary auxiliary quantity having the dimension of a mass. It appears in the intermediate parts of a calculation, but cannot ultimately influence relations between physical observables. Indeed, there exist in the literature a number of variations of the extension to $d \neq 4$ dimensions. These are able to yield consistent results because one is ultimately interested in only the physical limit of $d = 4$. Let us now return to the photon self-energy calculation to see how the dimensional regularization is implemented.

The self-energy of Eq. (1.20), now expressed as an integral in d dimensions, is

$$\Pi^{\alpha\beta}(q) = 4ie_0^2\mu^{2\epsilon} \int \frac{d^d p}{(2\pi)^d} \frac{p^\alpha(p-q)^\beta + p^\beta(p-q)^\alpha + g^{\alpha\beta}(m^2 - p\cdot(p-q))}{[p^2 - m^2 + i\epsilon][(p-q)^2 - m^2 + i\epsilon]}, \tag{1.22}$$

where we retain the same notation $\Pi^{\alpha\beta}(q)$ as for $d = 4$ and we have already computed the trace. Upon introducing the Feynman parameterization, Dirac relations, and integral identities of App. C–5, we can perform the integration over momentum to obtain

$$\Pi^{\alpha\beta}(q) = (q^\alpha q^\beta - q^2 g^{\alpha\beta}) \frac{e_0^2}{2\pi^2} \frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}} \mu^\epsilon \int_0^1 dx \frac{x(1-x)}{(m^2 - q^2 x(1-x))^\epsilon}. \tag{1.23}$$

We next expand $\Pi^{\alpha\beta}(q)$ in powers of ϵ and then pass to the limit $\epsilon \to 0$ of physical spacetime. In doing so, we use the familiar

$$a^\epsilon = e^{\ln a^\epsilon} = e^{\epsilon \ln a} = 1 + \epsilon \ln\, a + \cdots, \tag{1.24}$$

and take note of the combination

$$\frac{\Gamma(\epsilon)}{(4\pi)^{-\epsilon}} = \frac{1}{\epsilon} + \ln(4\pi) - \gamma + \mathcal{O}(\epsilon), \tag{1.25}$$

where $\gamma = 0.57221\ldots$ is the Euler constant. The presence of ϵ^{-1} makes it necessary to expand *all* the other ϵ-dependent factors in Eq. (1.23) and to take care in collecting quantities to a given order of ϵ. To order e^2, the vacuum polarization in Feynman gauge is then found to be

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$$\Pi(q) = \frac{e_0^2}{12\pi^2}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma \right.$$
$$\left. - 6\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2 - q^2 x(1-x)}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]$$
$$= \frac{e_0^2}{12\pi^2}\begin{cases} \dfrac{1}{\epsilon} + \ln(4\pi) - \gamma + \dfrac{5}{3} - \ln\dfrac{-q^2}{\mu^2} + \cdots & (|q^2| \gg m^2), \\ \dfrac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln\dfrac{m^2}{\mu^2} + \dfrac{q^2}{5m^2} + \cdots & (m^2 \gg |q^2|). \end{cases} \quad (1.26)$$

The above expression is an example of the general property in dimensional regularization that divergences from loop integrals take the form of poles in ϵ. These poles are absorbed by judiciously choosing the renormalization constants. Renormalization constants can also have finite parts whose specification depends on the particular renormalization scheme employed. One generally adopts a scheme which is tailored to facilitate comparison of theory with some set of physical amplitudes. In the *minimal subtraction* (MS) renormalization, the Z_i subtract off only the ϵ-poles, and thus have the very simple form,

$$Z_i^{(\mathrm{MS})} - 1 = \sum_{n=1}^{\infty} \frac{c_{i,n}}{\epsilon^n} \qquad (i = 1, 2, 3). \qquad (1.27)$$

Because the $\{Z_i^{(\mathrm{MS})} - 1\}$ have no finite parts, they are sensitive only to the ultraviolet behavior of the loop integrals, and the $c_{i,n}$ are independent of mass. The simple appearance of the MS scheme is somewhat deceptive since further (finite) renormalizations are required if the mass and coupling parameters of the theory are to be asociated with physical masses and couplings. A related renormalization scheme is the *modified minimal subtraction* ($\overline{\mathrm{MS}}$) in which renormalization constants are chosen to subtract off not only the ϵ-poles but also the omnipresent term $\ln(4\pi) - \gamma$ of Eq. (1.25). Minimal subtraction schemes are typically used in *QCD* where, due to the confinement phenomenon (cf. Sect. II–2), there is no renormalization scale that could naturally be associated with the mass of a freely propagating quark. Yet another approach is the *on-shell* (o-s) renormalization, where the renormalized mass and coupling parameters of the theory are arranged to coincide with their physical counterparts.

On-shell renormalization of the electric charge

The renormalization scale for electric charge is set by experimental determinations typically involving solid-state devices like Josephson junctions. These refer to probes of the electromagnetic vertex $-e\Gamma_\nu(p_2, p_1)$ of Fig. II–2(b) with on-shell electrons ($p_2^2 = p_1^2 = m_e{}^2$) and with $q^2 = (p_1 - p_2)^2 \simeq 0$. The value of the

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electromagnetic fine-structure constant $\alpha \equiv e^2/4\pi$ obtained under such conditions is given in rationalized units by

$$\alpha^{-1} = 137.035999074(44). \tag{1.28}$$

To interpret this in the context of the theoretical analysis performed thus far, recall from Eq. (1.18) how the photon propagator is modified by radiative corrections,

$$ie_0^2 D_{\mu\nu} = -\frac{i}{q^2} e_0^2\, g_{\mu\nu} \;\to\; ie^2 D'_{\mu\nu} = -\frac{i}{q^2}\frac{e_0^2}{1+\Pi(q)} g_{\mu\nu}. \tag{1.29}$$

We display only the $g_{\mu\nu}$ piece since, in view of current conservation, only it can contribute to the full amplitude upon coupling the propagator to electromagnetic vertices. The above suggests that we associate the physical, renormalized charge e with the bare charge parameter e_0 by

$$e^2 = \frac{e_0^2}{1+\Pi(0)} \simeq e_0^2[1-\Pi(0)]. \tag{1.30}$$

In this on-shell renormalization prescription, the $g_{\mu\nu}$ part of the photon propagator $iD'_{\mu\nu}(q)$ is seen to assume its unrenormalized form in the physical limit $q^2 \to 0$. The appellation 'on-shell' means that the physical kinematic point $q^2 = 0$ is used to implement the renormalization condition, and, by absorbing the singular vacuum polarization in the electric charge, one ensures that the photon has zero mass. Likewise, in the on-shell renormalization approach fermion propagators have poles at their physical masses.

Next, we show how to infer the form of the renormalization constant $Z_3^{(\text{o-s})}$ in the on-shell scheme. There is a relation, called the Ward identity, that implies $Z_1 = Z_2$ as a consequence of the gauge symmetry of the theory. From Eq. (1.14), this gives

$$e = \sqrt{Z_3^{(\text{o-s})}}\, e_0. \tag{1.31}$$

Use of the relation $e^2 \equiv Z_3^{(\text{o-s})} e_0^2$ then specifies the on-shell renormalization constant to be

$$Z_3^{(\text{o-s})} = 1 - \frac{e^2}{12\pi^2}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln\left(\frac{m^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{1.32}$$

One can similarly absorb the ϵ-pole in either the MS or $\overline{\text{MS}}$ schemes by adopting

$$Z_3^{(\text{MS})} = 1 - \frac{e^2}{12\pi^2}\frac{1}{\epsilon} + \mathcal{O}(e^4),$$
$$Z_3^{(\overline{\text{MS}})} = 1 - \frac{e^2}{12\pi^2}\left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right) + \mathcal{O}(e^4). \tag{1.33}$$

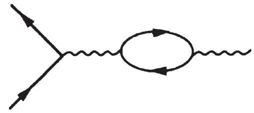

Fig. II–3 Virtual pair production in the vicinity of a charge.

Eqs. (1.32), (1.33) display how the various renormalization constants differ by finite amounts. The ϵ-poles in the fermion self-energy and the fermion-photon vertex can be dealt with in the same manner and we find, e.g., in MS renormalization (cf. Prob. II–3 and Sect. V–1),

$$Z_1^{(\text{MS})} = Z_2^{(\text{MS})} = 1 - \frac{e^2}{16\pi^2}\frac{1}{\epsilon} + \mathcal{O}(e^4), \tag{1.34}$$

$$\delta m^{(\text{MS})} = \frac{3e^2}{16\pi^2}\, m\, \frac{1}{\epsilon} + \mathcal{O}(e^4). \tag{1.35}$$

Electric charge as a running coupling constant

The concept of electric charge as a 'running' coupling constant is motivated by the following consideration. In the perturbative Feynman expansion for a given theory, the hope is that corrections to the lowest-order amplitudes will be small. However, potentially large corrections of the form $\ln q^2/q_0^2$ can arise if the theory is renormalized at scale q_0^2 but then applied at a very different scale q^2. It is convenient to deal with this problem by absorbing such logarithms into scale-dependent or 'running' renormalized coupling constants and masses.

To see why scale-dependent charge is not an unreasonable concept, consider the vacuum polarization process of Fig. II–3, which depicts virtual production of a fermion of charge $Q_i e$ together with its antiparticle near a charge source. Due to the source, each such vacuum fluctuation is polarized, and thus the source becomes screened. All charged fermion species contribute to the screening, and the larger the mass of the virtual pair, the closer they lie to the source. The effect is somewhat akin to concentric onion skins, with each virtual pair forming a layer, resulting in an effectively scale-dependent source charge.

Let us seek a method for specifying a running fine structure constant $\alpha(q)$ for nonzero momentum transfers, with $\alpha(0)$ to be identified with the α of Eq. (1.28). The interpretation of $e_0^2/(1 + \text{Re}\ \Pi(q))$ as a running charge is appealing since it would maintain the simple $-i/q^2$ structure of the lowest-order photon exchange amplitude. The fact that $\Pi(q)$ is divergent (see Eq. (1.26)) can be circumvented by subtracting off its value at $q^2 = 0$ to define a finite quantity $\overline{\Pi}(q) \equiv \Pi(q) - \Pi(0)$ and defining

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$$e^2(q) \equiv \frac{e^2}{1 + \mathrm{Re}\,\overline{\Pi}(q)} \simeq e^2[1 - \mathrm{Re}\,\overline{\Pi}(q)], \tag{1.36}$$

so that $\alpha(q) = e^2(q)/4\pi$. It is not difficult to deduce the behavior of $\overline{\Pi}(q)$ from the integral representation of Eq. (1.26), and we find

$$\overline{\Pi}(q) = \frac{\alpha}{3\pi} \begin{cases} \frac{5}{3} - \ln\frac{|q|^2}{m^2} + i\pi\theta(q^2) + \cdots & (|q^2| \gg m^2), \\ \frac{q^2}{5m^2} + \cdots & (m^2 \gg q^2). \end{cases} \tag{1.37}$$

Observe that the arbitrary energy scale μ is absent from $\overline{\Pi}(q)$, as would be expected since $\overline{\Pi}(q)$ is a physically measurable quantity.

The above formulae correspond to the loop correction of one fermion of mass m. Generally, loops from *all* available fermions must be included, although contributions of heavy ($m^2 \gg q^2$) fermions are seen to be suppressed. Important modern applications of the Standard Model engender phenomena at scales provided by the gauge-boson masses M_W, M_Z. To obtain an estimate for $\alpha(M_Z^2)$, we can apply Eq. (1.37) to find

$$\alpha^{-1}(M_Z^2) = \alpha^{-1}\left[1 - \frac{\alpha}{3\pi}\sum_i Q_i^2\left(\ln\frac{M_Z^2}{m_i^2} - \frac{5}{3}\right) + \cdots\right]. \tag{1.38}$$

If a sum over quark-loops (each being accompanied by the color factor $N_c = 3$) and lepton-loops is performed, then the mass values in Tables I–2, I–3 yield the approximate determination $\alpha^{-1}(M_Z^2) \simeq 130$. The main uncertainty in this approach arises from quarks. It is possible to perform a more accurate evaluation of $\alpha(M_Z^2)$ (cf. Sect. XVI–6) which avoids this difficulty.

Let us return to the question of how to define a momentum-dependent coupling. To emphasize the fact that a 'running fine-structure constant' is after all a matter of definition, let us consider a somewhat different derivation (and definition) of $\alpha(q^2)$. One is able to renormalize the electric charge in a mass-independent scheme [We 73] by calculating renormalization constants with $m = 0$. If we return to the vacuum polarization diagram, but with $m = 0$, we find

$$\begin{aligned}\Pi(q^2) &= \frac{e_0^2}{12\pi^2}\left(\frac{\mu^2}{-q^2}\right)^{\epsilon}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma + \frac{5}{3} + \mathcal{O}(\epsilon)\right] \\ &= \frac{e_0^2}{12\pi^2}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma + \frac{5}{3} - \ln\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]. \end{aligned} \tag{1.39}$$

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In order to apply the renormalization program, we must specify the value of the coupling at some renormalization point,[4] which we choose to be $q^2 = -\mu_R^2$, identifying

$$e^2(\mu_R^2) = \frac{e_0^2}{1 + \Pi(q^2)\big|_{-q^2=\mu_R^2}} \simeq e_0^2 \left[1 - \frac{e_0^2}{12\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{\mu_R^2}{\mu^2} + \cdots\right)\right]. \tag{1.40}$$

However, if we had chosen a different renormalization point $\mu_R^{2\,\prime}$, we would have obtained a different value,

$$e^2(\mu_R^{2\,\prime}) = e^2(\mu_R^2) + \frac{e_0^4}{12\pi^2}\ln\frac{\mu_R^{2\,\prime}}{\mu_R^2}. \tag{1.41}$$

The functional dependence of the charge on the renormalization scale is embodied in the so-called *beta* function of electrodynamics [GeL 54],

$$\beta_{QED}(e) \equiv \mu_R\frac{\partial e}{\partial\mu_R} = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5). \tag{1.42}$$

It can be shown [Po 74] that the leading and next-to-leading terms in a perturbative expansion of β_{QED} are independent of both renormalization and gauge choices.

The quantity $e^2(\mu_R^2)$ defined by integrating the beta function,

$$\frac{de}{\beta_{QED}(e)} = \frac{d\mu_R}{\mu_R}, \tag{1.43}$$

is not exactly the same quantity as the running coupling constant defined in Eq. (1.36), differing by a (small) finite renormalization. For example, the electron contribution to the running coupling in the range $m_e^2 \le \mu_R^2 \le M_Z^2$ is

$$\alpha^{-1}(\mu_R^2)\big|_{\mu_R^2=m_e^2} - \alpha^{-1}(\mu_R^2)\big|_{\mu_R^2=M_Z^2} = \frac{1}{3\pi}\ln\frac{M_Z^2}{m_e^2}, \tag{1.44}$$

which contains the dominant logarithmic dependence, but differs from Eq. (1.38) by a small additive term. However, complete calculations of *all* corrections to physical observables using the two schemes will yield the same answer. Since the running coupling constant is but a bookkeeping device, one's choice is a matter of taste or of convenience. Regardless of the specific definition employed for $\alpha(q^2)$, we see that as the energy scale is increased (or as distance is decreased), the running electric charge grows. This is anticipated from the screening of a test charge due to vacuum polarization (recall our explanation of Fig. II–3). As the momentum transfer of a photon probe is increased, the screening is penetrated and the effective charge increases.

