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CLIQUES OF IRREDUCIBLE REPRESENTATIONS, QUOTIENT GROUPS, AND BRAUER'S THEOREMS ON BLOCKS

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ABSTRACT. Assume k is an algebraically closed field of characteristic p and G is a finite group. If P is a p-subgroup of G such that $G = PC_G(P)$, and if H is a normal subgroup of G with $P \leq H$, then the number of H-cliques of irreducible $k[G]$-modules is the same as the number of H/P-cliques of irreducible $k[G/P]$-modules.

1. **Introduction.** Let k be an algebraically closed field of characteristic p and let G be a finite group. The paper [9] introduced for each normal subgroup H of G a partition of the set of irreducible $k[G]$-modules into equivalence classes called *H-cliques*. Irreducible $k[G]$-modules V and W belong to the same H-clique if there is an irreducible $k[G]^H$-module M which is isomorphic to a summand of $V_{k[G]^H}$ and to a summand of $W_{k[G]^H}$. (This is an equivalence relation because there is an analog (Theorem 2.5 in [9]) of Clifford's Theorem for the restriction of an irreducible $k[G]$-module V to $k[G]^H$; as a $k[G]^H$-module, $V_{k[G]^H}$ is semi-simple with all its summands conjugate in G and all the non-isomorphic conjugates of a given irreducible $k[G]^H$-module occurring the same number of times in a decomposition of $V_{k[G]^H}$.) When $H = G$, irreducible modules are in the same H-clique if and only if they belong to the same block. When $H = 1$, irreducible modules are in the same H-clique if and only if they are isomorphic. The partition of the set of irreducible $k[G]$-modules into H-cliques is a refinement of the partition of the set of irreducible $k[G]$-modules into blocks. If we imagine starting H off equal to 1 and "moving" H up along a chain of normal subgroups of G, we should see the blocks of G gradually emerge as we look at successively coarser partitions.

In [9], it was conjectured that for each H, the number of H-cliques of irreducible $k[G]$-modules is given by the sum $\sum_Q a(Q)$, where Q runs through a set of representatives for the G-conjugacy classes of p-subgroups of G, and where, for each Q, $a(Q)$ is the number of $\big(H \cap N_G(Q)\big)$-cliques of irreducible $k[N_G(Q)]$-modules that contain only modules with vertex Q. When $H = 1$ this formula is Alperin's Conjecture [1]; when $H = G$ this formula is essentially Brauer's First Main Theorem on Blocks. (Adding Brauer's First Main Theorem over all possible defect groups gives this formula.) For a proof of this conjecture for any H in the special case when G is p-solvable, see [10]. To test this conjecture, it is necessary to develop a theory of cliques analogous as much as possible to the theory of blocks. This paper is part of such a theory.

In the three stages of what Curtis and Reiner call Brauer's Extended First Main Theorem (61.7 in [4]), Brauer associates to a block of $k[G]$ with defect group D first a

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single block of $N_G(D)$ with defect group D, then a $N_G(D)$-class of blocks of $DC_G(D)$ with defect group D, and finally a $N_G(D)$-class of blocks of $\big(DC_G(D)\big)/D$ with trivial defect group. The formula above (if it turns out to be correct) may be regarded as a rough analog for cliques of the first stage of this theorem. In this paper we prove an analog of the third stage.

The correspondence of the third part of the Extended First Main Theorem in fact works for all blocks of $DC_G(D)$, even those whose defect group is not D. If $G = DC_G(D)$, then there is a natural one-to-one correspondence between the blocks of $k[G]$ and the blocks of $k[G/D]$, a correspondence that preserves defect groups in the sense that if Q is a defect group of a block of $k[G]$, then Q/D is a defect group of the corresponding block of $k[G/D]$. (See Lemma V4.5 in [11].) In this paper, we obtain an analogous correspondence for cliques. If $P \lhd G$ and V is an irreducible $k[G]$-module, then P is in the kernel of V so V has a well-defined structure as a $k[G/P]$-module. We call this module $V_{k[G/P]}$. The goal of this paper is the following theorem.

THEOREM 1.1. *Let k be an algebraically closed field of characteristic p. Let G be a finite group. Let P be a p-subgroup of G such that $G = PC_G(P)$. Let H be a normal subgroup of G with $P \leq H$. Let V and W be irreducible $k[G]$-modules. Then V and W belong to the same H-clique if and only if $V_{k[G/P]}$ and $W_{k[G/P]}$ belong to the same H/P-clique.*

If this theorem had been available at the time [9] was written, it would have simplified considerably the induction proof of the main theorem in that paper.

The proof of this theorem requires the use of the deepest theorems from Dade's Clifford Theory, developed in the papers [5], [6], and [7].

For the remainder of the introduction, let G, H, P, and k be as in Theorem 1.1. Let V be an irreducible $k[G]$-module. Let e be a centrally primitive idempotent of $k[H]$ such that $Ve \neq 0$. Let b be the block of $k[H]$ corresponding to e. Let Φ be the k-algebra homomorphism from $k[G]$ to $k[G/P]$ that sends each $g \in G$ to its natural projection in G/P. Let $\bar{e} = \Phi(e)$ and let $\bar{b} = \Phi(b)$.

By Theorem 2.5 of [9], the module $Ve_{k[G]^H e}$ is semi-simple. Since $\mathrm{Ker}(\Phi) \cap k[G]^H e \subseteq J(k[G]^H e)$, $k[G]^H e$ and $\Phi(k[G]^H e)$ have essentially the same irreducible modules. We can discuss $Ve_{k[G]^H e}$ or $Ve_{\Phi(k[G]^H e)}$ interchangeably.

The algebra $\Phi(k[G]^H e)$ is a subalgebra of $k[G/P]^{H/P}\bar{e}$. It is sometimes a proper subalgebra. (See Section 3.11 for an example.) The proof of Theorem 1.1 is easily reduced (in Lemma 4.1) to the problem of showing that nevertheless the module $(Ve)_{k[G/P]^{H/P}\bar{e}}$ is determined by the module $(Ve)_{\Phi(k[G]^H e)}$. Since these modules are semi-simple, this means we are reduced to studying the relationship between irreducible $\Phi(k[G]^H e)$-modules and irreducible $k[G/P]^{H/P}\bar{e}$-modules. Dade's theory in [7] is ideally suited to this purpose.

To study irreducible modules over the algebra $k[G]^H e$, we consider the direct sum decomposition

$$k[G]^H e = \bigoplus_{\sigma \in G/H} (k[G]^H e)_\sigma,$$

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where, for each $\sigma \in G/H$, $(k[G]^H e)_\sigma$ is the intersection of $k[G]^H e$ with the subspace of $k[G]$ generated by the elements of σ. This decomposition makes $k[G]^H e$ into a G/H-graded algebra, but not necessarily into a fully G/H-graded algebra (see Section 2 for definitions). Let $(G/H)[b] = \{\sigma \in G/H : (k[G]^H e)_\sigma \text{ contains a unit}\}$. Exploiting the fact that $(k[G]^H e)_1$ is the center of the block $k[H]e$ and is therefore a local ring, Dade proves three things about the group $(G/H)[b]$: if $\sigma \notin (G/H)[b]$, then the component $(k[G]^H e)_\sigma$ is in $J(k[G]^H e)$; if $\sigma \in (G/H)[b]$, then $(k[G]^H e)_\sigma / \Big(J\big((k[G]^H e)_1\big)(k[G]^H e)_\sigma\Big)$ is 1-dimensional, so the algebra

$$\bigoplus_{\sigma \in (G/H)[b]} (k[G]^H e)_\sigma / \Big(J\big((k[G]^H e)_1\big)(k[G]^H e)_\sigma\Big)$$

is a twisted group algebra for $(G/H)[b]$; this twisted group algebra is isomorphic to the quotient of $k[G]^H e$ modulo a two-sided ideal in the Jacobson radical of $k[G]^H e$. It follows that if we are only interested in semi-simple $k[G]^H e$-modules, we may as well study semi-simple modules over this twisted group algebra. (See Theorem 3.2 for details.) It is tremendously useful to do this, because while the algebra $k[G]^H e$ is somewhat mysterious, there is a rich theory that we can apply to the twisted group algebra.

