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ON RINGS GENERATING ATOMS OF LATTICES OF SPECIAL AND SUPERNILPOTENT RADICALS

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This note is to indicate a nonsemiprime ring R such that the smallest supernilpotent (respectively special) radical containing the ring R is an atom of the lattice of all supernilpotent (respectively special) radicals. This gives a positive answer to Puczylowski's and Roszkowska's question.

It is known [1] that the collections of all supernilpotent (that is hereditary and containing the prime radical β) and all special radicals of associative rings form complete lattices. We denote these lattices by K and Sp respectively. The smallest supernilpotent (respectively special) radical containing a ring A will be denoted by $\overline{1}A$ (respectively $\widehat{1}A$).

It is easy to check that if s is a supernilpotent radical then the class of all prime and s-semisimple rings is a special class and the upper radical $\widehat{s}$ determined by this class is the smallest special radical containing s.

A prime ring $R \neq 0$ is called a $*$-ring [2, 3, 4] if for every non-zero ideal I of R, the factor ring R/I is β-radical.

The problem of a description of atoms in K and Sp was raised in [1]. Then it was studied in [2, 3, 4, 5]. Among others, the following was proved:

PROPOSITION 1. [5, Proposition 12]. *If s is an atom of K then $\widehat{s}$ is an atom of Sp.*

PROPOSITION 2. [3, Theroem 1]. *If A is a *-ring then $\overline{1}A$ is an atom of K.*

PROPOSITION 3. [4, Theorem 1]. *If A is a *-ring then $\widehat{1}A$ is an atom of Sp.*

PROPOSITION 4. [3, Proposition 1]. *If A is a *-ring then a ring R is an $\overline{1}A$-radical if and only if every non-zero semiprime homomorphic image of R has a non-zero ideal isomorphic to an accessible subring of A.*

In view of the abovementioned results, Puczylowski and Roszkowska have put a natural question [5, Question 6] whether there exists a non-*-ring R such that $\overline{1}R$ is an atom of K or $\widehat{1}R$ is an atom of Sp. The aim of this note is to give a positive answer to this question. For this purpose we use the ring R described in the following:

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EXAMPLE: Let us consider the matrix ring

$$R = \left\{ \begin{pmatrix} r & a \\ 0 & r \end{pmatrix} \;\middle|\; r, a \in A \right\},$$

where A is a simple (that is, idempotent without nontrivial ideals) ring. Evidently, the subset

$$H = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \;\middle|\; a \in A \right\}$$

is a non-zero zero-ring ideal of R with $R/H \simeq A$. Thus R is neither semiprime nor β-radical. In particular, R is not a *-ring. Also the only ideals of R are $\{0\}$, H and R. Indeed, let $I \neq 0$ be an ideal of R and $0 \neq \begin{pmatrix} r & a \\ 0 & r \end{pmatrix} \in I$. If $r = 0$ then $a \neq 0$ and so AaA is a non-zero ideal of A, since A is semiprime. But A is simple, so $AaA = A$. Thus for every $x \in A$ there exist $y, z \in A$ such that $x = \sum yaz$. Hence for every $x \in A$, we have

$$\begin{aligned} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sum yaz \\ 0 & 0 \end{pmatrix} &= \sum \begin{pmatrix} 0 & yaz \\ 0 & 0 \end{pmatrix} \\ &= \sum \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in I, \end{aligned}$$

since $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in I$ and I is an ideal of R. This implies $H \subseteq I$. Similarly, if $r \neq 0$ then there exists an element $b \in A$ such that $rb \neq 0$, since A is semiprime. Consequently $ArbA$ is a non-zero ideal of a simple ring A and thus $ArbA = A$. Hence, for every $x \in A$ there exist $y, z \in A$ such that $x = \sum y(rb)z$ and so we have

$$\begin{aligned} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sum yrbz \\ 0 & 0 \end{pmatrix} &= \sum \begin{pmatrix} 0 & yrbz \\ 0 & 0 \end{pmatrix} \\ &= \sum \begin{pmatrix} y & a \\ 0 & y \end{pmatrix} \begin{pmatrix} r & a \\ 0 & r \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in I, \end{aligned}$$

since $\begin{pmatrix} r & a \\ 0 & r \end{pmatrix} \in I$ and I is an ideal of R. Thus, also in this case, $H \subseteq I$. This, however, implies $H = I$ or $I = R$, for otherwise I/H would be a non-trivial ideal of R/H, which is impossible since $R/H \simeq A$ and A is simple. Therefore $R/H \simeq A$ is the only non-zero semiprime homomorphic image of R and, obviously, A is an accessible subring of itself. Therefore, by Proposition 4, R is $\overline{1}A$-radical. Consequently $\overline{1}A \subseteq \overline{1}A$. On the other hand, $\beta \not\subseteq \overline{1}R$, since R is a non-zero $\overline{1}R$-ring and R is not a β-ring. This, and the fact that $\overline{1}A$ is an atom of K, by Proposition 2, implies $\overline{1}R = \overline{1}A$. Hence $\overline{1}R$ is an atom of K. But then Proposition 1 gives immediately that $\widehat{1}R$ is an atom of Sp which ends the proof. ∎

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