[4] Note that the renormalization point μ_R and the scale factor μ in dimensional regularization need not be identical. They are sometimes confused in the literature, and hence we use a different notation for the two quantities.

The use of a mass-independent scheme is convenient for identifying the high-energy scaling behavior of gauge theories. One useful feature is in the calculation of the one-loop beta function. Dimensional analysis requires that the one-loop charge renormalization be of the form,

$$g = g_0 \left[1 - g_0^2 b \left(\frac{\mu^2}{-q^2} \right)^{\epsilon} \left(\frac{1}{2\epsilon} + \text{finite terms} \right) + \mathcal{O}(g_0^4) \right], \tag{1.45}$$

where g is the 'charge' associated with the gauge theory being considered. Choosing the renormalization point as $q^2 = -\mu_R^2$ and forming the beta function as in Eq. (1.42), we see that $\beta = bg^3$. This allows the beta function to be simply identified with the coefficient of ϵ^{-1} to this order.

II–2 Quantum Chromodynamics

Chromodynamics, the nonabelian gauge description of the strong interactions, contains quarks and gluons instead of electrons and photons as its basic degrees of freedom [FrG 72, Co 11]. A hallmark of Quantum Chromodynamics (*QCD*) is asymptotic freedom [GrW 73a,b, Po 73], which reveals that only in the short-distance limit can perturbative methods be legitimately employed. The necessity to employ approaches alternative to perturbation theory for long-distance processes motivates much of the analysis in this book.

SU(3) gauge symmetry

Chromodynamics is the $SU(3)$ nonabelian gauge theory of color charge. The fermions which carry color charge are the *quarks*, each with field $\psi_j^{(\alpha)}$, where $\alpha = u, d, s, \ldots$ is the flavor label and $j = 1, 2, 3$ is the color index. The gauge bosons, which also carry color, are the *gluons*, each with field $A_\mu^a,\ a = 1, \ldots, 8$.[5] Classical chromodynamics is defined by the lagrangian

$$\mathcal{L}_{\text{color}} = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \sum_\alpha \overline{\psi}_j^{(\alpha)} (i \not{D}_{jk} - m^{(\alpha)} \delta_{jk}) \psi_k^{(\alpha)}, \tag{2.1}$$

where the repeated color indices are summed over. The gauge field strength tensor is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_3 f^{abc} A_\mu^b A_\nu^c, \tag{2.2a}$$

[5] In this section, it will be particularly important to explicitly display color indices. We shall reserve indices which begin the alphabet for gluon color indices (e.g., $a, b, c = 1, \ldots, 8$), use mid-alphabetic letters for quark color indices (e.g., $j, k, l = 1, 2, 3$), and employ greek symbols for flavor indices.

where g_3 is the $SU(3)$ gauge coupling parameter, and the quark covariant derivative is

$$\mathbf{D}_\mu \psi = \left(\mathbf{I}\partial_\mu + i g_3 A_\mu^a \frac{\boldsymbol{\lambda}_a}{2} \right) \psi. \tag{2.2b}$$

The lagrangian of Eq. (2.1) is invariant under local $SU(3)$ transformations of the color degree of freedom, under which the quark and gluon fields transform as given earlier in Eqs. (I–5.11), (I–5.17). Equations of motion for the quark and gluon fields are

$$\begin{aligned} (\, i \not{D} \; - \; m^{(\alpha)} \,)\psi^{(\alpha)} \; &= 0, \\ D^\mu F^a_{\mu\nu} \; &= \; g_3 \sum_\alpha \overline{\psi}^{(\alpha)} \frac{\lambda_a}{2} \gamma_\nu \psi^{(\alpha)}. \end{aligned} \tag{2.3}$$

In its quantum version, the $g_3 \to 0$ limit of $\mathcal{L}_{\text{color}}$ describes an exceedingly simple world. There exist only free massless spin one gluons and massive spin one-half quarks. However, the full theory is quite formidable. In particular, accelerator experiments reveal a particle spectrum which bears no resemblance to that of the noninteracting theory.

The group $SU(3)$ has an infinite number of irreducible representations R. The first several are $R = \mathbf{1}, \mathbf{3}, \mathbf{3}^*, \mathbf{6}, \mathbf{6}^*\ \mathbf{8}, \mathbf{10}, \mathbf{10}^*, \ldots$, where we label an irreducible representation in terms of its dimensionality. Quarks, antiquarks, and gluons are assigned to the representations $\mathbf{3}$, $\mathbf{3}^*$, $\mathbf{8}$ respectively. We denote the group generators for representation R by $\{\mathbf{F}_a(R)\}$ $(a = 1, \ldots, 8)$. The quantities $\boldsymbol{\lambda}/2$ are group generators for the $d = 3$ *fundamental* representation, i.e., $\mathbf{F}(3) = \boldsymbol{\lambda}/2$. They have the matrix representation

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\boldsymbol{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\lambda}_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

$$\boldsymbol{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.4}$$

As generators, they obey the commutation relations

$$[\boldsymbol{\lambda}_a, \boldsymbol{\lambda}_b] = 2i f_{abc} \boldsymbol{\lambda}_c \qquad (a, b, c = 1, \ldots, 8) \tag{2.5a}$$

Table II–1. *Nonvanishing f, d coefficients.*

abc	f_{abc}	abc	d_{abc}	abc	d_{abc}
123	1	118	$1/\sqrt{3}$	355	$1/2$
147	$1/2$	146	$1/2$	366	$-1/2$
156	$-1/2$	157	$1/2$	377	$-1/2$
246	$1/2$	228	$1/\sqrt{3}$	448	$-1/2\sqrt{3}$
257	$1/2$	247	$-1/2$	558	$-1/2\sqrt{3}$
345	$1/2$	256	$1/2$	668	$-1/2\sqrt{3}$
367	$-1/2$	338	$1/\sqrt{3}$	778	$-1/2\sqrt{3}$
458	$\sqrt{3}/2$	344	$1/2$	888	$-1/\sqrt{3}$
678	$\sqrt{3}/2$				

where the f-coefficients are totally antisymmetric structure constants of $SU(3)$. There exist corresponding anticommutation relations

$$\{\boldsymbol{\lambda}_a, \boldsymbol{\lambda}_b\} = \frac{4}{3}\delta_{ab}\,\mathbf{I} + 2d_{abc}\boldsymbol{\lambda}_c \qquad (a, b, c = 1, \ldots, 8) \tag{2.5b}$$

with d-coefficients which are totally symmetric. Values for f_{abc} and d_{abc} are given in Table II–1.

Useful trace relations obeyed by the $\{\boldsymbol{\lambda}_a\}$ are

$$\mathrm{Tr}\,\boldsymbol{\lambda}_a = 0 \qquad (a = 1, \ldots, 8) \tag{2.6}$$

from Eq. (2.4) and

$$\mathrm{Tr}\,\boldsymbol{\lambda}_a\boldsymbol{\lambda}_b = 2\delta_{ab} \qquad (a, b = 1, \ldots, 8) \tag{2.7}$$

from Eq. (2.5). The statement of completeness takes the form,

$$\lambda^a_{ij}\lambda^a_{kl} = -\frac{2}{3}\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk} \qquad (i, j, k, l = 1, 2, 3), \tag{2.8}$$

where $a = 1, \ldots, 8$ is summed over. Useful labels for the irreducible representations of $SU(3)$ are provided by the *Casimir invariants*. For any representation R, the quadratic Casimir invariant $C_2(R)$ is defined by squaring and summing the group generators $\{\mathbf{F}_a(R)\}$,

$$C_2(R)\mathbf{I} \equiv \sum_{a=1}^{8} \mathbf{F}_a^2(R). \tag{2.9}$$

There is also a third-order Casimir invariant,

$$C_3(R)\mathbf{I} \equiv \sum_{a,b,c=1}^{8} d_{abc}\mathbf{F}_a(R)\mathbf{F}_b(R)\mathbf{F}_c(R). \tag{2.10}$$

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The quark and antiquark states form the bases for the smallest nontrivial irreducible representations of $SU(3)$. It is possible to use products of them, say p factors of quarks and q factors of antiquarks, to construct all other irreducible tensors in $SU(3)$. Each irreducible representation R is then characterized by the pair (p, q). For example, we have the correspondences $\mathbf{1} \sim (0,0)$, $\mathbf{3} \sim (1,0)$, $\mathbf{3}^* \sim (0,1)$, $\mathbf{8} \sim (1,1)$, $\mathbf{10} \sim (3,0)$, etc. The (p, q) labeling scheme provides useful expressions for the dimension of a representation,

$$d(p,q) = (p+1)(q+1)(p+q+2)/2, \tag{2.11}$$

and of the two Casimir invariants,

$$\begin{aligned} C_2(p,q) &= (3p + 3q + p^2 + pq + q^2)/3, \\ C_3(p,q) &= (p-q)(2p+q+3)(2q+p+3)/18. \end{aligned} \tag{2.12}$$

From Eq. (2.12) we find $C_2(\mathbf{3}) = C_2(\mathbf{3}^*) = 4/3$ for the quark and antiquark representations. Equivalently, upon setting $j = k$ and summing in Eq. (2.8) we obtain

$$\lambda^a_{ij}\lambda^a_{jl} = \frac{16}{3}\delta_{il} = 4C_2(\mathbf{3})\delta_{il}. \tag{2.13}$$

Generators for the $d = 8$ *regular* (or *adjoint*) representation are determined from the structure constants themselves,

$$(F^a(\mathbf{8}))_{bc} = -if_{abc} \qquad (a, b, c = 1, \ldots, 8). \tag{2.14}$$

It follows directly from Eq. (2.14) and from using Eq. (2.12) to compute $C_2(\mathbf{8}) = 3$ that

$$f_{acd}f_{bcd} = C_2(\mathbf{8})\,\delta_{ab} = 3\,\delta_{ab}. \tag{2.15}$$

This result, in turn, enables us to determine

$$f_{abc}\boldsymbol{\lambda}_b\boldsymbol{\lambda}_c = \frac{1}{2}f_{abc}[\boldsymbol{\lambda}_b, \boldsymbol{\lambda}_c] = if_{abc}f_{bcd}\boldsymbol{\lambda}_d = iC_2(\mathbf{8})\boldsymbol{\lambda}_a. \tag{2.16}$$

As a final example involving $SU(3)$, we evaluate the quantity

$$\begin{aligned} \boldsymbol{\lambda}^b\boldsymbol{\lambda}^a\boldsymbol{\lambda}^b &= \frac{1}{2}\left(\boldsymbol{\lambda}^b[\boldsymbol{\lambda}^a, \boldsymbol{\lambda}^b] - [\boldsymbol{\lambda}^a, \boldsymbol{\lambda}^b]\boldsymbol{\lambda}^b + \boldsymbol{\lambda}^b\boldsymbol{\lambda}^b\boldsymbol{\lambda}^a + \boldsymbol{\lambda}^a\boldsymbol{\lambda}^b\boldsymbol{\lambda}^b\right) \\ &= 4C_2(\mathbf{3})\boldsymbol{\lambda}^a + if_{abc}[\boldsymbol{\lambda}^b, \boldsymbol{\lambda}^c] = 4\left(C_2(\mathbf{3}) - \frac{1}{2}C_2(\mathbf{8})\right)\boldsymbol{\lambda}^a. \end{aligned} \tag{2.17}$$

Shortly, we shall see how such combinations of color factors arise in various radiative corrections.

Including only gauge-invariant and renormalizable terms, we can write the most general form for a chromodynamic lagrangian as

$$\mathcal{L}_{\text{gen}} = -\frac{1}{4} Z F_a^{\mu\nu} F_{\mu\nu}^a + \overline{\psi}_L^\alpha Z_L^{\alpha\beta} i \not{D}\, \psi_L^\beta + \overline{\psi}_R^\alpha Z_R^{\alpha\beta} i \not{D}\, \psi_R^\beta - \overline{\psi}_L^\alpha M^{\alpha\beta} \psi_R^\beta$$
$$- \overline{\psi}_R^\alpha M^{\dagger\alpha\beta} \psi_L^\beta + \frac{g_3^2}{64\pi^2} \theta \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu}^a F_{\lambda\sigma}^a, \tag{2.18}$$

where the flavor matrices $Z_{L,R}$ are hermitian, color and flavor indices are as before, except that for simplicity we suppress quark color notation. The final contribution to Eq. (2.18) is called the θ-*term*. We can reduce $\mathcal{L}_{\text{gen}}$ to the form of $\mathcal{L}_{\text{color}}$ by first rescaling,

$$A_\mu^{\prime a} = Z^{1/2} A_\mu^a, \qquad g_3' = Z^{-1/2} g_3, \tag{2.19}$$

and then diagonalizing and rescaling with respect to quark flavors,

$$\psi'_{L,R} = U_{L,R} \psi_{L,R}, \quad U_{L,R} Z_{L,R} U_{L,R}^\dagger = \Lambda_{L,R}, \quad \psi''_{L,R} = \Lambda_{L,R}^{1/2} \psi', \tag{2.20}$$

where $\Lambda_{L,R}$ are diagonal. Finally we diagonalize the mass terms

$$\mathcal{L}_{\text{mass}} = -\overline{\psi}_L^{\prime\prime\alpha} M^{\prime\alpha\beta} \psi_R^{\prime\prime\beta} - \overline{\psi}_R^{\prime\prime\alpha} M^{\prime\dagger\alpha\beta} \psi_L^{\prime\prime\beta}, \tag{2.21}$$

where $M' = \Lambda_L^{-1/2} U_L M U_R^\dagger \Lambda_R^{-1/2}$, by means of yet another set of unitary transformations on the quark fields. Aside from the θ-term, this results in the canonical expression for $\mathcal{L}_{\text{color}}$ of Eq. (2.1).

We shall demonstrate later in Sect. IX–4 that the above quark mass diagonalization procedure induces a modification in the θ-parameter,

$$\theta \to \overline{\theta} = \theta + \arg\det M'. \tag{2.22}$$

This does not imply $\overline{\theta} = 0$ because both θ and the original quark mass matrices are arbitrary from the viewpoint of renormalizability and $SU(3)$ gauge invariance. In fact, the θ-term cannot be ruled out by any of the tenets which underlie the Standard Model. Moreover, although the θ-term can be expressed as a four-divergence

$$\mathcal{L}_\theta = \frac{g_3^2}{32\pi^2} \overline{\theta}\, \partial_\mu K^\mu, \tag{2.23}$$
$$K^\mu = \epsilon^{\mu\nu\lambda\sigma} A_\nu^a \left(F_{\lambda\sigma}^a + \frac{g_3}{3} f_{abc} A_\lambda^b A_\sigma^c \right), \tag{2.24}$$

analysis demonstrates that K^μ is a singular operator and that its divergence cannot be summarily discarded as was done in electrodynamics. This is a curious situation because the θ-term is CP-violating. Thus, one is faced with the specter of large CP-violating signals in the strong interactions. Yet such effects are not observed.

Indeed, it has been estimated that the θ-term generates a nonzero value for the neutron electric dipole moment $d_e(n) \simeq 5 \times 10^{-16}\,\overline{\theta}$ e-cm, but to date no signal has been observed experimentally, $d_e(n) < 2.9 \times 10^{-26}$ e-cm at C.L. 90% [RPP 12]. This provides the upper bound $\overline{\theta} < 5.8 \times 10^{-11}$. Perhaps Nature has dictated $\overline{\theta} \equiv 0$, albeit for reasons not yet understood.