Of course we can perform the same construction after applying the map Φ to everything, and thus we can replace the algebra $k[G/P]^{H/P}\bar{e}$ with a twisted group algebra for the group $\big((G/P)/(H/P)\big)[\bar{b}]$. We abbreviate the name of this group to $(\bar{G}/\bar{H})[\bar{b}]$. Applying Φ to the old twisted group algebra gives a subalgebra of the new twisted group algebra. We are thus reduced to studying the relationship between the two groups $(G/H)[b]$ and $(\bar{G}/\bar{H})[\bar{b}]$. All of Section 3, the bulk of the paper, is devoted to this study. After we identify $(G/H)[b]$ with its image under the natural isomorphism $G/H \longrightarrow (G/P)/(H/P)$, $(G/H)[b]$ is a normal subgroup of of $(\bar{G}/\bar{H})[\bar{b}]$ (see Section 3.1). It may be a proper subgroup (see the example in Section 3.11). However, in Theorem 3.5 we show that $(\bar{G}/\bar{H})[\bar{b}]/(G/H)[b]$ is a p-group. It is well known that if K is a normal subgroup of a group L with L/K a p-group, then irreducible $k[L]$-modules are determined by their restriction to $k[K]$; the same is true for twisted group algebras.(See Lemma 4.2.) Since we have replaced irreducible $\Phi(k[G]^H e)$-modules with irreducible modules over Dade's twisted group algebra for $(G/H)[b]$ and irreducible $k[G/P]^{H/P}\bar{e}$-modules with irreducible modules over Dade's twisted group algebra for $(\bar{G}/\bar{H})[\bar{b}]$, it follows that for any irreducible $k[G]$-module V, the module $(Ve)_{k[G/P]^{H/P}\bar{e}}$ is determined by the module $(Ve)_{\Phi(k[G]^H e)}$. This observation completes the proof of Theorem 1.1 (See Section 4.)

Studying Dade's group $(G/H)[b]$ directly is difficult. Fortunately, it is not necessary to do so. Let D be defect group of b. The tool we use for our comparison of $(G/H)[b]$ and $(\bar{G}/\bar{H})[\bar{b}]$ is a very difficult and deep theorem from Dade's paper [7] which shows how to compute $(G/H)[b]$ in terms of a Clifford extension for the conjugacy class of irreducible representations of $C_H(D)$ associated to b by Brauer's theory. This reduces the proof of our theorem to the comparatively mechanical job of comparing two closely related Clifford extensions.

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2. **Notation and assumptions.** Throughout the paper, p is a prime number and k is an algebraically closed field of characteristic p. All algebras in this paper are assumed to be algebras over k.

If K is a subgroup of the group G, $k[G]^K = \{a \in k[G] : a^x = a \text{ for all } x \in K\}$.

Let Γ be a (possibly infinite) group. A Γ-*graded algebra* is an algebra A together with a direct sum decomposition of subspaces

$$A = \bigoplus_{\sigma \in \Gamma} A_\sigma$$

such that for all σ and τ in Γ,

$$A_\sigma A_\tau \subseteq A_{\sigma\tau},$$

where $A_\sigma A_\tau$ is the subspace generated by all products xy with $x \in A_\sigma$ and $y \in A_\tau$. The subspaces A_σ are allowed to be 0.

A *fully* Γ-*graded algebra* is a Γ-graded algebra satisfying the additional condition

$$A_\sigma A_\tau = A_{\sigma\tau}$$

for all σ and τ in Γ.

Fully Γ-graded algebras are the same as the graded Clifford systems discussed in Dade's earlier papers. For a discussion of the reason for this change in terminology, see [8].

Let N be a normal subgroup of a finite group G. Let b be a block of $k[N]$; then G_b is defined to be the the inertia group in G of b; that is, $G_b = \{x \in G : b^x = b\}$. Let φ be a $k[N]$-module; then G_φ is defined to be the inertia group in G of φ; that is, $G_\varphi = \{x \in G : \varphi^x \cong \varphi\}$.

For all of the paper, assume the following hypothesis.

HYPOTHESIS 2.1. *1. G is a finite group.*

2. P is a p-subgroup of G such that $G = PC_G(P)$.

3. H is a normal subgroup of G.

4. $\bar{G} = G/P$.

5. for any $g \in G$, $\bar{g}$ is its image under the natural projection onto G/P.

6. for any subgroup K of G, $\bar{K}$ is its image under the natural projection onto G/P.

7. $\Phi: k[G] \longrightarrow k[\bar{G}]$ is the surjective algebra homomorphism which sends each g in G to $\bar{g}$.

8. $P \leq H$.

9. b is a block of $k[H]$ corresponding to the primitive central idempotent e.

10. $\bar{b} = \Phi(b)$.

11. $\bar{e} = \Phi(e)$.

Note that by a standard theorem about blocks (see for example V4.6 in Feit[11]), $\bar{e}$ is a primitive central idempotent of $\bar{G}$ corresponding to the block $\bar{b}$.

3. **Behavior of Dade's twisted group algebra under reduction modulo P.** For all of this section, assume Hypothesis 2.1.

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3.1. *Two twisted group algebras.* Our goal is essentially to compare the irreducible modules over the two algebras $\Phi(k[G]^H)$ and $k[\bar{G}]^{\bar{H}}$, which may not be the same. It will turn out to be easier to compare instead two slightly smaller algebras, whose importance was pointed out by Dade in [7]. In this subsection we define those algebras.

It is easily checked that $\Phi(k[G]^H) \subseteq k[\bar{G}]^{\bar{H}}$, so $\Phi(k[G]^H e) \subseteq k[\bar{G}]^{\bar{H}}\bar{e}$. See Section 3.11 for an example in which $\Phi(k[G]^H e) \neq k[\bar{G}]^{\bar{H}}\bar{e}$.

Now consider the algebra $k[G]^H e$.

$$k[G]^H e = \bigoplus_{\sigma \in G/H} (k[G]^H e)_\sigma,$$

where, for each $\sigma \in G/H$, $(k[G]^H e)_\sigma$ is the intersection of $k[G]^H e$ with the subspace of $k[G]$ generated by the elements of σ. This decomposition makes $k[G]^H e$ into a G/H-graded algebra, but not necessarily into a fully G/H-graded algebra.

Some of the components of $k[G]^H e$ may be equal to 0. For instance, we have the following lemma.

LEMMA 3.1. *Let σ be in G/H with σ not in G_b/H. Then $(k[G]^H e)_\sigma = 0$.*

PROOF. See Proposition 2.17 in [7]. ∎

In his paper Block Extensions [7], Dade identifies an important subgroup $(G/H)[b]$ of G_b/H, which is defined as follows.

$$(G/H)[b] = \{\sigma \in G/H : (k[G]^H e)_\sigma (k[G]^H e)_{\sigma^{-1}} = (k[G]^H e)_1\}.$$

By Proposition 2.17 in [7], the group $(G/H)[b]$ is a normal subgroup of G_b/H. This subgroup is important because of the following theorem.

THEOREM 3.2 (DADE). *The algebra*

$$\bigoplus_{\sigma \in (G/H)[b]} (k[G]^H e)_\sigma$$

is a fully $(G/H)[b]$-graded algebra. The subspace

$$I = \bigoplus_{\sigma \in (G/H)[b]} \Big(J\big((k[G]^H e)_1\big)\Big)(k[G]^H e)_\sigma \oplus \bigoplus_{\sigma \in (G/H)\setminus(G/H)[b]} (k[G]^H e)_\sigma$$

of $k[G]^H e$ is an ideal contained in $J(k[G]^H e)$. The algebra $k[G]^H e/I$, which is isomorphic to

$$\Big(\bigoplus_{\sigma \in (G/H)[b]} (k[G]^H e)_\sigma\Big) \Big/ \Big(\bigoplus_{\sigma \in (G/H)[b]} J\big((k[G]^H e)_1\big)(k[G]^H e)_\sigma\Big),$$

is a fully $(G/H)[b]$-graded algebra with each component of dimension 1 over k. (In other words it is a twisted group algebra.)

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PROOF. The algebra $\bigoplus_{\sigma\in(G/H)[b]}(k[G]^He)_\sigma$ is fully $(G/H)[b]$-graded by Theorem 2.10 in [7]. The algebra $\left(\bigoplus_{\sigma\in(G/H)[b]}(k[G]^He)_\sigma\right) \Big/ \left(\bigoplus_{\sigma\in(G/H)[b]} J\big((k[G]^He)_1\big)(k[G]^He)_\sigma\right)$, is fully G/H-graded and has all components of dimension 1 by 2.12 of [7]. The subspace I is an ideal contained in the Jacobson radical by Lemma 3.3 of [7]. ∎

For us, the value of this theorem is that it allows us to study irreducible modules over a twisted group algebra instead of irreducible modules over $k[G]^He$.

We will need the following lemma about $(G/H)[b]$.

LEMMA 3.3. *Let σ be an element of G/H. Then $\sigma \in (G/H)[b]$ if and only if $(k[G]^He)_\sigma$ contains a unit of $k[G]^He$.*

PROOF. This follows easily from Theorem 3.2. ∎

We can perform the same construction after reduction modulo P to get a subgroup $(\bar{G}/\bar{H})[\bar{b}]$ of $\bar{G}/\bar{H}$. Let $\Psi: G/H \longrightarrow \bar{G}/\bar{H}$ be the natural isomorphism. If $\sigma \in (G/H)[b]$, then $\Phi\big((k[G]^He)_\sigma\big)$ contains a unit of $k[\bar{G}]^{\bar{H}}\bar{e}$. Therefore

$$\Psi\big((G/H)[b]\big) \leq (\bar{G}/\bar{H})[\bar{b}].$$

Since $\Psi(G_b/H) = \bar{G}_{\bar{b}}/\bar{H}$ and $(G/H)[b] \lhd G_b/H$,

$$\Psi\big((G/H)[b]\big) \lhd (\bar{G}/\bar{H})[\bar{b}].$$

The aim of the first part of this paper is to show that the quotient of these two groups has order a power of p. See Section 3.11 for an example in which their quotient is not trivial.

We will need the following elementary lemma.

LEMMA 3.4. *Let D be a p-subgroup of G with $P \leq D$. Then $\overline{C_G(D)} \lhd C_{\bar{G}}(\bar{D})$ and the quotient of these two groups is a p-group.*

PROOF. Let a be an element of $C_{\bar{G}}(\bar{D})$, and let $x \in G$ be a member of the coset a. Then $x \in N_G(D)$, so $C_G(D)^x = C_G(D)$, so $\overline{C_G(D)}^a = \overline{C_G(D)}$. Thus $\overline{C_G(D)} \lhd C_{\bar{G}}(\bar{D})$.

Suppose there is an element of $C_{\bar{G}}(\bar{D})/\overline{C_G(D)}$ of order prime to p. Since $G = PC_G(P)$, there is an element $z \in C_G(P)$ of order prime to p such that $\bar{z} \in C_{\bar{G}}(\bar{D})$ but $z \notin C_G(D)$. Then $[D,z] \subseteq P$, and so $[D,z,z] \subseteq [P,z] = 1$. By 5.3.6 in Gorenstein's book [12], it follows that $[D,z] = 1$. This contradicts the fact that $z \notin C_G(D)$. ∎

THEOREM 3.5. *The order of $(\bar{G}/\bar{H})[\bar{b}]/\Psi\big((G/H)[b]\big)$ is a power of p.*

The proof of this theorem requires a careful analysis of several Clifford extensions. Before giving the proof, we will define the extensions and give a number of lemmas about the relationship between them.

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3.2. *Clifford extensions.* We need to recall the definition of Clifford extensions. This concept was introduced by Dade in [5].

Let K be a (possibly infinite) group and let N be a normal subgroup of K. Let φ be an irreducible (finite dimensional) $k[N]$-module. We will now define a central extension

$$1 \longrightarrow k^* \longrightarrow K\langle\varphi\rangle \longrightarrow K_\varphi/N \longrightarrow 1,$$

which is called the *Clifford extension* associated to K and φ.

Let $\mathcal{M}(\varphi)$ be the annihilator in $k[N]$ of φ. Conjugation by elements of K_φ fixes $\mathcal{M}(\varphi)$; thus $k[K_\varphi]\mathcal{M}(\varphi)$ is a two-sided ideal of $k[K_\varphi]$. We have the following direct sum decompositions.

$$\begin{aligned} k[K_\varphi] &= \bigoplus_{\sigma\in K_\varphi/N} (k[K_\varphi])_\sigma \\ k[K_\varphi]\mathcal{M}(\varphi) &= \bigoplus_{\sigma\in K_\varphi/N} (k[K_\varphi])_\sigma\mathcal{M}(\varphi) \\ k[K_\varphi]/k[K_\varphi]\mathcal{M}(\varphi) &= \bigoplus_{\sigma\in K_\varphi/N} (k[K_\varphi])_\sigma/\big((k[K_\varphi])_\sigma\mathcal{M}(\varphi)\big) \end{aligned}$$

The 1-component of this last algebra is $k[N]/\mathcal{M}(\varphi)$, which is a full $n\times n$ matrix algebra with entries in k, where n is the dimension of φ over k. Let C be the centralizer of the 1-component in $k[K_\varphi]/k[K_\varphi]\mathcal{M}(\varphi)$. Then it can easily be shown that in the grading

$$C = \bigoplus_{\sigma\in K_\varphi/N} C_\sigma,$$

each component contains a unit and each component is 1-dimensional over k. (Thus C is a twisted group algebra for K_φ/N.) We define $K\langle\varphi\rangle$ to be the group of all units of C that are contained in one of the subspaces of the grading. The embedding $\lambda\longmapsto\lambda 1$ of k^* in $K\langle\varphi\rangle$ makes this a central extension as in the above diagram.

Note that if L is a subgroup of K with $N\le L$, then $L\langle\varphi\rangle$ can be identified with the inverse image in $K\langle\varphi\rangle$ of L_φ/N.

Section 12 of Dade's paper Block Extensions [7] shows how to compute the group $(G/H)[b]$ in terms of the Clifford extensions of certain irreducible modules. To use this result, we will need to compare several Clifford extensions.

3.3. *The extension $N_G(D)\langle\varphi\rangle$ and the form ω.* Let D be a defect group of the block b. Let β be a block of $DC_H(D)$ that has defect group D and is sent to b by the Brauer correspondence. (The block β is unique up to conjugacy in $N_H(D)$.) There is a unique irreducible module in β; D is in the kernel of that module; let φ be the restriction to $k[C_H(D)]$ of that module. (Of course φ is irreducible.) One of the Clifford extensions we want to consider is

$$1 \longrightarrow k^* \longrightarrow N_G(D)\langle\varphi\rangle \longrightarrow N_G(D)_\varphi/C_H(D) \longrightarrow 1.$$

To this extension, we associate a "bilinear form" ω, which is defined as follows. Conjugation of elements of $C_G(D)\langle\varphi\rangle$ by elements of $N_H(D)\langle\varphi\rangle$ defines an action of

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$N_H(D)_\varphi$ on $C_G(D)\langle\varphi\rangle$ which is trivial on the central subgroup k^* and on the quotient $C_G(D)_\varphi / C_H(D)$. (See the last paragraph of page 201 of Dade's paper [7] for an explanation of why the action is trivial on the quotient.) We define ω by

$$\omega: N_H(D)_\varphi \times C_G(D)_\varphi / C_H(D) \longrightarrow k^*$$

where

$$(y_\tau)^\sigma = \omega(\sigma, \tau)y_\tau$$

for all $\sigma \in N_H(D)_\varphi, \tau \in C_G(D)_\varphi / C_H(D)$, and $y_\tau \in C_G(D)\langle\varphi\rangle$ such that y_τ projects onto τ. (It is easily checked that this does not depend on the choice of y_τ.) The form is "bilinear" in the following sense: for all σ_1 and σ_2 in $N_H(D)_\varphi$ and for all τ in $C_G(D)_\varphi / C_H(D)$, $\omega(\sigma_1\sigma_2, \tau) = \omega_1(\sigma_1, \tau)\omega_1(\sigma_2, \tau)$; for all σ in $N_H(D)_\varphi$ and for all τ_1 and τ_2 in $C_G(D)_\varphi / C_H(D)$, $\omega(\sigma, \tau_1\tau_2) = \omega(\sigma, \tau_1)\omega_1(\sigma, \tau_2)$.