QCD to one loop

To develop Feynman rules for *QCD*, we must first obtain an effective lagrangian which properly addresses the issue of $SU(3)$ gauge freedom. For the $U(1)$ gauge invariance of *QED*, this was accomplished by adding a gauge-fixing term to the classical lagrangian. The situation for $SU(3)$ is analogous, but somewhat more complicated due to its nonabelian structure. If we continue to use a Lorentz-invariant gauge-fixing procedure, the effective *QCD* lagrangian (for simplicity, consider just one quark flavor) can be expressed as

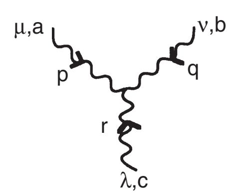
$$\begin{aligned}\mathcal{L}_{QCD} = &-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}_j(i\not{D} - m_0\mathbf{I})_{jk}\psi_k - \frac{1}{2\xi_0}(\partial_\mu A^\mu_a)^2 \\ &+ \partial_\mu\overline{c}_a\partial^\mu c_a + g_{3,0}f_{abe}A^\mu_a(\partial_\mu\overline{c}_b)c_e. \end{aligned} \tag{2.25}$$

Bare quantities carry the subscript '0' and the field strengths and covariant derivative are defined as in Eqs. (2.2a), (2.2b). The quantities $\{c_a(x)\}$ $(a = 1, \ldots, 8)$ are called *ghost fields*. As explained in App. A–5, they are anticommuting c-number quantities (i.e., *Grassmann variables*) which couple only to gluons. Ghosts occur only within loops, and never appear as asymptotic states. Each ghost-field loop contribution must be accompanied by an extra minus sign, analogous to that of a fermion–antifermion loop. Their presence is a consequence of the Lorentz-invariant gauge-fixing procedure. In alternative schemes such as axial or temporal gauge, ghost fields do not appear, but compensating unphysical singularities occur in Feynman integrals instead.

The Feynman rules for *QCD* are

three-gluon vertex:

$$-g_{3,0}f_{abc}[g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-r)_\mu + g_{\lambda\mu}(r-p)_\nu]$$



(2.26)

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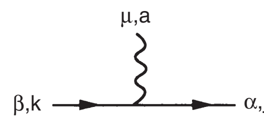
quark–gluon vertex:

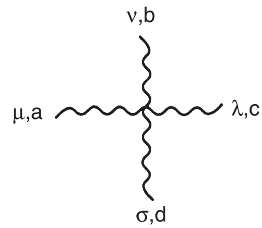
$$-ig_{3,0}(\gamma_\mu)_{\alpha\beta}\left(\frac{\lambda^a}{2}\right)_{jk}$$



(2.27)

four-gluon vertex:

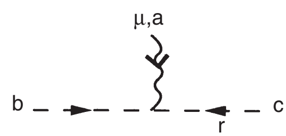
$$-ig_{3,0}^2\big[(f_{abe}f_{cde}(g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\lambda})$$
$$+f_{ace}f_{bde}(g_{\mu\nu}g_{\lambda\sigma}-g_{\mu\sigma}g_{\nu\lambda})$$
$$+f_{ade}f_{cbe}(g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\nu}g_{\lambda\sigma})\big]$$



(2.28)

ghost–gluon vertex:

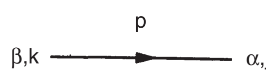
$$-g_{3,0}f_{abc}r_\mu$$



(2.29)

quark propagator $iS^{jk}_{\alpha\beta}(p)$:

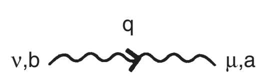
$$\frac{i\delta_{jk}\,(\not p+m_0)_{\alpha\beta}}{p^2-m_0^2+i\epsilon}$$



(2.30)

gluon propagator $iD^{ab}_{\mu\nu}(q)$:

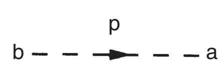
$$\frac{i\delta_{ab}}{q^2+i\epsilon}\left(-g^{\mu\nu}+(1-\xi_0)\frac{q^\mu q^\nu}{q^2+i\epsilon}\right)$$



(2.31)

ghost propagator:

$$\frac{i\delta_{ab}}{p^2+i\epsilon}$$



(2.32)

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The above rules involve a total of four distinct interaction vertices. Of these, the three-gluon and four-gluon self-vertices, and the ghost–gluon coupling have no counterpart in *QED*. That all four vertices are scaled by a single coupling strength g_3 is a consequence of gauge invariance. Also, chromodynamics exhibits a certain coupling-constant universality, called *flavor independence*, in the quark–gluon sector. All fields which transform according to a given representation of the $SU(3)$ of color have the same interaction structure, e.g., all triplets couple alike, all octets couple alike but differently from triplets, etc. Quarks are assigned solely to the color triplet representation. Thus, the quark–gluon interaction is independent of flavor.

The renormalization constants of *QCD* are

$$\begin{aligned} A^a_\mu &= Z_3^{1/2}(A^a_\mu)^r, & g_{3,0} &= Z_1 Z_3^{-3/2} g_3, \\ \psi &= Z_2^{1/2}\psi^r, & &= Z_4^{1/2} Z_3^{-1} g_3, \\ c^a &= \overline{Z}_3^{1/2}(c^a)^r, & &= Z_{1F} Z_2^{-1} Z_3^{-1/2} g_3, \\ \xi_0 &= Z_3 \xi, & &= \overline{Z}_1 \overline{Z}_3^{-1} Z_3^{-1/2} g_3, \\ m_0 &= m - \delta m, \end{aligned} \tag{2.33}$$

where the quantities Z_1, $\overline{Z}_1$, Z_{1F}, and Z_4 are defined by the above coupling, constant relations and can be determined from Z_2, Z_3, and $\overline{Z}_3$. In the following, working in $\xi_0 = 1$ gauge we shall compute the one-loop contributions to the gluon self-energy and to the quark–gluon vertex, and, by absorbing the ϵ-poles, thereby obtain expressions for Z_3 and Z_{1F} to leading order. Determination of the remaining renormalization constants, which can be computed from loop corrections to the quark and ghost propagators and the three-gluon, four–gluon, and ghost–gluon vertices will be left as exercises. However, it is clear from the definition of $g_{3,0}$ in Eq. (2.33) that the relations

$$\frac{Z_4}{Z_1} = \frac{Z_1}{Z_3} = \frac{\overline{Z}_1}{\overline{Z}_3} \tag{2.34}$$

must hold in any consistent renormalization scheme. These are the analogs of the Ward identities in *QED*. Physically, they ensure that the coupling-constant relations which appear in the *QCD* lagrangian (as a consequence of gauge invariance) are maintained in the full theory.

The *QCD* one-loop contribution to the quark–antiquark vacuum polarization amplitude of Fig. II–4(a),[6]

[6] To avoid notational clutter, we shall not put subscripts on the bare coupling for the remainder of this subsection.

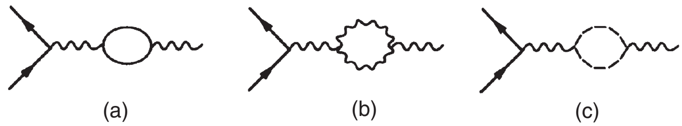

Fig. II–4 One-loop corrections to the gluon propagator: (a) quark–antiquark pair, (b) gluon pair, and (c) ghosts.

$$i\Pi^{ab}_{\alpha\beta}(q)\big|_{\text{quark}} = -\left(\frac{-ig_3}{2}\right)^2 \int \frac{d^4p}{(2\pi)^4} \times \text{Tr}\left[\gamma_\alpha(\lambda^a)_{kj}\frac{i}{\not p - m + i\epsilon}\gamma_\beta(\lambda^b)_{jk}\frac{i}{\not p - \not q - m + i\epsilon}\right], \quad (2.35)$$

differs from the *QED* self-energy only by the group factor $(\lambda^a)_{jk}(\lambda^b)_{kj} = \text{Tr}\,(\boldsymbol{\lambda}^a\boldsymbol{\lambda}^b) = 2\delta^{ab}$ (cf. Eq. (2.7)). Comparing with Eq. (1.39), we obtain

$$i\Pi^{ab}_{\alpha\beta}(q)\big|_{\text{quark}} = i\delta^{ab}(q_\alpha q_\beta - g_{\alpha\beta}q^2)\left(\frac{\mu^2}{-q^2}\right)^\epsilon\left[\frac{g_3^2}{24\pi^2}\frac{1}{\epsilon} + \cdots\right]. \quad (2.36)$$

This must be multiplied by the number of quark flavors n_f which contribute in the vacuum polarization loops.

The contribution from the gluon–gluon intermediate state of Fig. II–4(b) can be written

$$i\Pi^{ab}_{\alpha\beta}(q)\big|_{\text{gluon}} = \frac{1}{2}(-i)^2\int\frac{d^4k}{(2\pi)^4}\frac{N^{ab}_{\alpha\beta}}{[k^2+i\epsilon][(q-k)^2+i\epsilon]} \quad (2.37)$$

with

$$N^{ab}_{\alpha\beta} = g_3 f^{bcd}[-g_{\beta\mu}(q+k)_\nu + g_{\mu\nu}(2k-q)_\beta + g_{\nu\beta}(2q-k)_\mu] \times g_3 f^{acd}[g^\mu_{\ \alpha}(q+k)^\nu + g^{\mu\nu}(q-2k)_\alpha + g^\nu_{\ \alpha}(k-2q)^\mu]. \quad (2.38)$$

The prefactor $1/2$ in Eq. (2.37) is a Feynman symmetry factor associated with the identical intermediate-state gluons. To arrive at this expression, special care must be exercised with momentum flow in the three-gluon vertices. Upon extending the integration to d dimensions and using Eq. (2.15) to evaluate the color factor, we obtain

$$i\Pi^{ab}_{\alpha\beta}(q)\big|_{\text{gluon}} = -\frac{1}{2}C_2(\mathbf{8})\delta^{ab}g_3^{\,2}\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{N_{\alpha\beta}}{[k^2+i\epsilon][(q-k)^2+i\epsilon]} \quad (2.39)$$

with

$$N_{\alpha\beta} = (-5q^2 + 2q\cdot k - 2k^2)g_{\alpha\beta} + (6-d)q_\alpha q_\beta + (2d-3)(q_\alpha k_\beta + q_\beta k_\alpha) + (6-4d)k_\alpha k_\beta. \quad (2.40)$$

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Integration of Eq. (2.39) yields

$$i\Pi^{ab}_{\alpha\beta}(q)|_{\rm gluon} = -i\frac{g_3^2}{16\pi^2}C_2(\mathbf{8})\delta^{ab}\left(\frac{\mu^2}{-q^2}\right)^{\epsilon}\left[\frac{11}{3}q_\alpha q_\beta - \frac{19}{6}g_{\alpha\beta}q^2\right]\frac{1}{2\epsilon}+\cdots. \tag{2.41}$$

The final contribution to the gluon propagator is the ghost-loop amplitude of Fig. II–4(c),

$$i\Pi^{ab}_{\alpha\beta}(q)\Big|_{\rm ghost} = -\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{(k-q)^2+i\epsilon} \times [g_3 f^{bdc}(k-q)_\beta]\,\frac{i}{k^2+i\epsilon}\,[g_3 f^{acd}k_\alpha]. \tag{2.42}$$

The bracketed quantities arise from the gluon–ghost vertices, and the minus prefactor must accompany any ghost loop. Following the standard steps to a d-dimensional form, we arrive at

$$i\Pi^{ab}_{\alpha\beta}(q)\Big|_{\rm ghost} = -{g_3}^2C_2(\mathbf{8})\delta^{ab}\,\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{k_\alpha(k-q)_\beta}{[(k-q)^2+i\epsilon][k^2+i\epsilon]}, \tag{2.43}$$

which becomes to leading order in ϵ,

$$i\Pi^{ab}_{\alpha\beta}(q)\Big|_{\rm ghost} = i\delta^{ab}\frac{g_3^2}{16\pi^2}C_2(\mathbf{8})\left(\frac{\mu^2}{-q^2}\right)^{\epsilon}\left[\frac{1}{3}q_\alpha q_\beta + \frac{1}{6}g_{\alpha\beta}q^2\right]\frac{1}{2\epsilon}. \tag{2.44}$$

The sum of gluon and ghost contributions takes the gauge-invariant form

$$i\Pi^{ab}_{\alpha\beta}(q)\Big|_{\rm gl+gh} = -i\delta^{ab}\frac{g_3^2}{8\pi^2}\,C_2(\mathbf{8})\,\frac{5}{3}\left(\frac{\mu^2}{-q^2}\right)^{\epsilon}\left[q_\alpha q_\beta - g_{\alpha\beta}q^2\right]\frac{1}{2\epsilon}+\cdots. \tag{2.45}$$

Finally, adding the quark contribution for n_f flavors gives the total result

$$\Pi^{ab}_{\alpha\beta}(q) = i\delta^{ab}(q_\alpha q_\beta - g_{\alpha\beta}q^2)\frac{g_3^2}{8\pi^2}\left(\frac{\mu^2}{-q^2}\right)^{\epsilon}\left[\frac{2n_f}{3} - \frac{5}{3}C_2(\mathbf{8})\right]\frac{1}{2\epsilon}+\cdots. \tag{2.46}$$

Renormalizing at $q^2 = -\mu_R^2$, we find[7]

$$Z_3 = 1 - \frac{g_3^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{2\epsilon}\left[\frac{2n_f}{3} - \frac{5}{3}C_2(\mathbf{8})\right]\frac{1}{2\epsilon} + \mathcal{O}(g_3^4). \tag{2.47}$$

Proceeding next to the quark–gluon vertex, written through first order as

$$-i\frac{g_3}{2}(\Gamma^a_\nu)_{ji}(p_2,p_1) = -i\frac{g_3}{2}\gamma_\nu(\lambda^a)_{ji} - ig_3(\Lambda^a_\nu)_{ji}(p_2,p_1)+\cdots, \tag{2.48}$$

[7] For notational simplicity, we discontinue displaying the superscript (MS) on renormalization constants.