By the main theorem of Section 12 of Dade's paper [7] (see (0.3b) in the introduction of [7]), we can use the form ω to compute $(G/H)[b]$. We define the group $C_G(D)_\omega$ to be the preimage in $C_G(D)_\varphi$ of the right kernel of ω; that is, $C_G(D)_\omega = \{a \in C_G(D)_\varphi : \omega\big(\sigma, aC_H(D)\big) = 1 \text{ for all } \sigma \in N_H(D)_\varphi\}$. Then, by Corollary 12.6 (or statement(0.3b)) of [7],

$$(G/H)[b] = C_G(D)_\omega H/H.$$

3.4. *The extension $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle$.* Let $\bar{\beta} = \Phi(\beta)$ and let $\bar{\varphi}$ be the irreducible module over $k[\overline{C_H(D)}]$ corresponding to φ. Note that $N_{\bar{G}}(\bar{D}) = \overline{N_G(D)}$ and that $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} = \overline{N_G(D)_\varphi}$. The second Clifford extension we wish to consider is

$$1 \longrightarrow k^* \longrightarrow N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle \longrightarrow N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)} \longrightarrow 1.$$

We now compare the Clifford extensions $N_G(D)\langle\varphi\rangle$ and $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle$.

LEMMA 3.6. *There is a commutative diagram as follows.*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & N_G(D)\langle\varphi\rangle & \longrightarrow & N_G(D)_\varphi / C_H(D) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle & \longrightarrow & N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)} & \longrightarrow & 1,
\end{array}$$

where the vertical arrow on the right is the natural map.

It should be noted that the second and third vertical arrows are not isomorphisms; they both have kernels isomorphic to $P/\big(P \cap Z(D)\big)$.

PROOF. The map Φ sends the algebra $k[N_G(D)_\varphi]$ onto the algebra $k[N_{\bar{G}}(\bar{D})_{\bar{\varphi}}]$. The algebra $k[N_G(D)_\varphi]$ is fully $N_G(D)_\varphi / C_H(D)$-graded and hence is also $N_G(D)_\varphi / PC_H(D)$-graded; the algebra $k[N_{\bar{G}}(\bar{D})_{\bar{\varphi}}]$ is fully $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$-graded; the map Φ induces an isomorphism from $N_G(D)_\varphi / PC_H(D)$ to $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$ and respects the gradings.

The map Φ sends the annihilator $\mathcal{M}(\varphi)$ of φ in $k[C_H(D)]$ into the annihilator $\mathcal{M}(\bar{\varphi})$ of $\bar{\varphi}$ in $k[\overline{C_H(D)}]$. Thus Φ induces a map

$$k[N_G(D)_\varphi]/k[N_G(D)_\varphi]\mathcal{M}(\varphi) \longrightarrow k[N_{\bar{G}}(\bar{D})_{\bar{\varphi}}]/k[N_{\bar{G}}(\bar{D})_{\bar{\varphi}}]\mathcal{M}(\bar{\varphi}).$$

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The 1-component of the left hand side with respect to the $N_G(D)_\varphi / C_H(D)$-grading is a full $n \times n$ matrix algebra with entries in k, where n is the dimension of φ over k. The 1-component of the left hand side with respect to the $N_G(D)_\varphi / C_H(D)$-grading must be mapped into the 1-component of the right hand side with respect to the $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$-grading. The 1-component of the right hand side with respect to the $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$-grading is $k[\overline{C_H(D)}] / \mathcal{M}(\bar{\varphi})$, which is isomorphic to the k-endomorphism algebra of $\bar{\varphi}$. Since the dimension over k of $\bar{\varphi}$ is the same as the dimension over k of φ, the 1-component of the right hand side with respect to the $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$-grading is also a full $n \times n$ matrix algebra with entries in k. Since a full matrix algebra is simple, the 1-component of the left hand side with respect to the $N_G(D)_\varphi / C_H(D)$-grading must be mapped *onto* the 1-component of the right hand side with respect to the $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$-grading.

Therefore Φ induces a map that respects gradings from the centralizer in $k[N_G(D)_\varphi] / k[N_G(D)_\varphi]\mathcal{M}(\varphi)$ of the 1-component with respect to the $N_G(D)_\varphi / C_H(D)$-grading onto the centralizer in $k[N_{\bar{G}}(\bar{D})_{\bar{\varphi}}] / k[N_{\bar{G}}(\bar{D})_{\bar{\varphi}}]\mathcal{M}(\bar{\varphi})$ of the 1-component with respect to the $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} / \overline{C_H(D)}$-grading. The restriction of this map to the appropriate groups of units is the center vertical arrow of the diagram. ∎

3.5. *The extension $N_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$ and the form ω_2.*

LEMMA 3.7. *There is a unique block $\hat{\beta}$ of $k[\bar{D}C_{\bar{H}}(\bar{D})]$ which covers $\bar{\beta}$. The block $\hat{\beta}$ has the following properties.*

1. *$\hat{\beta}$ has defect group $\bar{D}$.*
2. *$\hat{\beta}^{\bar{H}} = \bar{b}$.*
3. *Let $\hat{\varphi}$ be the restriction to $k[C_{\bar{H}}(\bar{D})]$ of the unique irreducible module in $\hat{\beta}$. Then $\hat{\varphi}$ is the unique irreducible $k[C_{\bar{H}}(\bar{D})]$-module covering $\bar{\varphi}$.*

PROOF. Since $\bar{D}C_{\bar{H}}(\bar{D}) / \overline{DC_H(D)}$ is a p-group (by Lemma 3.4), it follows from V3.5 of Feit's book [11] that there is exactly one block of $\bar{D}C_{\bar{H}}(\bar{D})$ covering $\bar{\beta}$. Let this block be $\hat{\beta}$.

First, we show that $\hat{\beta}$ has properties 1 and 2. Recall b is a block of $k[H]$ with defect group D. By Brauer's First Main Theorem on Blocks, there is a unique block B of $k[N_H(D)]$ with defect group D such that $B^H = b$. The block β was chosen from the unique $N_H(D)$-conjugacy class of blocks of $DC_H(D)$ covered by B. Necessarily D is a defect group of β. Now consider blocks of $k[\bar{H}]$. The block $\bar{b}$ has defect group $\bar{D}$ and the block $\bar{B}$ ($= \Phi(B)$) is the unique block of $k[N_{\bar{H}}(\bar{D})]$ with defect group $\bar{D}$ such that $\bar{B}^{\bar{H}} = \bar{b}$. Since B covers β, $\bar{B}$ covers $\bar{\beta}$. It follows that at least one of the members of the $N_{\bar{H}}(\bar{D})$-conjugacy class of blocks of $k[\bar{D}C_{\bar{H}}(\bar{D})]$ covered by $\bar{B}$ must cover $\bar{\beta}$. This block must be $\hat{\beta}$, so $\hat{\beta}$ has defect group $\bar{D}$ and $\hat{\beta}^{\bar{H}} = \bar{b}$.

Now we show 3. Let $\hat{\varphi}$ be defined as in 3. Because $\bar{D}$ is in the kernel of the unique irreducible module in $\hat{\beta}$, $\hat{\varphi}$ is irreducible. Clearly $\hat{\varphi}$ covers $\bar{\varphi}$. Since $C_{\bar{H}}(\bar{D}) / \overline{C_H(D)}$ is a p-group (by Lemma 3.4), III3.15 in Feit's book [11] shows that there is no other irreducible $k[C_{\bar{H}}(\bar{D})]$-module covering $\bar{\varphi}$. ∎

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Since $\hat{\varphi}$ is the unique irreducible $k[C_{\bar{H}}(\bar{D})]$-module covering $\bar{\varphi}$, $N_{\bar{H}}(\bar{D})_{\bar{\varphi}} \leq N_{\bar{H}}(\bar{D})_{\hat{\varphi}}$ and $N_{\bar{G}}(\bar{D})_{\bar{\varphi}} \leq N_{\bar{G}}(\bar{D})_{\hat{\varphi}}$. By (0.1b) of [5] (or by an easy calculation), $N_{\bar{H}}(\bar{D})_{\hat{\varphi}} = N_{\bar{H}}(\bar{D})_{\bar{\varphi}} C_{\bar{H}}(\bar{D})$ and $N_{\bar{G}}(\bar{D})_{\hat{\varphi}} = N_{\bar{G}}(\bar{D})_{\bar{\varphi}} C_{\bar{H}}(\bar{D})$.

The third Clifford extension we want to consider is

$$1 \longrightarrow k^* \longrightarrow N_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle \longrightarrow N_{\bar{G}}(\bar{D})_{\hat{\varphi}}/C_{\bar{H}}(\bar{D}) \longrightarrow 1.$$

To this extension we associate a "bilinear form" ω_2, which is defined as follows: Conjugation of elements of $C_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$ by elements of $N_{\bar{H}}(\bar{D})\langle\hat{\varphi}\rangle$ defines an action of $N_{\bar{H}}(\bar{D})_{\hat{\varphi}}$ on $C_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$ which is trivial on the central subgroup k^* and on the quotient $C_{\bar{G}}(\bar{D})_{\hat{\varphi}}/C_{\bar{H}}(\bar{D})$. (See the last paragraph of page 201 of Dade's paper [7] for an explanation of why the action is trivial on the quotient.) We define ω_2 by

$$\omega_2 : N_{\bar{H}}(\bar{D})_{\hat{\varphi}} \times C_{\bar{G}}(\bar{D})_{\hat{\varphi}}/C_{\bar{H}}(\bar{D}) \longrightarrow k^*$$

where

$$(z_\gamma)^\alpha = \omega_2(\alpha, \gamma) z_\gamma$$

for all $\alpha \in N_{\bar{H}}(\bar{D})_{\hat{\varphi}}, \gamma \in C_{\bar{G}}(\bar{D})_{\hat{\varphi}}/C_{\bar{H}}(\bar{D})$, and z_γ in $C_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$ such that z_γ projects onto γ.

Again, by the main theorem of Section 12 of [7] (see (0.3b) in the introduction of [7]), we can use the form ω_2 to compute $(G/H)[b]$.

We define the group $C_{\bar{G}}(\bar{D})_{\omega_2}$ to be the preimage in $C_{\bar{G}}(\bar{D})_{\hat{\varphi}}$ of the right kernel of ω; that is, $C_{\bar{G}}(\bar{D})_{\omega_2} = \{a \in C_{\bar{G}}(\bar{D})_{\hat{\varphi}} : \omega_2(\sigma, aC_{\bar{H}}(\bar{D})) = 1 \text{ for all } \sigma \in N_{\bar{H}}(\bar{D})_{\hat{\varphi}}\}$. Then, as before,

$$(\bar{G}/\bar{H})[\bar{b}] = C_{\bar{G}}(\bar{D})_{\omega_2}\bar{H}/\bar{H}.$$

3.6. *The isomorphic extensions $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle/Q$ and $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle\langle\psi\rangle$.* The group $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle$ has a subgroup $C_{\bar{H}}(\bar{D})\langle\bar{\varphi}\rangle$ which is a central extension of $C_{\bar{H}}(\bar{D})_{\bar{\varphi}}/\overline{C_H(D)}$ by k^*. Since $C_{\bar{H}}(\bar{D})_{\bar{\varphi}}/\overline{C_H(D)}$ is a p-group and k is of characteristic p, it follows from III10.2 in [2] or from Remark 7 in [3] that this central extension splits; the group $C_{\bar{H}}(\bar{D})\langle\bar{\varphi}\rangle$ is an internal direct product $k^* \times Q$ where Q is a p-group isomorphic to $C_{\bar{H}}(\bar{D})_{\bar{\varphi}}/\overline{C_H(D)}$.

The fourth extension we want to consider is

$$1 \longrightarrow k^* \longrightarrow N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle/Q \longrightarrow N_{\bar{G}}(\bar{D})_{\bar{\varphi}}/C_{\bar{H}}(\bar{D})_{\bar{\varphi}} \longrightarrow 1.$$

Let ψ be the 1-dimensional $k[k^* \times Q]$-module on which Q acts trivially and elements of the group k^* act by ordinary scalar multiplication. Since ψ is stable in $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle$, we have the Clifford extension

$$1 \longrightarrow k^* \longrightarrow N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle\langle\psi\rangle \longrightarrow N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle/C_{\bar{H}}(\bar{D})\langle\bar{\varphi}\rangle \longrightarrow 1.$$

Since $C_{\bar{H}}(\bar{D})\langle\bar{\varphi}\rangle$ is the preimage of $C_{\bar{H}}(\bar{D})_{\bar{\varphi}}$ in $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle$ under the map $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle \longrightarrow N_{\bar{G}}(\bar{D})_{\bar{\varphi}}$ in the diagram above, there is a natural isomorphism $N_{\bar{G}}(\bar{D})\langle\bar{\varphi}\rangle/C_{\bar{H}}(\bar{D})\langle\bar{\varphi}\rangle \cong N_{\bar{G}}(\bar{D})_{\bar{\varphi}}/C_{\bar{H}}(\bar{D})_{\bar{\varphi}}$.

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LEMMA 3.8. *Consider the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle/Q & \longrightarrow & N_{\bar G}(\bar D)_{\bar\varphi}/C_{\bar H}(\bar D)_{\bar\varphi} & \longrightarrow & 1 \\ & & \downarrow & & \vdots & & \downarrow & & \\ 1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle/C_{\bar H}(\bar D)\langle\bar\varphi\rangle & \longrightarrow & 1 \end{array}$$

where the arrow on the right is the natural isomorphism described above. There is an isomorphism to replace the dotted arrow so that the diagram commutes.

PROOF. Let M be the maximal ideal of $k[k^*\times Q]$ corresponding to ψ. Since $k[k^*\times Q]/M$ is 1-dimensional, the centralizer in $k[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]/\big(Mk[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]\big)$ of $k[k^*\times Q]/M$ is all of $k[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]/\big(Mk[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]\big)$. Thus $N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle$ is just the group consisting of all those units of $k[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]/\big(Mk[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]\big)$ that are in one of the subspaces of the grading

$$k[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]/\big(Mk[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]\big)=\bigoplus_{\sigma\in N_{\bar G}(\bar D)\langle\bar\varphi\rangle/C_{\bar H}(\bar D)\langle\bar\varphi\rangle} k[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]/\big(Mk[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]\big)_\sigma.$$

The map $N_{\bar G}(\bar D)\langle\bar\varphi\rangle\longrightarrow N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle$ that sends each group element to its projection modulo $Mk[N_{\bar G}(\bar D)\langle\bar\varphi\rangle]$ is a group homomorphism with Q in its kernel. The corresponding group homomorphism $N_{\bar G}(\bar D)\langle\bar\varphi\rangle/Q\longrightarrow N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle$ makes the diagram of the lemma commutative if we put it in the place of the dotted arrow. By the Short Five Lemma (Lemma I3.1 in Maclane's book[13]), this group homomorphism is in fact an isomorphism. ∎