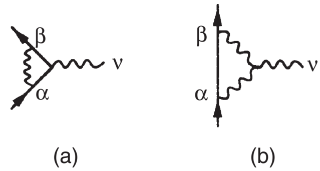

Fig. II–5 One-loop corrections to the quark–gluon vertex.

we see from Fig. II–5 that there are radiative corrections from both quark and gluon intermediate states. The quark contribution is

$$-ig_3[\Lambda^a_\nu(p_2, p_1)]_{ji}\Big|_{\rm quark} = \left(\frac{-ig_3}{2}\right)^3 \int \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\alpha\beta}}{k^2+i\epsilon}\,(\lambda^b)_{jn}\gamma_\alpha \times \frac{i}{\not p_2 - \not k - m + i\epsilon}(\lambda^a)_{nl}\gamma_\nu\,\frac{i}{\not p_1 - \not k - m + i\epsilon}\,(\lambda^b)_{li}\gamma_\beta. \quad (2.49)$$

Aside from the replacement $e \to g_3$ and a color factor $\lambda^b\lambda^a\lambda^b/8$, which is evaluated in Eq. (2.17), the remaining expression is the *QED* vertex, which will be analyzed in detail in Sect. V–1. Thus we anticipate from Eq. (V–1.19) that at $p_1 = p_2 = p$ and $|p|^2 \gg m^2$,

$$[\Lambda^a_\nu(p,p)]_{ji}\Big|_{\rm quark} = (C_2(\mathbf{3}) - \frac{1}{2}C_2(\mathbf{8}))\,\frac{g_3^2}{8\pi^2}\,\frac{1}{2\epsilon}\,\left(\frac{\mu^2}{-p^2}\right)^\epsilon (\lambda^a/2)_{ji}\gamma_\nu \; + \cdots . \quad (2.50)$$

The two-gluon intermediate state, which has no counterpart in *QED*, has the form

$$-ig_3(\Lambda^a_\nu(p_2,p_1))_{ji}\Big|_{\rm gluon} = if_{abc}(\lambda^c\lambda^b)_{ji}\,\frac{g_3^3}{4}\int\frac{d^4k}{(2\pi)^4}\,\gamma^\beta(\not k + m)\gamma^\alpha \times \frac{g_{\nu\beta}(2p_2 - k - p_1)_\alpha + g_{\beta\alpha}(2k - p_1 - p_2)_\nu + g_{\alpha\nu}(2p_1 - k - p_2)_\beta}{[k^2 - m^2 + i\epsilon][(p_1-k)^2 + i\epsilon]^2[(p_2-k)^2 + i\epsilon]}. \quad (2.51)$$

By a now-standard set of steps, it is not difficult to extract the ϵ-pole from the extension of the above to d dimensions, and we find

$$(\Lambda^a_\nu(p,p))_{ji}\Big|_{\rm gluon} = (\lambda^a/2)_{ji}\gamma_\nu\,\frac{3}{2}C_2(\mathbf{8})\,\frac{g_3^2}{8\pi^2}\left(\frac{\mu^2}{-p^2}\right)^\epsilon\frac{1}{2\epsilon}\; + \cdots, \quad (2.52)$$

implying a total vertex correction of the form,

$$(\Lambda^a_\nu(p,p))_{ji}\Big|_{\rm tot} = (\lambda^a/2)_{ji}\gamma_\nu\,[C_2(\mathbf{3}) + C_2(\mathbf{8})]\,\frac{g_3^2}{8\pi^2}\left(\frac{\mu^2}{-p^2}\right)^\epsilon\,\frac{1}{2\epsilon}\; + \cdots . \quad (2.53)$$

We thus determine the renormalization constant for the quark–gluon vertex at $p_i^2 = -\mu_R^2$ to be

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$$Z_{1F} = 1 - [C_2(\mathbf{3}) + C_2(\mathbf{8})]\frac{g_3^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{2\epsilon}\frac{1}{2\epsilon} + \cdots . \tag{2.54}$$

There remains the task of determining Z_2. We shall leave this for an exercise (cf. Prob. II–3) and simply quote the result

$$Z_2 = 1 - C_2(\mathbf{3})\frac{g_3^2}{8\pi^2}\left(\frac{\mu}{\mu_R}\right)^{2\epsilon}\frac{1}{2\epsilon} + \cdots . \tag{2.55}$$

Asymptotic freedom and renormalization group

A striking property of *QCD* is *asymptotic freedom* [GrW 73a,b, Po 73]. This is the statement that, unlike the electric charge, the coupling constant $g_3(\mu_R)$ of color decreases as the scale of renormalization μ_R is increased. To demonstrate this, we first combine our results for Z_1, Z_2 and Z_3 to obtain the coupling renormalization constant Z_g,

$$\begin{aligned} g_{3,0} &= Z_{1F} Z_2^{-1} Z_3^{-1/2} g_3 \equiv Z_g g_3, \\ Z_g &= 1 - \frac{\alpha_s}{4\pi}\left(11 - \frac{2n_f}{3}\right)\left(\frac{\mu}{\mu_R}\right)^{2\epsilon}\frac{1}{2\epsilon} + \cdots , \end{aligned} \tag{2.56}$$

where $\alpha_s \equiv g_3^2/(4\pi)$. From the ϵ^{-1} coefficient of Z_g, we learn that

$$\mu_R \frac{\partial g_3}{\partial \mu_R} = -\left[\frac{11}{3}C_2(\mathbf{8}) - \frac{n_f}{2}C_2(\mathbf{3})\right]\frac{g_3^3}{16\pi^2} + \mathcal{O}(g_3^5), \tag{2.57a}$$

or equivalently,

$$\beta_{QCD} = -\left(11 - \frac{2n_f}{3}\right)\frac{g_3^3}{16\pi^2} + \mathcal{O}(g_3^5) \equiv -\beta_0 \frac{g_3^3}{16\pi^2} + \mathcal{O}(g_3^5). \tag{2.57b}$$

The sign of the leading term in β_{QCD} is negative for the six-flavor world $n_f = 6$, becoming positive only if the number of quark flavors exceeds 16. As we have already seen, the *QED* vacuum acts as a dielectric medium with dielectric constant $\epsilon_{QED} > 1$ because spontaneous creation of charged fermion–antifermion pairs results in screening (i.e., vacuum polarization) of electric charge. The dielectric property $\epsilon_{QED} > 1$ means that the *QED* vacuum has magnetic susceptibility $\mu_{QED} < 1$, and thus is a diamagnetic medium. The *QCD* vacuum is the recipient of similar effects from virtual quark–antiquark pairs, but *these are overwhelmed by contributions from virtual gluons*. As a result, the *QCD* vacuum is a paramagnetic medium ($\mu_{QCD} > 1$) and antiscreens ($\epsilon_{QCD} < 1$) color charge [Hu 81].

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The effect of asymptotic freedom can be displayed most clearly by performing a renormalization group (RG) analysis on the $1PI$ amplitudes of the theory. A connected[8] renormalized Green's function is defined in coordinate space as

$$G^{(n_F,n_B)}(\{x\}) = \langle 0|T\left(\overline{\psi}^r(x_1)\ldots A^r(x_n)\right)|0\rangle_{\text{conn}} \tag{2.58}$$

where the numbers of quark and gluon fields are $n_F,\ n_B$, respectively, and for convenience we suppress color and Lorentz indices. We employ the same symbol $G^{(n_F,n_B)}$ for the momentum Green's function

$$(2\pi)^4\delta^4(p_1+\cdots+p_n)G^{(n_F,n_B)}(\{p\}) = \int\prod_{j=1}^{n}(d^4x_j\, e^{-ip_j\cdot x_j})\, G^{(n_F,n_B)}(\{x\}) \tag{2.59}$$

where $n = n_F + n_B$. The $1PI$ amplitudes $\Gamma^{(n_F,n_B)}$ are obtained by removing the external-leg propagators from $G^{(n_F,n_B)}_{1PI}$,

$$G^{(n_F,n_B)}_{1PI} = \prod_{i'}D(p_{i'})\prod_{j'}S(p_{j'})\,\Gamma^{(n_F,n_B)}(\{p\})\prod_{i}D(p_i)\prod_{j}S(p_j), \tag{2.60}$$

where unprimed (primed) momenta represent initial (final) states. The relations of Eq. (2.33) imply for any renormalization scheme, which we need not specify yet, that

$$\begin{aligned} G^{(n_F,n_B)} &= Z_2^{-n_F/2}Z_3^{-n_B/2}\,G_0^{(n_F,n_B)},\\ D = Z_3^{-1}\,D_0,&\qquad S = Z_2^{-1}\,S_0, \end{aligned} \tag{2.61}$$

where the zero subscript denotes unrenormalized quantities. From this, we have

$$\Gamma^{(n_F,n_B)} = Z_2^{n_F/2}Z_3^{n_B/2}\Gamma_0^{(n_F,n_B)}, \tag{2.62}$$

and the combination of terms

$$Z_2^{-n_F/2}(\mu_R)\,Z_3^{-n_B/2}(\mu_R)\,\Gamma^{(n_F,n_B)}(\{p\},g_3(\mu_R),m(\mu_R),\xi(\mu_R);\mu_R) \tag{2.63}$$

is therefore independent of the renormalization scale μ_R.

Let us now ascertain the behavior of $\Gamma^{(n_F,n_B)}$ in the deep Euclidean kinematic limit where all momenta $\{p\}$ are both spacelike (in order to avoid singularities) and very large compared to any other mass scale in the theory. To keep the situation as simple as possible, we omit the dependence of $\Gamma^{(n_F,n_B)}$ on both the quark-mass $m(\mu_R)$ and gauge $\xi(\mu_R)$ parameters.[9] Then from Eq. (2.59) we find in response to a scale transformation $p \to \lambda p$ that

[8] All the fields participating in a *connected* Green's function are affected by interactions; in a *disconnected* Green's function, one or more of the field quanta propagate freely.

[9] We shall define a 'running mass parameter' later, in Chap. XIV.

$$G^{(n_F,n_B)}(\{\lambda p\}, g_3(\mu_R); \mu_R) = \lambda^{4-n_B-3n_F/2}\, G^{(n_F,n_B)}(\{p\}, g_3(\mu_R); \mu_R/\lambda). \quad (2.64)$$

This behavior is almost that of a homogeneous function occurring in a scale-invariant theory. Canonical dimensions of the fields appear in the exponent of the scaling factor along with an additive factor of four arising from the four-momentum delta function in Eq. (2.59). However, in $G^{(n_F,n_B)}$, there is also an implicit dependence on λ due to the presence of the renormalization scale μ_R. The corresponding scaling property of the $1PI$ amplitude is found from Eqs. (2.63), (2.64) to be

$$\Gamma^{(n_F,n_B)}(\{\lambda p\}, g_3(\mu_R); \mu_R) = \lambda^{4-n_B-3n_F/2}\Gamma^{(n_F,n_B)}(\{p\}, g_3(\mu_R); \mu_R/\lambda) \quad (2.65)$$

or

$$\Gamma^{(n_F,n_B)}(\{\lambda p\}, g_3(\mu_R); \mu_R) = \lambda^{4-n_B-3n_F/2}\left(\frac{Z_3(\lambda\mu_R)}{Z_3(\mu_R)}\right)^{-n_B/2} \times \left(\frac{Z_2(\lambda\mu_R)}{Z_2(\mu_R)}\right)^{-n_F/2}\Gamma^{(n_F,n_B)}(\{p\}, g_3(\lambda\mu_R); \mu_R). \quad (2.66)$$

This functional relationship can be converted to a differential RG equation by taking the λ-derivative of both sides and then setting $\lambda = 1$,

$$\left(\sum_i^n p_i\frac{\partial}{\partial p_i} + n_B(1+\gamma_B) + n_F\left(\frac{3}{2}+\gamma_F\right) - 4 - \beta_{QCD}\frac{\partial}{\partial g_3}\right)\Gamma^{(n_F,n_B)} = 0, \quad (2.67)$$

where

$$\gamma_F = \mu_R\frac{\partial}{\partial \mu_R}\ln Z_2^{1/2}, \qquad \gamma_B = \mu_R\frac{\partial}{\partial \mu_R}\ln Z_3^{1/2} \quad (2.68)$$

are called the *anomalous dimensions* of the respective fields and β_{QCD} is as in Eq. (2.57).

Let us now see how to obtain leading-order estimates for the above anomalous dimensions. To this order, the result for β_{QCD} is both gauge and renormalization scheme-independent. To start, we can use the result of Eq. (2.55) to determine γ_F,

$$\gamma_F = \frac{1}{2}\,\mu_R\,\frac{\partial \ln Z_2}{\partial \mu_R} = \frac{g_3^2}{16\pi^2}C_2(\mathbf{3}) \;+\; \mathcal{O}(g_3^4), \quad (2.69)$$

and analogously for γ_B. To solve the RG equation, we employ the variable $t = \ln\lambda$, where λ is the scaling parameter appearing in Eqs. (2.64)–(2.66), and introduce the *running coupling constant* $\overline{g}_3(t)$,

$$\frac{\partial \overline{g}_3}{\partial t} = \beta(\overline{g}_3), \qquad \overline{g}_3(0) = g_3. \quad (2.70)$$

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Then it is straightforward to verify that the solution to Eq. (2.67) is

$$\Gamma(\{e^t p\}, g_3(\mu_R); \mu_R) = e^{t(4-n_B-3n_F/2)}\mathcal{D}(t)\Gamma(\{p\}, \overline{g}_3(t); \mu_R), \tag{2.71}$$

where

$$\mathcal{D}(t) = \exp\left(-\int_0^t dt' \left[n_B\gamma_B\left(\overline{g}_3(t')\right) + n_F\gamma_F\left(\overline{g}_3(t')\right)\right]\right) \tag{2.72}$$

is the anomalous dimension factor. The scaling behavior of the $1PI$ amplitude is seen to have field dimensions with anomalous contributions in addition to the canonical values.

Despite naive expectations, the interaction strength at the scaled momentum is not the constant g_3, but rather the running coupling constant $\overline{g}_3$ whose magnitude decreases as the momentum is increased. Employing the lowest-order contribution for β_{QCD} in Eq. (2.57b), we can integrate Eq. (2.70) over the interval $t_1 < t < t_2$ to obtain

$$(\overline{g}_3(t_2))^{-2} - (\overline{g}_3(t_1))^{-2} = 2\left(11 - 2n_f/3\right)(t_2 - t_1)/16\pi^2, \tag{2.73}$$

where n_f is the number of quark flavors having mass less than $\sqrt{t_2}$. It is conventional to express this relation in a somewhat different form. Defining a scale Λ at which $\overline{g}_3$ diverges and letting $\alpha_s(q^2) \equiv \overline{g}_3^2(q^2)/4\pi$, we have to lowest order,

$$\alpha_s(q^2) = \frac{4\pi}{(11 - 2n_f/3)}\,\frac{1}{\ln(q^2/\Lambda^2)} + \cdots, \tag{2.74}$$

where n_f is the number of quark flavors with mass less than $\sqrt{q^2}$. Higher order contributions are discussed at the end of this section.

If $\alpha_s(q^2)$ continues to grow as q^2 is lowered, any perturbative representation of β_{QCD} ultimately becomes a poor approximation, and we can no longer integrate Eq. (2.70) with confidence. Although unproven, a popular working hypothesis is that the *QCD* coupling indeed continues to grow as the energy is lowered, leading to the phenomenon of quark confinement. In *QED*, the free parameter $\alpha(q \simeq 0) \simeq 1/137$ is quite small and expansions in powers of α converge rapidly. However *QCD* behaves differently. In particular, it is clear from Eq. (2.74) that α_s is not really a free parameter, but is instead inexorably related to some mass scale, e.g., Λ. This phenomenon, called *dimensional transmutation*, means that an energy such as Λ can effectively serve to replace the dimensionless quantity α_s in the formulae of *QCD*. Specifying *QCD* operationally requires not only a lagrangian but also a value for Λ. For example, *QCD* perturbation theory is useful only if 'large' mass scales M (i.e. those with $(\Lambda/M)^2 \ll 1$) are probed. Because the complexity of low-energy *QCD* has thus far prevented direct analytic solution of the theory, there have been substantial efforts to develop alternative approaches. These include

Table II–2. *Determinations of* $\alpha_s(M_Z)$.

Experiment	q [GeV]	$\alpha_s(q^2)$	$\alpha_s(M_Z)$
τ decays	1.777	0.330 ± 0.014	0.1197 ± 0.0016
DIS $[F_2]$	$2 \to 15$	$---$	0.1142 ± 0.0023
DIS $[e + p \to \text{jets}]$	$6 \to 100$	$---$	0.1198 ± 0.0032
$Q\bar{Q}$ states	7.5	0.1923 ± 0.0024	0.1183 ± 0.0008
Υ decays	9.46	$0.184^{+0.015}_{-0.014}$	$0.1190^{+0.006}_{-0.005}$
e^+e^- jets & shapes	$14 \to 44$	$---$	0.1172 ± 0.0051
e^+e^- [ew]	91.17	0.1193 ± 0.0028	0.1193 ± 0.0028
e^+e^- jets & shapes	$91 \to 208$	$---$	0.1224 ± 0.0039

attempts to solve *QCD* numerically (lattice-gauge theory), phenomenological study of various theoretical constructs (potential, bag, Skyrme models), exploitation of the invariances contained in $\mathcal{L}_{QCD}$ (notably chiral and flavor symmetries), and consideration of the infinite color limit $N_c \to \infty$ as a first approximation to *QCD* (N_c^{-1} expansion). The first of these topics is beyond the scope of this book (e.g. see [GaL 10, DeD 10]), but the others will form the basis for much of our discussion.