3.7. *The isomorphism between $N_{\bar G}(\bar D)\langle\hat\varphi\rangle$ and $N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle$.* Now we will use the main theorem from Dade's paper [5] to compare $N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle$ and $N_{\bar G}(\bar D)\langle\hat\varphi\rangle$. Note that $\overline{C_H(D)}$ and $C_{\bar H}(\bar D)$ are normal subgroups of $N_{\bar G}(\bar D)$ with $\overline{C_H(D)}\lhd C_{\bar H}(\bar D)$, and recall that $\hat\varphi$ is the unique irreducible $k[C_{\bar H}(\bar D)]$-module that covers the $k[\overline{C_H(D)}]$-module $\bar\varphi$. Also note that ψ is the unique irreducible $k[k^*\times Q]$-module on which the elements of the group k^* act by scalar multiplication and recall $C_{\bar H}(\bar D)\langle\varphi\rangle=k^*\times Q$. Therefore ψ corresponds to $\hat\varphi$ under the Clifford correspondence. (The Clifford correspondence is a one-to-one correspondence, described in the introduction and in Section 8 of [5], between the set of irreducible $k[C_{\bar H}(\bar D)]$-modules covering $\bar\varphi$ and the set of irreducible representations of the Clifford extension $C_{\bar H}(\bar D)\langle\varphi\rangle$ on which elements of the central subgroup k^* act by scalar multiplication.) By (0.1b) in [5], $N_{\bar G}(\bar D)_{\hat\varphi}=N_{\bar G}(\bar D)_{\bar\varphi}C_{\bar H}(\bar D)$. Thus there is a natural isomorphism $N_{\bar G}(\bar D)_{\hat\varphi}/C_{\bar H}(\bar D)\longrightarrow N_{\bar G}(\bar D)\langle\bar\varphi\rangle/C_{\bar H}(\bar D)\langle\bar\varphi\rangle$. (Recall that $N_{\bar G}(\bar D)\langle\bar\varphi\rangle/C_{\bar H}(\bar D)\langle\bar\varphi\rangle$ is naturally isomorphic to $N_{\bar G}(\bar D)_{\bar\varphi}/C_{\bar H}(\bar D)_{\bar\varphi}=N_{\bar G}(\bar D)_{\bar\varphi}/\big(C_{\bar H}(\bar D)\cap N_{\bar G}(\bar D)_{\bar\varphi}\big)$.) The main theorem from [5], described on page 236 (the first page of the paper), tells us that there is an isomorphism $N_{\bar G}(\bar D)\langle\hat\varphi\rangle\longrightarrow N_{\bar G}(\bar D)\langle\hat\varphi\rangle\langle\psi\rangle$ such that the following diagram is commutative (where of course the arrow on the right is the natural isomorphism).

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\hat\varphi\rangle & \longrightarrow & N_{\bar G}(\bar D)_{\hat\varphi}/C_{\bar H}(\bar D) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle\langle\psi\rangle & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle/C_{\bar H}(\bar D)\langle\bar\varphi\rangle & \longrightarrow & 1 \end{array}$$

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3.8. *The big diagrams.* Assembling the diagrams from Sections 3.4, 3.6, and 3.7, we obtain a commutative diagram as follows.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & N_G(D)\langle\varphi\rangle & \longrightarrow & N_G(D)_\varphi/C_H(D) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle & \longrightarrow & N_{\bar G}(\bar D)_{\bar\varphi}/\overline{C_H(D)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\bar\varphi\rangle/Q & \longrightarrow & N_{\bar G}(\bar D)_{\bar\varphi}/C_{\bar H}(\bar D)_{\bar\varphi} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & N_{\bar G}(\bar D)\langle\hat\varphi\rangle & \longrightarrow & N_{\bar G}(\bar D)_{\hat\varphi}/C_{\bar H}(\bar D) & \longrightarrow & 1.
\end{array}$$

Next, we will consider a diagram that is a subdiagram of this one in the sense that all the groups are subgroups of the groups of this diagram and all the arrows are restrictions of the arrows in this diagram.

LEMMA 3.9. *There is a commutative diagram as follows, in which all the vertical arrows are isomorphisms and all the arrows are restrictions of the arrows in the diagram above.*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^* & \longrightarrow & C_G(D)\langle\varphi\rangle & \longrightarrow & C_G(D)_\varphi/C_H(D) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & \overline{C_G(D)}\langle\bar\varphi\rangle & \longrightarrow & \overline{C_G(D)}_{\bar\varphi}/\overline{C_H(D)} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & \overline{C_G(D)}\langle\bar\varphi\rangle Q/Q & \longrightarrow & \overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)_{\bar\varphi}/C_{\bar H}(\bar D)_{\bar\varphi} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & k^* & \longrightarrow & \big(\overline{C_G(D)}C_{\bar H}(\bar D)\big)\langle\hat\varphi\rangle & \longrightarrow & \overline{C_G(D)}_{\hat\varphi}C_{\bar H}(\bar D)/C_{\bar H}(\bar D) & \longrightarrow & 1.
\end{array}$$

PROOF. First, we show that all the maps in the right column are isomorphisms. They are all clearly surjective; we need to check that they are injective. First, consider the top map. The natural map $N_G(D)_\varphi \longrightarrow N_{\bar G}(\bar D)_{\bar\varphi}/\overline{C_H(D)}$ has kernel $PC_H(D)$, so the kernel of the natural map $C_G(D)_\varphi \longrightarrow \overline{C_G(D)}_{\bar\varphi}/\overline{C_H(D)}$ is $PC_H(D)\cap C_G(D)_\varphi$; to see that $PC_H(D)\cap C_G(D)_\varphi = C_H(D)$, we observe that $P \leq H$ and so $PC_H(D)\cap C_G(D)_\varphi \subseteq PC_H(D)\cap C_G(D) \subseteq H\cap C_G(D) = C_H(D)$. Next we examine the middle map. The kernel of the natural map $\overline{C_G(D)}_{\bar\varphi} \longrightarrow \overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)_{\bar\varphi}/C_{\bar H}(\bar D)_{\bar\varphi}$ is $\overline{C_G(D)}_{\bar\varphi}\cap C_{\bar H}(\bar D)_{\bar\varphi}$. Suppose $a \in \overline{C_G(D)}_{\bar\varphi}\cap C_{\bar H}(\bar D)_{\bar\varphi}$. Then there is an $x \in C_G(D)_\varphi$ which projects onto a. Since x projects onto an element of $C_{\bar H}(\bar D)_{\bar\varphi}$, x certainly projects onto an element of $\bar H$; since $P \leq H$, it follows that $x \in H$; since also $x \in C_G(D)_\varphi$, it follows that $x \in C_H(D)$. Therefore $a \in \overline{C_H(D)}$. It follows that the middle map of the right column is injective. Next we examine the bottom map of the right column. To see that it is injective, it is only necessary to observe that $\overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)_{\bar\varphi}\cap C_{\bar H}(\bar D) = C_{\bar H}(\bar D)_{\bar\varphi}$.

Next, we show that all the maps in the center column are isomorphisms. It is first necessary to check that the appropriate maps in the first big diagram actually send $C_G(D)\langle\varphi\rangle$ into $\overline{C_G(D)}\langle\bar\varphi\rangle$ and $\overline{G_G(D)}\langle\bar\varphi\rangle Q/Q$ into $\big(\overline{C_G(D)}C_{\bar H}(\bar D)\big)\langle\hat\varphi\rangle$. This follows easily from the commutativity of the first big diagram and the fact that $C_{\bar G}(\bar D)\langle\bar\varphi\rangle Q/Q$ is the preimage in $N_{\bar G}(\bar D)\langle\bar\varphi\rangle/Q$ of $\overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)_{\bar\varphi}/C_{\bar H}(\bar D)_{\bar\varphi}$. The maps in the center column are isomorphisms by the Short Five-Lemma. ∎

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3.9. *The forms Ω and Ω_2, and their relationship to the forms ω and ω_2.* We use the top and bottom rows of the two big diagrams above to define two forms.

First, we consider the top rows of the diagrams.

The group $N_H(D)\langle\varphi\rangle$ acts by conjugation on $C_G(D)\langle\varphi\rangle$ to produce an action of $N_H(D)_\varphi / C_H(D)$ on $C_G(D)\langle\varphi\rangle$. Recall it is the associated action of $N_H(D)_\varphi$ on $C_G(D)\langle\varphi\rangle$ that produces the form $\omega: N_H(D)_\varphi \times C_G(D)_\varphi / C_H(D) \longrightarrow k^*$.

LEMMA 3.10. *Let K be the preimage in $N_G(D)$ of $C_{\bar{H}}(\bar{D})$. Then $K_\varphi / C_H(D)$ is in the kernel of the action of $N_H(D)_\varphi / C_H(D)$ on $C_G(D)\langle\varphi\rangle$.*

PROOF. By Lemma 3.4, $C_{\bar{H}}(\bar{D}) / \overline{C_H(D)}$ is a p-group; therefore, $K_\varphi / C_H(D)$ is a p-group. Let y be an element of $C_G(D)\langle\varphi\rangle$ and let τ be an element of the subgroup $K_\varphi / C_H(D)$ of $N_H(D)_\varphi / C_H(D)$. Then $y^\tau = \alpha y$ for some $\alpha \in k^*$. Since $K_\varphi / C_H(D)$ is a p-group, the order of τ is a power of p; thus $\alpha^{p^n} = 1$ for some n so $\alpha = 1$. ∎

As a consequence of this lemma, the conjugation action of $N_H(D)\langle\varphi\rangle$ on $C_G(D)\langle\varphi\rangle$ gives rise to an action of $N_H(D)_\varphi / K_\varphi$ on $C_G(D)\langle\varphi\rangle$, an action which is trivial on the central subgroup k^* and on $C_G(D)_\varphi / C_H(D)$. Thus we get a "bilinear form"

$$\Omega: N_H(D)_\varphi / K_\varphi \times C_G(D)_\varphi / C_H(D) \longrightarrow k^*$$

given by

$$(y_\tau)^\rho = \Omega(\rho, \tau) y_\tau$$

for all $\rho \in N_H(D)_\varphi / K_\varphi, \tau \in C_G(D)_\varphi / C_H(D)$, and $y_\tau \in C_G(D)\langle\varphi\rangle$ such that y_τ projects onto τ.

The form Ω differs from the form ω only in the inessential way that we have obtained Ω from ω by taking the quotient of the left hand side by a subgroup in the left kernel of ω. In particular, the two forms have the same right kernel.

Next, we examine the bottom rows of the big diagrams. The reader should now look again at the definition of ω_2 above. The form ω_2 arises from the conjugation action of $N_{\bar{H}}(\bar{D})\langle\hat{\varphi}\rangle$ on $C_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$. The group k^* is in the kernel of this action so there is an associated action of $N_{\bar{H}}(\bar{D})_{\hat{\varphi}} / C_{\bar{H}}(\bar{D})$ on $C_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$. By (0.1b) of [5], $N_{\bar{H}}(\bar{D})_{\hat{\varphi}} = N_{\bar{H}}(\bar{D})_{\bar{\varphi}} C_{\bar{H}}(\bar{D})$. Thus, via the natural isomorphism, there is an associated action of $N_{\bar{H}}(\bar{D})_{\bar{\varphi}} / \big(N_{\bar{H}}(\bar{D})_{\bar{\varphi}} \cap C_{\bar{H}}(\bar{D})\big) = N_{\bar{H}}(\bar{D})_{\bar{\varphi}} / C_{\bar{H}}(\bar{D})_{\bar{\varphi}}$ on $C_{\bar{G}}(\bar{D})\langle\hat{\varphi}\rangle$. As in the definition of ω_2, this action gives rise to a "bilinear form"

$$\overline{\omega_2}: N_{\bar{H}}(\bar{D})_{\bar{\varphi}} / C_{\bar{H}}(\bar{D})_{\bar{\varphi}} \times C_{\bar{G}}(\bar{D})_{\hat{\varphi}} / C_{\bar{H}}(\bar{D}) \longrightarrow k^*.$$

The form $\overline{\omega_2}$ differs from the original form ω_2 only in the inessential way that we have replaced the left hand side by a group that is naturally isomorphic to the quotient of the original side modulo a subgroup in the left kernel of the original form. In particular, the forms $\overline{\omega_2}$ and ω_2 have the same right kernel.

Now we define a form associated to the last rows of the big diagrams above. Note that $\overline{C_G(D)}_{\bar{\varphi}} C_{\bar{H}}(\bar{D}) / C_{\bar{H}}(\bar{D})$ is a subgroup of $C_{\bar{G}}(\bar{D})_{\hat{\varphi}} / C_{\bar{H}}(\bar{D})$. Let

$$\Omega_2: N_{\bar{H}}(\bar{D})_{\bar{\varphi}} / C_{\bar{H}}(\bar{D})_{\bar{\varphi}} \times \overline{C_G(D)}_{\bar{\varphi}} C_{\bar{H}}(\bar{D}) / C_{\bar{H}}(\bar{D}) \longrightarrow k^*$$

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be the restriction of $\overline{\omega_2}$.

The forms Ω and Ω_2 have left hand sides and right hand sides that are naturally isomorphic. Because of the commutativity of the big diagrams, they can be obtained from each other via those natural isomorphisms. The forms Ω and ω have the same right kernels. The right kernel of Ω_2 is the intersection of the right kernel of ω_2 with the right hand side of Ω_2. Note that therefore *if g is in $C_G(D)_\varphi$ then its image in $\overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)/C_{\bar H}(\bar D)$ is in the right kernel of ω_2 if and only if its image in $C_G(D)_\varphi/C_H(D)$ is in the right kernel of ω.*

3.10. *Proof of Theorem 3.5.*

PROOF. We will show that every element of $(\bar G/\bar H)[\bar b]/\Psi\big((G/H)[b]\big)$ has order a power of p. Recall from Sections 3.3 and 3.5 that $(G/H)[b] = C_G(D)_\omega H/H$ and that $(\bar G/\bar H)[\bar b] = C_{\bar G}(\bar D)_{\omega_2}\bar H/\bar H$.

Let $a \in C_{\bar G}(\bar D)_{\omega_2}\bar H/\bar H$. Since $C_{\bar G}(\bar D)_{\hat\varphi} = C_{\bar G}(\bar D)_{\bar\varphi}C_{\bar H}(\bar D)$, and since $C_{\bar G}(\bar D)_{\omega_2} \subseteq C_{\bar G}(\bar D)_{\hat\varphi}$ and $C_{\bar H}(\bar D) \subseteq \bar H$, the coset a has a representative x in $C_{\bar G}(\bar D)_{\hat\varphi}$, an element x whose natural image in $C_{\bar G}(\bar D)_{\bar\varphi}C_{\bar H}(\bar D)/C_{\bar H}(\bar D)$ is in the right kernel of ω_2. Since $C_{\bar G}(\bar D)/\overline{C_H(D)}$ is a p-group, there is a positive integer n such that $x^{p^n} \in \overline{C_G(D)}_{\bar\varphi}$. Of course the natural image of x^{p^n} in $\overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)/C_{\bar H}(\bar D)$ is also in the right kernel of ω_2. Let g be an element of $C_G(D)_\varphi$ that projects onto x^{p^n}. Then the natural image of g in $\overline{C_G(D)}_{\bar\varphi}C_{\bar H}(\bar D)/C_{\bar H}(\bar D)$ is in the right kernel of ω_2, so by the last sentence of Section 3.9, the natural image of g in $C_G(D)_\varphi/C_H(D)$ is in the right kernel of ω. Therefore $g \in C_G(D)_\omega$. Since $\Psi(gH/H) = x^{p^n}\bar H/\bar H$, it follows that $x^{p^n}\bar H/\bar H \in \Psi\big(C_G(D)_\omega H/H\big) = \Psi\big((G/H)[b]\big)$, so $a^{p^n} \in \Psi\big((G/H)[b]\big)$. ∎

3.11. *An example.* We give an example in which $\Phi(k[G]^H e) \neq k[\bar G]^{\bar H}\bar e$ and $\Psi\big((G/H)[b]\big) \neq (\bar G/\bar H)[\bar b]$.

LEMMA 3.11. *Assume in addition to Hypothesis 2.1 that H is a p-group. Let b be the unique block of H. Then $(G/H)[b] = C_G(H)H/H$.*

PROOF. Since b is the unique block of $k[H]$, the corresponding central idempotent e is equal to 1, so $k[G]^H e = k[G]^H$. There is a basis for $k[G]^H$ consisting of all the elements of $k[G]$ of the form $\sum_{x\in C} x$, where C is an H-conjugacy class in G. Let M be an irreducible $k[G]$-module and let C be an H-conjugacy class in G. Since H is a p-group, H is in the kernel of M;hence every element of C acts in the same way on M. Since the order of C is a power of p, it follows that $\sum_{x\in C} x$ acts as 0 on M unless the order of C is 1, *i.e.*, unless $C = \{x\}$ with $x \in C_G(H)$. Theorem 2.5 in [9] (the analog of Clifford's Theorem for the restriction of irreducible $k[G]$-modules to $k[G]^H$) implies that $J(k[G]^H) = J(k[G]) \cap k[G]^H$; therefore $\sum_{x\in C} x$ is in the Jacobson radical of $k[G]^H$ unless $C = \{x\}$ with $x \in C_G(H)$.