Attempts to infer $\alpha_s(q^2)$ from experimental data are typically carried out under kinematic conditions for which a perturbative analysis of *QCD* presumably makes sense. Systems commonly used for this purpose include decays of the τ lepton, deep-inelastic scattering (DIS) structure functions, Υ decay, and hadronic event shapes and jet production in e^+e^- annihilation. Suppose, as is generally the case, a given process is computed to some order in *QCD* perturbation theory and regularized in the $\overline{\text{MS}}$ scheme. If such a theoretical expression is then used to fit the data with a q-value characteristic of the given process employed, an expression such as Eq. (2.74) can be used to determine Λ and $\alpha_s(q^2)$ can be evolved to different q. Since this operation depends on both the regularization procedure and the number of quark flavors n_f used in Eq. (2.74), a notation like $\Lambda^{(n_f)}_{\overline{\text{MS}}}$ would be precise. Unfortunately there is no uniformity in the rate of convergence of *QCD* perturbation theory from process to process. Thus, determinations of $\alpha_s(q^2)$ are affected by both theoretical and experimental uncertainties, and a scatter of quoted values results. Nonetheless, an impressive consistency now exists between determinations carried out for a variety of conditions. Table II–2 lists values of $\alpha_s(M_Z)$ as inferred from a diverse set of experimental inputs [Be 09], and Figure II–6, which has attained the status of a *QCD* icon, displays the same. The current world average at the Z-boson mass scale is [Be *et al.* 11]

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007, \tag{2.75}$$

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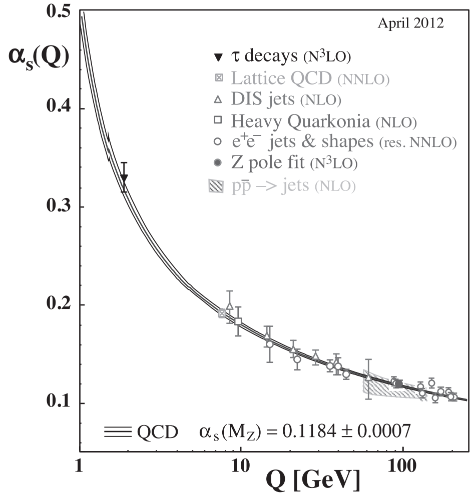

Fig. II–6 Energy dependence of $\alpha_s(Q)$, from [RPP 12] (used with permission).

which implies the value $\Lambda^{(5)}_{\overline{\rm MS}} = 213 \pm 8$ MeV for the five-flavor sector of *QCD*. Determinations of $\alpha_s(q^2)$ have been found to be qualitatively in accord consistent with the predicted q^2 dependence of *QCD*. Taken over the full range of available data, values in the range $0.2 \leq \Lambda(\text{GeV}) \leq 0.4$ are not uncommon, e.g., $\Lambda^{(3)}_{\overline{\rm MS}} = 339 \pm 10$ MeV and $\Lambda^{(4)}_{\overline{\rm MS}} = 296 \pm 10$ MeV as cited in [Be *et al.* 11].

To conclude this section, we briefly comment on the status of higher-order contributions to the running of the strong fine structure constant. To date, analytic calculations on $\alpha_s(\mu)$ have been performed up to the four-loop level,

$$\mu^2 \frac{\partial}{\partial \mu^2} a_s = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + \cdots , \tag{2.76}$$

in which $a_s \equiv \alpha_s/(4\pi)$ is the expansion parameter and exact expressions for the coefficients β_0, β_1, β_2 and β_3 appear in [RiVL 97]. The following useful approximations are also provided,

$$\begin{aligned}
\beta_0 &\simeq 11 - 0.66667 n_f \\
\beta_1 &\simeq 102 - 12.6667 n_f \\
\beta_2 &\simeq 1428.50 - 279.61 n_f + 6.01852 n_f^2 \\
\beta_3 &\simeq 29243.0 - 6946.30 n_f + 405.089 n_f^2 + 1.49931 n_f^3 ,
\end{aligned} \tag{2.77}$$

where as usual n_f denotes the number of active flavors. The four-loop running of α_s can then be expressed as [ChKS 98],

$$\alpha_s(\mu_R^2) \simeq \frac{4\pi}{\beta_0 t}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln t}{t} + \frac{\beta_1^2(\ln^2 t - \ln t - 1) + \beta_0\beta_2}{\beta_4 t^2}\right.$$
$$\left.\frac{\beta_1^3\left(\ln^3 t - 2.5\ln^2 t - 2\ln t + 0.5\right) + 3\beta_0\beta_1\beta_2\ln t - 0.5\beta_0^2\beta_3}{\beta_0^6 t^3}\right], \tag{2.78}$$

where $t \equiv \ln(\mu_R^2/\Lambda^2)$. As an example, let us use this (taking $n_f = 5$) to determine $\alpha_s(\mu)$ at three mass scales involving respectively the b quark, the Z boson, and the Higgs boson, i.e., $\mu_b = 4.18$ GeV, $\mu_Z = 91.1876$ GeV and $\mu_H = 125.5$ GeV,

$$\alpha_s(m_b) \simeq 0.2266, \qquad \alpha_s(M_Z) \simeq 0.1184, \qquad \alpha_s(M_H) \simeq 0.1129. \tag{2.79}$$

These values reflect the behavior expected from asymptotic freedom, as discussed earlier.[10] They will later be of use in discussing running quark mass (Chap. XIV) and Higgs-boson phenomenology (Chap. XV).

II–3 Electroweak interactions

The Weinberg–Salam–Glashow model [Gl 61, We 67b, Sa 69] is a gauge theory of the electroweak interactions whose input fermionic degrees of freedom are massless spin one-half chiral particles. It has the group structure $SU(2)_L \times U(1)$, where the $SU(2)_L$, $U(1)$ represent *weak isospin* and *weak hypercharge* respectively. The subscript 'L' on $SU(2)_L$ indicates that, among fermions, only left-handed states transform nontrivially under weak isospin.

Weak isospin and weak hypercharge assignments

First, we shall discuss how the fermionic weak isospin ($T_{\rm w}$, $T_{\rm w3}$) and weak hypercharge ($Y_{\rm w}$) quantum numbers are assigned. The fermion generations are taken to obey a 'template' pattern – we assume that each succeeding generation differs from the first only in mass. Thus, it will suffice to consider just the lightest fermions for the remainder of this section. The first-generation electroweak assignments are displayed in Table II–3.

For weak isospin, experience gained from charged weak current interactions such as nuclear beta decay dictates that left-handed fermions belong to weak isodoublets while right-handed fermions be placed in weak isosinglets, as in

[10] Using the exact relations for $\beta_0, \ldots, \beta_4$ yields the same results to the stated level of accuracy.

Table II–3. $SU(2)_L \times U(1)$ *fermion assignments*

Particle	$T_{\rm w}$	$T_{\rm w3}$	$Y_{\rm w}$
$\nu_{e,L}$	$1/2$	$1/2$	-1
e_L	$1/2$	$-1/2$	-1
$\nu_{e,R}$	0	0	0
e_R	0	0	-2
u_L	$1/2$	$1/2$	$1/3$
d_L	$1/2$	$-1/2$	$1/3$
u_R	0	0	$4/3$
d_R	0	0	$-2/3$

$$\begin{aligned} &\text{leptons :} \qquad \ell_L \equiv \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad \nu_{e,R} \quad e_R, \\ &\text{quarks :} \qquad q_L \equiv \begin{pmatrix} u \\ d \end{pmatrix}_L \qquad u_R \quad d_R. \end{aligned} \tag{3.1}$$

In view of nonzero neutrino mass, we include a right-handed neutrino. Each of the degrees of freedom displayed above must be assigned a weak hypercharge. There are *a priori* six in all,[11]

$$\begin{aligned} &Y(q_L) \equiv Y_q, \quad Y(u_R) \equiv Y_u, \quad Y(d_R) \equiv Y_d, \\ &Y(\ell_L) \equiv Y_\ell, \quad Y(e_R) \equiv Y_e\ , \quad Y(\nu_R) \equiv Y_\nu. \end{aligned} \tag{3.2}$$

In the Standard Model one identifies the electromagnetic current, following spontaneous symmetry breaking in the electroweak sector, by its coupling to the linear combination of neutral gauge bosons having zero mass. The electric charge Q carried by a particle is thus *linearly* related to the $SU(2)_L \times U(1)_Y$ quantum numbers $T_{\rm w3}$ and $Y_{\rm w}$,

$$aQ = T_{\rm w3} + bY_{\rm w}, \tag{3.3}$$

where a, b are constants. We can use the freedom in assigning the scale of the electric charge Q to choose $a = 1$. At this point, let us not assume any knowledge of the fermion electric charge values. Ultimately, however, the left-handed and right-handed components of the charged chiral fermions must unite to form the physical states themselves. Consistency demands that the electric charges of the chiral components of each such charged fermion be the same, whatever value that charge might have. Using Eq. (3.3), we find

[11] The reason that weak hypercharge engenders so many free parameters in contrast to weak isospin lies in the difference between an abelian gauge structure (like weak hypercharge) and one which is nonabelian (like weak isospin). Thus all doublets have the same weak isospin properties irrespective of their other properties, analogous to flavor independence in *QCD*. For the abelian group of weak hypercharge, the group structure by itself provides no guidelines for assigning the weak hypercharge quantum number. Like the electric charge, it is *a priori* an arbitrary quantity.

$$Y_q = Y_u - \frac{1}{2b} = Y_d + \frac{1}{2b}, \quad Y_\ell = Y_e + \frac{1}{2b} = Y_\nu - \frac{1}{2b}. \tag{3.4}$$

Additional information is contained in axial anomaly cancelation conditions, to be discussed in detail in Sect. III–3 (see especially Eq. (III–3.60b) and subsequent discussion). In particular, the cancelation requirement implies the conditions

$$\mathrm{Tr}\, {F_3}^2 Y_{\rm w} = 0, \tag{3.5a}$$
$$\mathrm{Tr}\, T_{\rm w3}^2 Y_{\rm w} = 0, \tag{3.5b}$$
$$\mathrm{Tr}\, {Y_{\rm w}}^3 = 0, \tag{3.5c}$$

where 'Tr' represents a sum over fermions and in Eq. (3.5a) F_3 is the third generator of the octet of color charges. These constraints imply

$$2Y_q - Y_u - Y_d = 0, \tag{3.6a}$$
$$3Y_q + Y_\ell = 0, \tag{3.6b}$$
$$2(3{Y_q}^3 + {Y_\ell}^3) - 3({Y_u}^3 + {Y_d}^3) - {Y_e}^3 - Y_\nu^3 = 0, \tag{3.6c}$$

where the factors of '3' are color related and the minus signs arise from chirality dependence of the anomalies. Then, insertion of Eq. (3.4) into Eqs. (3.6b), (3.6c) yields

$$\left(bY_\ell + \frac{1}{2}\right)^3 - Y_\nu^3 = 0. \tag{3.7}$$

If neutrinos are Majorana particles (i.e. identical to their antiparticles), then they cannot carry electric charge and by Eq. (3.3), one has $Y_\nu = 0$. If so, Eq. (3.7) implies $bY_\ell = -1/2$, which fixes the remaining Y_i via Eq. (3.4). Thus, provided neutrinos are Majorana particles, once the weak isospin is chosen as in Eqs. (3.1), (3.2) and all possible chiral anomalies are arranged to cancel, one obtains a prediction for the fermion electric charge. We also learn that any attempt to determine weak hypercharge values from the known fermion electric charges is affected by an arbitrariness associated with the value of 'b'. This accounts for the variety of conventions seen in the literature. For definiteness, we have taken $b = 1/2$ in Eq. (3.3) and thus the relationship among the various quantum numbers in Table II–3 is

$$Y_{\rm w} = 2(Q - T_{\rm w3}). \tag{3.8}$$

On the other hand, if neutrinos are Dirac particles, it follows from Eq. (3.4) that Eq. (3.7) becomes a trivality and we learn nothing of weak hypercharge assignments from anomaly cancelation arguments. In this instance, one assigns the weak hypercharge by inserting the observed fermion electric charges into Eq. (3.8). The ability to *predict* $\{Q_i\}$ values has been lost.

$SU(2)_L \times U(1)_Y$ gauge-invariant lagrangian

Having assigned quantum numbers, we turn next to the electroweak interactions. The Weinberg–Salam lagrangian divides naturally into three additive parts, gauge (G), fermion (F), and Higgs (H),

$$\mathcal{L}_{WS} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H . \tag{3.9}$$

Throughout this section we shall concentrate on establishing the general form of the electroweak sector, referring at times to only a few tree-level amplitudes. We shall return in Chap. V to the subject of electromagnetic radiative corrections, and present the electroweak Feynman rules along with various radiative corrections in Chap. XVI.

The gauge-boson fields, which couple to the weak isospin and weak hypercharge are, respectively, $\overrightarrow{W}_\mu = (W^1_\mu,\ W^2_\mu,\ W^3_\mu)$ and B_μ. These contribute to the purely gauge part of the lagrangian as

$$\mathcal{L}_G = -\frac{1}{4} F_i^{\mu\nu} F^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}, \tag{3.10}$$

where $F^i_{\mu\nu}$ (i = 1,2,3) is the $SU(2)$ field strength,

$$F^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g_2 \epsilon^{ijk} W^j_\mu W^k_\nu , \tag{3.11}$$

and $B_{\mu\nu}$ is the $U(1)$ field strength,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu . \tag{3.12}$$

The fermionic sector of the lagrangian density includes both the left-handed and right-handed chiralities. Summing over left-handed weak isodoublets ψ_L and right-handed weak isosinglets ψ_R, we have

$$\mathcal{L}_F = \sum_{\psi_L} \overline{\psi}_L \, i \not{D} \, \psi_L \; + \sum_{\psi_R} \overline{\psi}_R \, i \not{D} \, \psi_R . \tag{3.13}$$

Since a right-handed chiral fermion does not couple to weak isospin, its covariant derivative has the simple form

$$D_\mu \psi_R = (\partial_\mu + i \frac{g_1}{2} Y_{\rm w} B_\mu) \psi_R . \tag{3.14}$$

This expression serves to define the $U(1)$ coupling g_1. Its normalization is dictated by our convention for weak hypercharge $Y_{\rm w}$. The corresponding covariant derivative for the $SU(2)_L$ doublet ψ_L is

$$D_\mu \psi_L = \left(\mathbf{I} \left(\partial_\mu + i \frac{g_1}{2} Y_{\rm w} B_\mu \right) + i g_2 \frac{\vec{\boldsymbol{\tau}}}{2} \vec{W}_\mu \right) \psi_L , \tag{3.15}$$

given in terms of the $SU(2)$ gauge coupling constant g_2 and the 2×2 matrices $\mathbf{I}$, $\vec{\boldsymbol{\tau}}$.

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We shall not display the quark color degree of freedom in this section for reasons of notational simplicity. However, it is understood that all situations in which quark internal degrees of freedom are summed over, as in Eq. (3.13), must include a color sum. Similarly, relations like Eq. (3.14) or Eq. (3.15) hold for each distinct internal color state when applied to quark fields.

The above equations define a mathematically consistent gauge theory of weak isospin and weak hypercharge. However, it is not a physically acceptable electroweak theory of Nature because the fermions and gauge bosons are massless. A Higgs sector must be added to the above lagrangians to arrive at the full Weinberg–Salam model. Thus, we introduce into the theory a complex doublet

$$\Phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \tag{3.16}$$

of spin-zero Higgs fields with electric charge assignments as indicated. The quanta of these fields then each carry one unit of weak hypercharge. The Higgs lagrangian $\mathcal{L}_H$ is the sum of two kinds of terms, $\mathcal{L}_{HG}$ and $\mathcal{L}_{HF}$, which contain the Higgs–gauge and Higgs–fermion couplings respectively. The former is written as

$$\mathcal{L}_{HG} = (D^\mu \Phi)^* D_\mu \Phi - V(\Phi), \tag{3.17}$$

where

$$D_\mu \Phi = \left(\mathbf{I} \left[\partial_\mu + i \frac{g_1}{2} B_\mu \right] + i g_2 \frac{\vec{\boldsymbol{\tau}}}{2} \cdot \vec{W}_\mu \right) \Phi, \tag{3.18}$$

and V is the Higgs self-interaction,

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \tag{3.19}$$

The parameters μ^2 and λ are positive but otherwise arbitrary. For simplicity, we write the Higgs–fermion interaction in this section for just the first generation of fermions. Denoting the left-handed quark and lepton doublets respectively as q_L and ℓ_L, we have

$$\mathcal{L}_{HF} = -g_u \bar{q}_L \widetilde{\Phi} u_R - g_d \bar{q}_L \Phi d_R - g_\nu \bar{\ell}_L \widetilde{\Phi} \nu_{e,R} - g_e \bar{\ell}_L \Phi e_R + \text{h.c.}, \tag{3.20}$$

where the coupling constants g_u, g_d, g_e and g_ν are arbitrary and we employ the charge conjugate to Φ,

$$\widetilde{\Phi} = i \tau_2 \Phi^*. \tag{3.21}$$

In a sense the Higgs potential V and Higgs–fermion coupling $\mathcal{L}_{HF}$ lie outside our guiding principle of gauge invariance because neither contains a gauge field. However, there is no principle which forbids such contributions, and their presence is phenomenologically required. Moreover, note that each is written in $SU(2)_L \times U(1)$ invariant form.

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Spontaneous symmetry breaking

Mass generation for fermions and gauge bosons proceeds by means of spontaneous breaking of the $SU(2)_L \times U(1)$ symmetry. To begin, we obtain the ground-state Higgs configuration by minimizing the potential V to give

$$\Phi(-\mu^2 + 2\lambda\Phi^\dagger\Phi) = 0. \tag{3.22}$$

We interpret this ground-state relation in terms of vacuum expectation values, denoted by a zero subscript. Eq. (3.22) has two solutions, the trivial solution $\langle\Phi\rangle_0 = 0$ and the nontrivial solution,

$$\langle\Phi^\dagger\Phi\rangle_0 = \frac{v^2}{2}, \tag{3.23}$$

with

$$v \equiv \sqrt{\frac{\mu^2}{\lambda}}. \tag{3.24}$$

Let us consider the latter alternative. A nontrivial vacuum Higgs configuration, which obeys the constraint Eq. (3.23), respects conservation of electric charge, and describes the spontaneous symmetry breaking of the original $SU(2)_L \times U(1)$ symmetry is

$$\langle\Phi\rangle_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{3.25}$$

In one interpretation, it is the order parameter for the Weinberg–Salam model, playing a role analogous to the magnetization in a ferromagnet. Group theoretically, it is seen to transform as a component of a weak isodoublet. The energy scale, v, of the effect is not predicted by the model and must be inferred from experiment.

The fermion and gauge-boson masses are determined by employing Eq. (3.25) for the Higgs field everywhere in the lagrangian $\mathcal{L}_H$. We first define charged fields $W_\mu^\pm$,

$$W_\mu{}^\pm = \sqrt{\frac{1}{2}}(W_\mu^1 \mp iW_\mu^2). \tag{3.26}$$

corresponding to the gauge bosons $W^\pm$. By substitution, we find for the mass contribution to the lagrangian

$$\begin{aligned}\mathcal{L}_{\text{mass}} = &-\frac{v}{\sqrt{2}}(g_u\bar{u}u + g_d\bar{d}d + g_e\bar{e}e) + \left(\frac{vg_2}{2}\right)^2 W_\mu^+ W_-^\mu \\ &+ \frac{v^2}{8}(W_\mu^3\ B_\mu)\begin{pmatrix} g_2^2 & -g_1g_2 \\ -g_1g_2 & g_1^2 \end{pmatrix}\begin{pmatrix} W_3^\mu \\ B^\mu \end{pmatrix}.\end{aligned} \tag{3.27}$$

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The fermion masses are given by

$$m_f = \frac{v}{\sqrt{2}} g_f \quad (f = u,\ d,\ e,\ \ldots). \tag{3.28}$$

Although the theory can accommodate fermions of any mass, it does not predict the mass values. Instead, the measured fermion masses are used to fix the arbitrary Higgs–fermion couplings. The charged W-boson masses can be read off directly from Eq. (3.27),

$$M_W = \frac{v}{2}\, g_2, \tag{3.29}$$

but the symmetry breaking induces the neutral gauge bosons to undergo mixing. Their mass matrix is not diagonal in the basis of W^3, B states. Diagonalization occurs in the basis

$$\begin{aligned} Z_\mu &= \cos\theta_\mathrm{w}\ W_\mu^3 \ -\ \sin\theta_\mathrm{w}\ B_\mu, \\ A_\mu &= \sin\theta_\mathrm{w}\ W_\mu^3 \ +\ \cos\theta_\mathrm{w}\ B_\mu, \end{aligned} \tag{3.30}$$

where the weak mixing angle (or *Weinberg angle*) θ_w is defined by

$$\tan\theta_\mathrm{w} = \frac{g_1}{g_2}. \tag{3.31}$$

The neutral gauge-boson masses are found to be

$$M_\gamma = 0, \qquad M_Z = \frac{v}{2}\sqrt{g_1^2 + g_2^2}, \tag{3.32}$$

and the fields A_μ and Z_μ correspond to the massless photon and massive Z^0-boson, respectively. Observe that the $W^\pm$-to-Z^0 mass ratio is fixed by

$$\frac{M_W}{M_Z} = \cos\theta_\mathrm{w}. \tag{3.33}$$

Electroweak currents

Now that we have determined the mass spectrum of the theory in terms of the input parameters, we must next study the various gauge–fermion interactions. The traditional description of electromagnetic and low-energy charged weak interactions of spin one-half particles is expressed as

$$\mathcal{L}_\mathrm{int} = -eA_\mu J_\mathrm{em}^\mu - \frac{G_F}{\sqrt{2}} J_\mathrm{ch}^{\mu\dagger} J_\mu^\mathrm{ch}, \tag{3.34}$$

where J_em^μ is the electromagnetic current

$$J_\mathrm{em}^\mu = -\bar{e}\gamma^\mu e + \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d + \cdots, \tag{3.35}$$

J_ch^μ is the charged weak current (ignoring quark mixing)

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$$J^\mu_{\rm ch} = \overline{\nu_e}\gamma^\mu(1+\gamma_5)e + \overline{u}\gamma^\mu(1+\gamma_5)d \ + \cdots , \tag{3.36}$$

and $G_F \simeq 1.166 \times 10^{-5}$ GeV^{-2} is the Fermi constant (cf. Sect. V–2).

Alternatively, we can use Eqs. (3.13)–(3.15) to obtain the charged and neutral interactions in the $SU(2)_L \times U(1)_Y$ description,

$$\mathcal{L}'_{\rm int} = -\frac{g_2}{\sqrt{8}}\left(W^+_\mu J^\mu_{\rm ch} + W^-_\mu J^{\mu\dagger}_{\rm ch}\right) - g_2 W^3_\mu J^\mu_{\rm w3} - g_1 B_\mu (J^\mu_{\rm em} - J^\mu_{\rm w3}), \tag{3.37}$$

where $J^\mu_{\rm w3}$ is the third component of the weak isospin current,

$$\vec{J}^\mu_{\rm w} = \sum_{\psi_L} \overline{\psi}_L \gamma^\mu \frac{\vec{\boldsymbol{\tau}}}{2}\psi_L, \tag{3.38}$$

summed over all left-handed fermion weak isodoublets. Substituting for B_μ and W^3_μ in Eq. (3.37) in terms of A_μ and Z_μ yields

$$\mathcal{L}'_{\rm int} = -\frac{g_2}{\sqrt{8}}\left(W^+_\mu J^\mu_{\rm ch} + W^-_\mu J^{\mu\dagger}_{\rm ch}\right) - g_1 \cos\theta_{\rm w} A_\mu J^\mu_{\rm em} + \mathcal{L}_{\rm ntl-wk}, \tag{3.39}$$

where $J^\mu_{\rm ch} = 2J^\mu_{{\rm w},1+i2}$ is given in Eq. (3.36) and the neutral weak interaction $\mathcal{L}_{\rm ntl-wk}$ for fermion f is[12]

$$\begin{aligned}\mathcal{L}^{(f)}_{\rm ntl-wk} &= -\frac{g_2}{2\cos\theta_{\rm w}} Z^\mu \ \bar{f}(g^{(f)}_{\rm v}\gamma_\mu + g^{(f)}_{\rm a}\gamma_\mu\gamma_5) f, \\ g^{(f)}_{\rm v} &\equiv T^{(f)}_{\rm w3} - 2\sin^2\theta_{\rm w}\, Q^{(f)}_{\rm el}, \qquad g^{(f)}_{\rm a} \equiv T^{(f)}_{\rm w3}.\end{aligned} \tag{3.40}$$

Specifically, we have for the vector and axial-vector couplings

$$\begin{aligned} g^{(e,\mu,\tau)}_{\rm v} &= -\frac{1}{2} + 2\sin^2\theta_{\rm w}, & g^{(e,\mu,\tau)}_{\rm a} &= -\frac{1}{2}, \\ g^{(u,c,t)}_{\rm v} &= \frac{1}{2} - \frac{4}{3}\sin^2\theta_{\rm w}, & g^{(u,c,t)}_{\rm a} &= \frac{1}{2}, \\ g^{(d,s,b)}_{\rm v} &= -\frac{1}{2} + \frac{2}{3}\sin^2\theta_{\rm w}, & g^{(d,s,b)}_{\rm a} &= -\frac{1}{2}, \\ g^{(\nu_e,\nu_\mu,\nu_\tau)}_{\rm v} &= \frac{1}{2}, & g^{(\nu_e,\nu_\mu,\nu_\tau)}_{\rm a} &= \frac{1}{2}. \end{aligned} \tag{3.41}$$

Observe the structure of the neutral weak couplings $g^{(f)}_{\rm v,a}$. If $\theta_{\rm w}$ were to vanish, neutral weak interactions would be given strictly in terms of $T_{\rm w3}$, the third component of weak isospin. However in the real world, phenomena like low-energy neutrino

[12] One should be careful not to confuse Eq. (3.40) with the alternate form

$$\begin{aligned}\mathcal{L}^{(f)}_{\rm ntl-wk} &= -eZ^\mu \bar{\psi}_f(v_f\gamma_\mu + a_f\gamma_\mu\gamma_5)\psi_f, \\ v_f &= \frac{T^{(f)}_{\rm w3} - 2\sin^2\theta_{\rm w}\, Q^{(f)}_{\rm el}}{2\sin\theta_{\rm w}\cos\theta_{\rm w}}, \qquad a_f = \frac{T^{(f)}_{\rm w3}}{2\sin\theta_{\rm w}\cos\theta_{\rm w}},\end{aligned}$$

which also appears in the literature.

interactions, M_W/M_Z, deep inelastic lepton scattering data, etc. all depend on the value of $\theta_{\rm w}$. In addition, we note that because $\sin^2\theta_{\rm w} \simeq 0.23$, the leptonic vector coupling constants $g_{\rm v}^{(e,\mu,\tau)}$ are substantially suppressed relative to the axial-vector couplings.

Comparison of Eq. (3.34) with Eq. (3.39) yields

$$e = g_1 \cos\theta_{\rm w} = g_2 \sin\theta_{\rm w}. \tag{3.42}$$

The Fermi interaction of Eq. (3.34) corresponds in the Weinberg–Salam model to a second-order interaction mediated by W-exchange and evaluated in the limit of small momentum transfer ($1 \gg q^2/M_W^2$),

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}. \tag{3.43}$$

Together, these relations provide a tree-level expression for the W-boson mass,

$$M_W^2 = \frac{1}{\sin^2\theta_{\rm w}} \frac{\pi\alpha}{\sqrt{2}G_F} \simeq \left(\frac{37.281 \text{ GeV}}{\sin\theta_{\rm w}}\right)^2. \tag{3.44}$$

Also, Eqs. (3.29), (3.43) imply

$$v = 2^{-1/4} G_F^{-1/2} \simeq 246.221(2) \text{ GeV}. \tag{3.45}$$

It is the quantity v which sets the scale of spontaneous symmetry breaking in the $SU(2)_L \times U(1)$ theory, and all masses in the Standard Model are proportional to it, although with widely differing coefficients.

We shall resume in Chaps. XV, XVI discussion of a number of topics introduced in this section, among them the Higgs scalar, the $W^\pm$ and Z^0 gauge bosons, and phenomenology of the neutral weak current. Also included will be a description of quantization procedures for the electroweak sector, including the issue of radiative corrections. First, however, in the intervening chapters we shall encounter a number of applications involving light fermions undergoing electroweak interactions at very modest energies and momentum transfers. For these it will suffice to work with just tree-level $W^\pm$ and/or Z^0 exchange, and to consider only photonic or gluonic radiative corrections. We shall also neglect the gauge-dependent longitudinal polarization contributions to the gauge-boson propagators (analogous to the $q^\mu q^\nu$ term in the photon propagator in Eq. (1.18)), as well as effects of the Higgs degrees of freedom. For photon propagators, the $q^\mu q^\nu$ terms do not contribute to physical amplitudes because of current conservation. Although current conservation is generally not present for the weak interactions, both the $q^\mu q^\nu$ propagator terms and Higgs contributions are suppressed by powers of $(m_f/M_W)^2$ for an external fermion of mass m_f.

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II–4 Fermion mixing

In our discussion of the Weinberg–Salam model, we limited the number of fermion generations to one. We now lift that restriction and consider the implication of having n generations. Although the existing experimental situation supports the value $n = 3$, we shall take n arbitrary in our initial analysis.