It follows that in the grading

$$k[G]^H = \bigoplus_{\sigma\in G/H} (k[G]^H)_\sigma,$$

the component $(k[G]^H)_\sigma$ contains a unit if and only if the coset σ contains an element of $C_G(H)$. Now Lemma 3.3 completes the proof. ∎

Let $p = 2$, $G = \langle x,y : x^2 = y^4 = 1, xyx = y^3\rangle$, $H = \langle y\rangle$,and $P = \langle y^2\rangle$. Then $P = Z(G)$ so certainly $G = PC_G(P)$. Let b be the unique block of $k[H]$. By Lemma 3.11, $(G/H)[b] = C_G(H)H/H = H/H = 1$, while $\bar{G}(= G/P)$ is abelian so $(\bar{G}/\bar{H})[\bar{b}] = \bar{G}/\bar{H}$, which is of order 2.

An elementary calculation shows that in the same example $\Phi(k[G]^He) \neq k[\bar{G}]^{\bar{H}}\bar{e}$.

4. **Behavior of cliques under reduction modulo** P. Let G be a finite group and let H be a normal subgroup of G. Let M be an irreducible $k[G]^H$-module, and let e be a centrally primitive idempotent of $k[H]$. Then multiplication by e is a kG^H-endomorphism of M, so $Me = M$ or $Me = 0$. Since the distinct centrally primitive idempotents of $k[H]$ are mutually orthogonal, it follows that there is a unique centrally primitive idempotent e of $k[H]$ with $Me = M$.

LEMMA 4.1. *Let V and W be irreducible $k[G]$-modules. Let e be a primitive central idempotent of $k[H]$ such that $Ve \neq 0$. Then there is a (non-zero) irreducible $k[G]^H$-module X such that $X | V_{k[G]^H}$ and $X | W_{k[G]^H}$ if and only if there is a (non-zero) irreducible $k[G]^He$-module Y such that $Y | Ve_{k[G]^He}$ and $Y | We_{k[G]^He}$.*

PROOF. Suppose there is an irreducible $k[G]^H$-module X such that $X | V_{k[G]^H}$ and $X | W_{k[G]^H}$. Recall that $V_{k[G]^H}$ is semi-simple, that all G-conjugates of X are summands of $V_{k[G]^H}$, and that every irreducible summand of $V_{k[G]^H}$ is conjugate in G to X. Since $Ve \neq 0$, there is a conjugate Z of X such that $Ze \neq 0$. Then $Ze = Z$, $Z_{k[G]^He}$ is an irreducible $k[G]^He$-module, and $Z_{k[G]^He} | Ve_{k[G]^He}$. Since also $W_{k[G]^H}$ is semi-simple and all G-conjugates of X are summands of $W_{k[G]^H}$, and since $X | W_{k[G]^H}$, it follows that $Z | W_{k[G]^H}$. Hence $We \neq 0$ and $Z_{k[G]^He} | We_{k[G]^He}$.

Conversely, suppose there is an irreducible $k[G]^He$-module Y such that $Y | Ve_{k[G]^He}$ and $Y | We_{k[G]^He}$. Since e is a central idempotent of $k[G]^H$, we have the decomposition $k[G]^H = k[G]^He \oplus k[G]^H(1-e)$, where the two summands on the right are two-sided ideals. It follows that we can make Y into a $k[G]^H$-module by simply having all elements of $k[G]^H(1-e)$ act as 0; call this $k[G]^H$-module $Y_{k[G]^H}$. Since $V_{k[G]^H} = Ve_{k[G]^H} \oplus V(1-e)_{k[G]^H}$, and $W_{k[G]^H} = We_{k[G]^H} \oplus W(1-e)_{k[G]^H}$, it follows that $Y_{k[G]^H} | V_{k[G]^H}$ and $Y_{k[G]^H} | W_{k[G]^H}$. ∎

LEMMA 4.2. *Let Γ and Σ be finite groups with $\Sigma \lhd \Gamma$ and Γ/Σ a p-group. Let A be a twisted group algebra for Γ with a grading $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ in which each subspace A_γ is 1-dimensional over k. Let $A' = \bigoplus_{\sigma\in\Sigma} A_\sigma$. Let V and W be irreducible A-modules. Then V and W are isomorphic if and only if $V_{A'}$ and $W_{A'}$ are isomorphic.*

PROOF. Let $\Gamma' = \{u \in A : u$ is a unit and $u \in A_\gamma$ for some $\gamma \in \Gamma\}$. Let $\Sigma' = \{u \in A : u$ is a unit and $u \in A_\sigma$ for some $\sigma \in \Sigma\}$. Then $\Sigma' \lhd \Gamma'$ and $\Gamma'/\Sigma' \cong \Gamma/\Sigma$.

Let M be an irreducible A'-module. Let ψ be the associated irreducible representation of Σ'. There is a one-to-one correspondence, called the *Clifford Correspondence* (described in the introduction and in Section 8 of [5]), between the set of irreducible A-modules lying over M and the set of irreducible representations of the Clifford extension

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$\Gamma'\langle\psi\rangle$ on which elements of the central subgroup k^* act by scalar multiplication. Note that $\Gamma'\langle\psi\rangle$ is a central extension of the p-group Γ'_ψ/Σ' by k^*; therefore, by III10.2 in [2], $\Gamma'\langle\psi\rangle$ splits as the direct product of k^* and a p-group Q isomorphic to Γ'_ψ/Σ'. Since $Q$is a p-group and k is of characteristic p, Q acts trivially on every irreducible representation of $\Gamma'\langle\psi\rangle$, so there is exactly one irreducible representation of $\Gamma'\langle\psi\rangle$ on which k^* acts by scalar multiplication. Therefore, by the Clifford Correspondence, there is exactly one irreducible A-module lying over M. The lemma follows. ∎

THEOREM 4.3. *Let V and W be irreducible $k[G]$-modules. Then V and W belong to the same H-clique if and only if $V_{k[\bar{G}]}$ and $W_{k[\bar{G}]}$ belong to the same $\bar{H}$-clique.*

PROOF. Since $\Phi(k[G]^H) \subseteq k[\bar{G}]^{\bar{H}}$, it is clear that irreducible $k[G]$-modules in the same $\bar{H}$-clique belong to the same H-clique. The remainder of the proof is devoted to demonstrating the converse.

Let

$$A' = \Big(\bigoplus_{\sigma\in\Psi((G/H)[b])} \Phi(k[G]^H e)_\sigma\Big) \Big/ \Big(\bigoplus_{\sigma\in\Psi((G/H)[b])} J\big(\Phi(k[G]^H e)_1\big)\Phi(k[G]^H e)_\sigma\Big)$$

and let

$$A = \Big(\bigoplus_{\sigma\in(\bar{G}/\bar{H})[\bar{b}]} (k[\bar{G}]^{\bar{H}}\bar{e})_\sigma\Big) \Big/ \Big(\bigoplus_{\sigma\in(\bar{G}/\bar{H})[b]} J\big((k[\bar{G}]^{\bar{H}}\bar{e})_1\big)(k[\bar{G}]^{\bar{H}}\bar{e})_\sigma\Big).$$

By Theorems 3.2 and 3.5, the algebras A and A' are related in the same way as the algebras of Lemma 4.2.

Assume that V and W belong to the same H-clique. Let b be a block of $k[H]$ covered by the block B of $k[G]$ that contains V and W; let e be the central primitive idempotent of $k[H]$ corresponding to b. By Lemma 4.1, $Ve_{k[G]^He}$ and $We_{k[G]^He}$ share an (isomorphism type of) irreducible submodule. By Theorem 3.2 $Ve_{A'}$ and $We_{A'}$ share an irreducible submodule. By Lemma 4.2, $V\bar{e}_A$ and $W\bar{e}_A$ share an irreducible submodule. By Theorem 3.2, $V\bar{e}_{k[\bar{G}]^{\bar{H}}\bar{e}}$ and $W\bar{e}_{k[\bar{G}]^{\bar{H}}\bar{e}}$ share an irreducible submodule. By Lemma 4.1, $V_{k[\bar{G}]}$ and $W_{k[\bar{G}]}$ are in the same $\bar{H}$-clique. ∎

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