Diagonalization of mass matrices

To begin, it is necessary to generalize the Higgs–fermion lagrangian $\mathcal{L}_{HF}$ of Eq. (3.20) to

$$\begin{aligned} -\mathcal{L}_{HF} = g_u^{\alpha\beta}\, \bar{q}'_{L,\alpha} \widetilde{\Phi} u'_{R,\beta} + g_d^{\alpha\beta}\, \bar{q}'_{L,\alpha} \Phi d'_{R,\beta} + g_\nu^{\alpha\beta}\, \bar{\ell}'_{L,\alpha} \widetilde{\Phi} \nu'_{R,\beta} \\ + g_e^{\alpha\beta}\, \bar{\ell}'_{L,\alpha} \Phi e'_{R,\beta} + \text{h.c.}, \end{aligned} \tag{4.1}$$

where we employ the summation convention $\alpha, \beta = 1, \ldots, n$, and adopt the notation

$$\begin{aligned} \vec{u}\,' &= (u', c', t', \ldots), \\ \vec{d}\,' &= (d', s', b', \ldots), \\ \vec{\nu}\,' &= (\nu_e, \nu_\mu, \nu_\tau, \ldots), \\ \vec{e}\,' &= (e', \mu', \tau', \ldots), \\ \vec{q}\,' &= \left(\begin{pmatrix} u' \\ d' \end{pmatrix}, \begin{pmatrix} c' \\ s' \end{pmatrix}, \begin{pmatrix} t' \\ b' \end{pmatrix}, \ldots \right), \\ \vec{\ell}\,' &= \left(\begin{pmatrix} \nu_e \\ e' \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu' \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau' \end{pmatrix}, \ldots \right). \end{aligned} \tag{4.2}$$

Observe that we denote the individual neutrino flavor eigenstates as ν_e, ν_μ, ν_τ, with no primes. The states which appear in the original gauge-invariant lagrangian are generally *not* the mass eigenstates. That is, there is no reason why the $n \times n$ generational coupling matrices $\mathbf{g}_u$, $\mathbf{g}_d$, $\mathbf{g}_\nu$, $\mathbf{g}_e$ should be diagonal. Following spontaneous symmetry breaking, we obtain the generally nondiagonal $n \times n$ mass matrices $\mathbf{m}'_u$, $\mathbf{m}'_d$, $\mathbf{m}'_\nu$, $\mathbf{m}'_e$ from the analog of Eq. (3.28),

$$\mathbf{m}'_f = \frac{v}{\sqrt{2}}\, \mathbf{g}_f \qquad (f = u, d, \nu, e). \tag{4.3}$$

Although not diagonal in the *flavor basis*, these matrices can be brought to diagonal form in the *mass basis*. The transformation from flavor eigenstates to mass eigenstates is accomplished by means of the steps

$$
\begin{aligned}
-\mathcal{L}_{F,\ \mathrm{mass}} &= \bar{\vec{u}}_L^{\,\prime}\, \mathbf{m}_u' \vec{u}_R^{\,\prime} + \bar{\vec{d}}_L^{\,\prime}\, \mathbf{m}_d' \vec{d}_R^{\,\prime} + \bar{\vec{\nu}}_L^{\,\prime}\, \mathbf{m}_\nu' \vec{\nu}_R^{\,\prime} + \bar{\vec{e}}_L^{\,\prime}\, \mathbf{m}_e' \vec{e}_R^{\,\prime} + \text{h.c.}, \\
&= \bar{\vec{u}}_L^{\,\prime}\, \mathbf{S}_L^u \mathbf{S}_L^{u\dagger} \mathbf{m}_u' \mathbf{S}_R^u \mathbf{S}_R^{u\dagger}\, \vec{u}_R^{\,\prime} + \bar{\vec{d}}_L^{\,\prime}\, \mathbf{S}_L^d \mathbf{S}_L^{d\dagger} \mathbf{m}_d' \mathbf{S}_R^d \mathbf{S}_R^{d\dagger}\, \vec{d}_R^{\,\prime} \\
&\quad + \bar{\vec{\nu}}_L^{\,\prime}\, \mathbf{S}_L^\nu \mathbf{S}_L^{\nu\dagger} \mathbf{m}_\nu' \mathbf{S}_R^\nu \mathbf{S}_R^{\nu\dagger}\, \vec{\nu}_R^{\,\prime} + \bar{\vec{e}}_L^{\,\prime}\, \mathbf{S}_L^e \mathbf{S}_L^{e\dagger} \mathbf{m}_e' \mathbf{S}_R^e \mathbf{S}_R^{e\dagger}\, \vec{e}_R^{\,\prime} + \text{h.c.} \\
&= \bar{\vec{u}}_L\, \mathbf{m}_u\, \vec{u}_R + \bar{\vec{d}}_L\, \mathbf{m}_d\, \vec{d}_R + \bar{\vec{\nu}}_L\, \mathbf{m}_\nu\, \vec{\nu}_R + \bar{\vec{e}}_L\, \mathbf{m}_e\, \vec{e}_R + \text{h.c.} \\
&= \bar{\vec{u}}\, \mathbf{m}_u\, \vec{u} + \bar{\vec{d}}\, \mathbf{m}_d\, \vec{d} + \bar{\vec{\nu}}\, \mathbf{m}_\nu\, \vec{\nu} + \bar{\vec{e}}\, \mathbf{m}_e\, \vec{e}.
\end{aligned} \tag{4.4}
$$

The $n \times n$ unitary matrices $S_{L,R}^{\alpha}$ $(\alpha = u,\ d,\ \nu,\ e)$ relate the basis states,

$$
\begin{aligned}
\vec{u}_L^{\,\prime} &= \mathbf{S}_L^u\, \vec{u}_L, \quad \vec{d}_L^{\,\prime} = \mathbf{S}_L^d\, \vec{d}_L, \quad \vec{\nu}_L^{\,\prime} = \mathbf{S}_L^\nu\, \vec{\nu}_L, \quad \vec{e}_L^{\,\prime} = \mathbf{S}_L^e\, \vec{e}_L, \\
\vec{u}_R^{\,\prime} &= \mathbf{S}_R^u\, \vec{u}_R, \quad \vec{d}_R^{\,\prime} = \mathbf{S}_R^d\, \vec{d}_R, \quad \vec{\nu}_R^{\,\prime} = \mathbf{S}_R^\nu\, \vec{\nu}_R, \quad \vec{e}_R^{\,\prime} = \mathbf{S}_R^e\, \vec{e}_R,
\end{aligned} \tag{4.5}
$$

and induce the biunitary diagonalizations

$$
\mathbf{m}_\alpha' = \mathbf{S}_L^\alpha\, \mathbf{m}_\alpha\, \mathbf{S}_R^{\alpha\dagger}, \qquad (\alpha = u, d, \nu, e), \tag{4.6}
$$

thus yielding the diagonal quark mass matrices

$$
\mathbf{m}_u = \begin{pmatrix} m_u & 0 & 0 & \dots \\ 0 & m_c & 0 & \dots \\ 0 & 0 & m_t & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{m}_d = \begin{pmatrix} m_d & 0 & 0 & \dots \\ 0 & m_s & 0 & \dots \\ 0 & 0 & m_b & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.7a}
$$

and the diagonal lepton mass matrices

$$
\mathbf{m}_\nu = \begin{pmatrix} m_1 & 0 & 0 & \dots \\ 0 & m_2 & 0 & \dots \\ 0 & 0 & m_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{m}_e = \begin{pmatrix} m_e & 0 & 0 & \dots \\ 0 & m_\mu & 0 & \dots \\ 0 & 0 & m_\tau & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.7b}
$$

Although the Weinberg–Salam model is first written down in terms of the flavor basis states, actual calculations which confront theory with experiment are performed using the mass basis states. We must then transform from one to the other. This turns out to have no effect on the structure of the electromagnetic and neutral weak currents. One simply omits writing the primes, which would otherwise appear. The reason is that (aside from mass) each generation is a replica of the others, and products of the unitary transformation matrices always give rise to the unit matrix in flavor space. Thus, at the lagrangian level, there are no *flavor-changing neutral currents* in the theory.

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As an example of this, consider the leptonic contribution to the electromagnetic current,

$$\begin{aligned} J^{\mu}_{\text{em}}(\text{lept}) &= -\bar{e}'_{\alpha}\gamma^{\mu}e'_{\alpha} \;=\; -\bar{e}'_{L,\alpha}\gamma^{\mu}e'_{L,\alpha} \;-\; \bar{e}'_{R,\alpha}\gamma^{\mu}e'_{R,\alpha} \\ &= -(\bar{e}_L\mathbf{S}^{e\dagger}_L)_{\alpha}\gamma^{\mu}(\mathbf{S}^e_L e_L)_{\alpha} \;-\; (\bar{e}_R\mathbf{S}^{e\dagger}_R)_{\alpha}\gamma^{\mu}(\mathbf{S}^e_R e_R)_{\alpha} \\ &= -\bar{e}_{L,\alpha}\gamma^{\mu}e_{L,\alpha} - \bar{e}_{R,\alpha}\gamma^{\mu}e_{R,\alpha} \;=\; -\bar{e}_{\alpha}\gamma^{\mu}e_{\alpha}, \end{aligned} \tag{4.8}$$

where we sum over family index $\alpha = 1, \ldots, n$ and invoke the unitarity of matrices $\mathbf{S}^e_{L,R}$. Note that there is no difficulty in passing the $\mathbf{S}^e_{L,R}$ through γ^{μ} because the former matrices act in flavor space whereas the latter matrix acts in spin space.

Quark mixing

Thus far, the distinction between flavor basis states and mass eigenstates has been seen to have no apparent effect. However, mixing between generations does manifest itself in the system of quark charged weak currents,

$$J^{\mu}_{\text{ch}}(\text{qk}) = 2\bar{u}'_{L,\alpha}\gamma^{\mu}d'_{L,\alpha} = 2\bar{u}_{L,\alpha}\gamma^{\mu}\mathbf{V}_{\alpha\beta}d_{L,\beta}, \tag{4.9}$$

where

$$\mathbf{V} \equiv \mathbf{S}^{u\dagger}_L\mathbf{S}^d_L. \tag{4.10}$$

The quark-mixing matrix $\mathbf{V}$, being the product of two unitary matrices, is itself unitary. The Standard Model does not predict the content of $\mathbf{V}$. Rather, its matrix elements must be phenomenologically extracted from data. For the two-generation case, $\mathbf{V}$ is called the *Cabibbo* matrix [Ca 63]. For three generations, it has been referred to as the *Kobayashi–Maskawa* (KM) matrix [KoM 73] after its originators, but is now usually denoted by the abbreviation 'CKM'. We shall analyze properties of such mixing matrices for the remainder of this section.

An $n \times n$ unitary matrix is characterized by n^2 real-valued parameters. Of these, $n(n-1)/2$ are angles and $n(n+1)/2$ are phases. Not all the phases have physical significance, because $2n-1$ of them can be removed by *quark rephasing*. The effect of quark rephasing

$$u_{L,\alpha} \to e^{i\theta^u_{\alpha}}u_{L,\alpha}, \qquad d_{L,\alpha} \to e^{i\theta^d_{\alpha}}d_{L,\alpha} \qquad (\alpha = 1, \ldots, n) \tag{4.11}$$

on an element of the mixing matrix is

$$\mathbf{V}_{\alpha\beta} \to \mathbf{V}_{\alpha\beta}e^{i(\theta^d_{\beta}-\theta^u_{\alpha})} \qquad (\alpha, \beta = 1, \ldots, n). \tag{4.12}$$

Since an overall common rephasing does not affect $\mathbf{V}$, only the $2n-1$ remaining transformations of the type in Eq. (4.11) are effective in removing complex phases. This leaves $\mathbf{V}$ with $(n-1)(n-2)/2$ such phases. One must be careful to also

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transform the right-chirality fields of a given flavor in like manner to keep masses real. If so, all terms in the lagrangian other than **V** are unaffected by this procedure.

For two generations, there are no complex phases. The only parameter is commonly taken to be the *Cabibbo angle* θ_C and we write

$$\mathbf{V} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}. \tag{4.13}$$

A common notation for the $n = 2$ mixed states is

$$\begin{pmatrix} d_C \\ s_C \end{pmatrix} \equiv \mathbf{V} \begin{pmatrix} d \\ s \end{pmatrix}. \tag{4.14}$$

Within the two-generation approximation, weak interaction decay data imply the numerical value, $\sin\theta_C \simeq 0.226$.

The three-generation case involves the 3×3 matrix

$$\mathbf{V} = \begin{pmatrix} V_{\rm ud} & V_{\rm us} & V_{\rm ub} \\ V_{\rm cd} & V_{\rm cs} & V_{\rm cb} \\ V_{\rm td} & V_{\rm ts} & V_{\rm tb} \end{pmatrix}, \tag{4.15}$$

which is a form that emphasizes the physical significance of each matrix element. The $n = 3$ mixing matrix can be expressed in terms of four parameters, of which one is a complex phase. The presence of a complex phase is highly significant because it signals the existence of *CP* violation in the theory. We shall return to this point shortly. The KM representation employs three mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ and a complex phase δ. It can be viewed as the following Eulerian construction of three matrices,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.16}$$

where $s_{\alpha\beta} \equiv \sin\theta_{\alpha\beta}$, $c_{\alpha\beta} \equiv \cos\theta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$. In combined form this becomes

$$\mathbf{V} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{4.17}$$

By means of quark rephasing, it can be arranged that the angles $\{\theta_{\alpha\beta}\}$ all lie in the first quadrant. In the limit $\theta_{23} = \theta_{13} = 0$, KM mixing reduces to Cabibbo mixing with the identification $\theta_{12} = \theta_C$.

An alternative approach for describing the quark mixing matrix, the *Wolfenstein* parameterization [Wo 83], expresses the mixing matrix as the unit 3×3 matrix together with a perturbative hierarchical structure organized by a smallness parameter λ. In the updated version, the Wolfenstein representation contains four parameters λ, A, ρ, η defined by

$$s_{12} \equiv \lambda, \quad s_{23} \equiv A\lambda^2, \quad s_{13}e^{-i\delta} \equiv A\lambda^3(\rho - i\eta). \tag{4.18}$$

These definitions hold to all orders in λ. Since many phenomenological applications require accuracy to the level of order λ^5, we write

$$\mathbf{V} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{4} & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda + \frac{A^2\lambda^5}{2}(1 - 2(\rho + i\eta)) & 1 - \frac{\lambda^2}{2} - \frac{\lambda^4}{8}(1 + 4A^2) & \lambda^2 A \\ \lambda^3 A(1 - \bar{\rho} - i\bar{\eta}) & -\lambda^2 A + \frac{A\lambda^4}{2}(1 - 2(\rho + i\eta)) & 1 - \frac{A^2\lambda^4}{2} \end{pmatrix}. \tag{4.19}$$

Observe that the matrix element V_{td} is expressed in terms of $\bar{\rho} \equiv \rho(1 - \lambda^2/2)$ and $\bar{\eta} \equiv \eta(1 - \lambda^2/2)$. These quantities, which are useful in generalizing the so-called unitarity triangle (cf. Sect. XIV–5) beyond leading order, are directly cited in modern fits of the CKM matrix.

Attempts to theoretically predict the content of the CKM matrix have not borne fruit. The CKM matrix elements have come to be thought of as basic quantities, much like particle masses and interaction coupling constants. As such, each matrix element must be determined experimentally (with several experiments per matrix element). This endeavor, which has been a preoccupation of 'Flavor Physics' for many years, has finally reached an acceptable level of sensitivity, particularly with the operation of several B-factories (cf. Chap. XIV). Current values [RPP 12] for the Wolfenstein parameters are

$$\begin{aligned} \lambda &= 0.2257^{+0.0008}_{-0.0010}, \qquad & A &= 0.814^{+0.021}_{-0.022}, \\ \bar{\rho} &= 0.135^{+0.031}_{-0.016}, & \bar{\eta} &= 0.349^{+0.015}_{-0.017}. \end{aligned} \tag{4.20a}$$

Alternatively, we have for the original parameter set,

$$\begin{aligned} s_{12} &= 0.2257^{+0.0008}_{-0.0010}, \qquad & s_{23} &= 0.0415^{+0.0014}_{-0.0015}, \\ s_{13} &= 0.0036^{+0.0004}_{-0.0003}, & \delta &= \left(68.9^{+3.0}_{-5.4}\right)^o. \end{aligned} \tag{4.20b}$$

Neutrino mixing

Flavor mixing affects not only the quarks, but also the leptons, in the form of neutrino mixing. Just as the 3×3 quark mixing matrix $\mathbf{V}$ is associated with the acronym 'CKM', there will be a 3×3 lepton mixing matrix $\mathbf{U}$ for neutrinos. Its

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standard acronym 'PMNS', acknowledges the early work of Pontecorvo [Po 68] and of Maki, Nakagawa, and Sakata [MaNS 62]. As we will show in Sect. VI–2, when we include the possibility of a Majorana nature of neutrino mass, lepton mixing has a form very similar to quark mixing,

$$\mathbf{U} = \mathbf{V}^{(\nu)}\mathcal{P}_\nu, \tag{4.21}$$

where $\mathbf{V}^{(\nu)}$ has the same mathematical content as the quark mixing matrix $\mathbf{V}$ of Eq. (4.17) except that the mixing angles $\{\theta_{ij}\}$ and phase δ now pertain to the neutrino sector and

$$\mathcal{P}_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_1/2} & 0 \\ 0 & 0 & e^{i\alpha_2/2} \end{pmatrix}. \tag{4.22}$$

Here, the $\{\alpha_i\}$ are so-called Majorana phases. They are physical, i.e., observable, if the Majorana neutrino option is chosen by Nature. Although not contributing to neutrino oscillations, they will occur in the neutrinoless double beta decay (cf. Sect. VI–5) of certain nuclei.

Thus far, information about the lepton mixing matrix has come from fits to neutrino oscillation data (although highly anticipated searches for neutrinoless double beta decay are underway, cf. Sect. VI–5). There is no evidence at this time for *CP* violation in the lepton sector, so one cannot yet distinguish between the Dirac and Majorana cases described above. For either, the leptonic mixing angles are measured to be [RPP 12][13]

$$\begin{aligned} \sin^2(2\theta_{12}) &= 0.857 \pm 0.024, \\ \sin^2(2\theta_{23}) &\geq 0.95 \quad (\text{at } 90\%\text{C.L.}), \\ \sin^2(2\theta_{13}) &= 0.098 \pm 0.013, \end{aligned} \tag{4.23}$$

which translates into angles (here we take the value for θ_{23} from Table 13.7 of [RPP 12])

$$\theta_{12} = (33.9 \pm 1.0)^{\text{o}}, \quad \theta_{23} = \left(40.4^{+4.6}_{-1.8}\right)^{\text{o}}, \quad \theta_{13} = (9.1 \pm 0.6)^{\text{o}}. \tag{4.24}$$

These values are quite different from the quark mixing angles inferred from Eq. (4.20b). Whereas the quark mixing matrix is a 'zeroth-order' unit 3×3 matrix modified by perturbative entries proportional to powers of the smallness parameter λ, current data for lepton mixing are consistent with the 'zeroth-order' representation[14]

[13] See also [FoTV 12] and [FoLMMPR 12].
[14] This form, referred to as tri-bimaximal mixing [HaPS 02], is used here as simply a numerical convenience.

$$\mathbf{V}_0^{(\nu)} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.25}$$

Perturbative modifications of this can be introduced as

$$\sin\theta_{13} = \frac{r}{\sqrt{2}}, \qquad \sin\theta_{12} = \frac{1+s}{\sqrt{3}}, \qquad \sin\theta_{23} = \frac{1+a}{\sqrt{2}}, \tag{4.26}$$

where the r, s, a parameters are sensitive, in part, to reactor, solar, and atmospheric data, yielding $\mathbf{V}_0^{(\nu)} \rightarrow \mathbf{V}^{(\nu)}$, where [KiL 13]

$$\mathbf{V}^{(\nu)} = \begin{pmatrix} \frac{2}{\sqrt{6}}\left(1-\frac{s}{2}\right) & \frac{1}{\sqrt{3}}(1+s) & \frac{r}{\sqrt{2}}e^{-i\delta} \\ -\frac{1}{\sqrt{6}}\left(1+s-a+re^{i\delta}\right) & \frac{1}{\sqrt{3}}\left(1-\frac{s}{2}-a-\frac{1}{2}re^{i\delta}\right) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}\left(1+s+a-re^{i\delta}\right) & -\frac{1}{\sqrt{3}}\left(1+\frac{s}{2}+a+\frac{r}{2}e^{i\delta}\right) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix}. \tag{4.27}$$

In the above δ is the phase parameter which reflects the possibility of *CP* violation in the lepton sector but for which there is, as of yet, no evidence. For the others, the current limits

$$r = 0.22 \pm 0.01, \qquad s = -0.03 \pm 0.03, \qquad a = 0.10 \pm 0.05, \tag{4.28}$$

imply that we are not very far from this tri-bimaximal form.

Quark CP violation and rephasing invariants

There is no unique parameterization for three-generation quark mixing. Any scheme which is convenient to the situation at hand may be employed as long as it is used consistently and adheres to the underlying principles. There is, however, a somewhat different logical position to adopt, that of working solely with *rephasing invariants*. After all, only those functions of $\mathbf{V}$ which are invariant under the rephasing operation in Eq. (4.18) can be observable. An obvious set of quadratic invariants are the squared moduli $\Delta_{ij}^{(2)} \equiv |V_{ij}|^2$ where $i, j = 1, 2, 3$. The unitarity conditions $\mathbf{V}^\dagger\mathbf{V} = \mathbf{V}\mathbf{V}^\dagger = \mathbf{I}$ constrain the number of independent squared-moduli to four. They are of course all real-valued. In addition there are quartic functions $\Delta_{ab}^{(4)} \equiv V_{ij}V_{kl}V_{il}^*V_{kj}^*$, where we suspend the summation convention for repeated indices and, to avoid redundant factors of squared-moduli, take a, i, k (b, j, l) cyclic. There are yet higher-order invariants, but they are all expressible in terms of the quadratic and quartic functions. The nine quantities $\Delta_{ab}^{(4)}$ are generally

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complex-valued. A unique measure of CP violations for three generations is provided by the rephasing invariant $\mathrm{Im}\Delta^{(4)}_{ab}$,

$$\mathrm{Im}\,[V_{ij}V_{kl}V^*_{il}V^*_{kj}] = J\sum_{m,n}\epsilon_{ikm}\,\epsilon_{jln}, \tag{4.29}$$

where J is the so-called Jarlskog invariant [Ja 85],

$$J = c_{12}c^2_{13}c_{23}s_{12}s_{13}s_{23}s_\delta = \lambda^6 A^2\bar{\eta} + \mathcal{O}(\lambda^8) = \left(2.96^{+0.20}_{-0.16}\right)\times 10^{-5}. \tag{4.30}$$

This combination of quark mixing parameters will always appear in calculations of CP-violating phenomena. To have nonzero CP violating effects, the KM angles must avoid the values $\theta_{ij} = 0,\ \pi/2$, and $\delta = 0,\ \pi$. The CP-violating invariant J achieves its maximum value for $c_{13} = 2/\sqrt{3},\ c_{12} = c_{23} = 1/\sqrt{2},\ s_\delta = 1$ at which it equals $1/6\sqrt{3}$. This set of circumstances is very unlike the real-world value in Eq. (4.30).

The consideration of rephasing invariants need not involve just the mixing matrix $\mathbf{V}$, but can also be applied to the $Q = 2/3,\ -1/3$ nondiagonal mass matrices $\mathbf{m}'_u,\ \mathbf{m}'_d$ themselves. In particular, the determinant of their commutator is found to provide an invariant measure of CP violations [Ja 85]. If, for simplicity, we work in a basis where $\mathbf{m}_u{}'$, $\mathbf{m}'_d$ are hermitian, it can be shown that $\mathbf{S}^{u,d}_L = \mathbf{S}^{u,d}_R \equiv \mathbf{S}^{u,d}$. Thus we have

$$[\mathbf{m}'_u, \mathbf{m}'_d] = \mathbf{S}^u\,[\mathbf{m}_u, \mathbf{V}\mathbf{m}_d\mathbf{V}^\dagger]\,\mathbf{S}^{u\dagger} = \mathbf{S}^u\mathbf{V}\,[\mathbf{V}^\dagger\mathbf{m}_u\mathbf{V}, \mathbf{m}_d]\,\mathbf{V}^\dagger\mathbf{S}^{u\dagger}, \tag{4.31}$$

from which it follows that

$$\det\,[\mathbf{m}'_u, \mathbf{m}'_d] = \det\,[\mathbf{m}_u, \mathbf{V}\mathbf{m}_d\mathbf{V}^\dagger] = \det\,[\mathbf{V}^\dagger\mathbf{m}_u\mathbf{V}, \mathbf{m}_d]. \tag{4.32}$$

The two commutators on the right-hand sides of this relation are skew-hermitian and each of their matrix elements is multiplied by a $Q = 2/3, -1/3$ quark mass difference, respectively. The determinant is thus proportional to the product of all $Q = 2/3,\ -1/3$ quark mass differences, and explicit evaluation reveals

$$\det\,[\mathbf{m}'_u, \mathbf{m}'_d] = 2i\ \mathrm{Im}\Delta^{(4)}\prod_{\alpha>\beta}(m_{u,\alpha} - m_{u,\beta})(m_{d,\alpha} - m_{d,\beta}). \tag{4.33}$$

This provides a more extensive list of necessary conditions for CP violations to be present. Not only are the mixing angles constrained as discussed above, but also the quark masses within a given charge sector must not exhibit degeneracies.

Our discussion of the $n = 2, 3$ generation cases suggests how larger systems $n = 4, \ldots$ can be addressed, although the number of parameters becomes formidable, e.g., four generations require six mixing angles and three complex phases. However, existing data indicate the existence of just three fermion generations, e.g., measurements from Z^0-decay fail to see additional neutrinos, and there is no

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evidence from e^+e^- or $\overline{p}p$ collisions for additional quarks or leptons lighter than roughly 50 GeV. Moreover, if there were more than three quark generations the full quark-mixing matrix would be unitary, but one would expect to see violations of unitarity in any submatrix. Yet to the present level of sensitivity, the 3×3 KM mixing matrix obeys the unitarity constraint. The most accurate data occur in the $(V^\dagger V)_{11} = 1$ sector. Here, the contribution from $V_{\rm ub}$ is negligible and one finds [Ma 11, HaT 10]

$$(V^\dagger V)_{11} = |V_{\rm ud}|^2 + |V_{\rm us}|^2 + |V_{\rm ub}|^2 = 0.9999(6). \tag{4.34}$$

Problems

(1) **SU(3)**

(a) Starting from the general form $\lambda^a_{ij}\lambda^a_{kl} = A\delta_{ij}\delta_{kl} + B\delta_{il}\delta_{jk} + C\delta_{ik}\delta_{jl}$ ($a = 1, \ldots, 8$ is summed), determine A, B, C by using the trace relations of Eqs. (II–2.5a, 2.5b), etc.

(b) Determine $\mathrm{Tr}\,\lambda^a\lambda^b\lambda^c$.

(c) Determine $\epsilon_{ijk}\epsilon_{lmn}\lambda^a_{mp}\lambda^b_{pk}\lambda^a_{ni}\lambda^b_{lj}$.

(d) Consider the 8×8 matrices $(F_a)_{bc} = -if_{abc}$, where the $\{f_{abc}\}$ are $SU(3)$ structure constants ($a, b, c = 1, \ldots, 8$). Show that these matrices (the *regular* or *adjoint* representation) obey the Lie algebra of $SU(3)$, and determine $\mathrm{Tr}\,F_aF_b$.

(2) **Gauge invariance and the *QCD* interaction vertices**

(a) Define constants f, g such that the covariant derivative of quark q is $D_\mu q = (\partial_\mu + ifA_\mu)q$ and the *QCD* gauge transformations are $A_\mu \to UA_\mu U^{-1} + ig^{-1}U\partial_\mu U^{-1}$ and $q \to Uq$, where A_μ are the gauge fields in matrix form (cf. Eq. (I–5.15)). Show that $\bar{q}\not{D}\,q$ is invariant under a gauge transformation only if $f = g$.

(b) Define a constant h such that the *QCD* field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ih[A_\mu, A_\nu]$. Let the gauge transformation for A_μ be as in (a). Show that $F_{\mu\nu}$ transforms as $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$ only if $h = g$.

(3) **Fermion self-energy in *QED* and *QCD***

(a) Express the fermion *QED* self-energy, $-i\Sigma(p)$, of Fig. II–2(b) as a Feynman integral and use dimensional regularization to verify the forms of $Z_2^{(\rm MS)}$, $\delta m^{(\rm MS)}$ appearing in Eqs. (1.34), (1.35).

(b) Proceed analogously to determine Z_2 for *QCD* and thus verify Eq. (2.55).

(4) **Gravity as a gauge theory**

The only force which remains outside of the present Standard Model is gravity. General relativity is also a gauge theory, being invariant under local-coordinate

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transformations. The full theory is too complex for presentation here, but we can study the weak field limit. In general relativity the metric tensor becomes a function of spacetime, with the weak field limit being an expansion around flat space, $g^{\mu\nu}(x) = g^{\mu\nu} + h^{\mu\nu}(x)$, with $1 \gg h^{\mu\nu}$. Let us consider weak field gravity coupled to a scalar field, with lagrangian $\mathcal{L} = \mathcal{L}_{\rm grav} + \mathcal{L}_{\rm matter}$ defined as

$$\mathcal{L}_{\rm grav} = \frac{1}{64\pi G_N}\left[\partial_\lambda h_{\mu\nu}\partial^\lambda \bar{h}^{\mu\nu} - 2\partial^\lambda \bar{h}_{\mu\lambda}\partial_\sigma \bar{h}^{\mu\sigma}\right],$$
$$\mathcal{L}_{\rm matter} = \frac{1}{2}\left(1 - \frac{1}{2}h^\lambda_\lambda\right)\left[(g^{\mu\nu} + h^{\mu\nu})\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right],$$

where $\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - g^{\mu\nu}h^\lambda_\lambda/2$, all indices are raised and lowered with the flat space metric $g^{\mu\nu}$, and G_N is the Cavendish constant.

(a) Show that the action is invariant under the action of an infinitesimal coordinate translation, $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$ $(1 \gg \epsilon^\mu(x))$, together with a gauge change on $h^{\mu\nu}$,

$$\varphi(x) \to \varphi'(x') = \varphi(x),$$
$$h^{\mu\nu}(x) \to h'^{\mu\nu}(x') = h^{\mu\nu}(x) + \partial^\mu\epsilon^\nu(x) + \partial^\nu\epsilon^\mu(x).$$

Note: both ϵ^μ and $h^{\mu\nu}$ are infinitesimal and should be treated to first order only.

(b) Obtain the equations of motion for φ and $h^{\mu\nu}$. The source term for $h^{\mu\nu}$ is $T^{\mu\nu}$, the energy-momentum tensor for φ. Use the equation of motion for $h^{\mu\nu}$ to show that $T^{\mu\nu}$ is conserved. Simplify the equations with the choice of 'harmonic gauge', $\partial_\nu\bar{h}^{\mu\nu} = 0$.

(c) Solve for $h^{\mu\nu}$ near a point mass at rest, corresponding to $T^{00} = M\delta^{(3)}(x)$ and $T^{0i} = T^{ij} = 0$. Perform a nonrelativistic reduction for φ, i.e., $\varphi(x, t) = e^{-imt}\tilde{\varphi}(x, t)$, in order to obtain a Schrödinger equation for $\tilde{\varphi}$ in the gravitational field. Verify that Newtonian gravity is reproduced.

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