Can. J. Math. Vol. XXVIII, No. 6, 1976, pp. 1153–1161

SYSTEMS OF MAGIC LATIN k-CUBES

JOSEPH ARKIN, VERNER E. HOGGATT, JR. AND E. G. STRAUS

1. Introduction. A Latin k-cube A of order n is a k-dimensional array $A = (a_{i_1 i_2 \ldots i_k})$, $0 \leqq i_j \leqq n - 1$ where

$$a_{i_1 \ldots i_k} \in \{0, 1, \ldots, n-1\} \text{ and } a_{i_1 \ldots i_{r-1} j i_{r+1} \ldots i_k}$$

runs through the distinct elements $0, 1, \ldots, n - 1$ as j runs from 0 to $n - 1$.

A k-tuple of Latin k-cubes, $A^{(1)}, A^{(2)}, \ldots, A^{(k)}$ is *orthogonal* if, upon superposition, the k-tuples of entries $(a_{i_1}{}^{(1)}{}_{\ldots i_k}, a_{i_1}{}^{(2)}{}_{\ldots i_k}, \ldots, a_{i_1}{}^{(k)}{}_{\ldots i_k})$ run through all ordered k-tuples $(0, \ldots, 0)$ to $(k - 1, \ldots, k - 1)$. A *system of* $r \geqq k$ *Latin* k*-cubes* is *orthogonal* if every k of its cubes are orthogonal. A *major diagonal* of a k-cube of order n are the entries $a_{i_1 \ldots i_k}$ where r of the indices run simultaneously from 0 to $n - 1$ while the remaining $k - r$ indices run from $n - 1$ to 0. There are thus 2^{k-1} major diagonals. A *minor diagonal* is obtained by holding m indices fixed $(0 < m < k)$ while letting the other indices run simultaneously from 0 to $n - 1$ or $n - 1$ to 0.

A Latin k-cube is *magic* if the sum of the elements in each major diagonal equals the sum, $n(n - 1)/2$, of the elements of a row in each of the directions of the cube. In particular, if all the entries in the major diagonals are distinct, a case which we shall call *strongly Latin*, then the k-cube is magic. However it is easy to construct magic k-cubes which are not strongly Latin. If we have an orthogonal system of k magic Latin k-cubes and consider the ordered k-tuples of their superposition as integers expressed in base n, then this superposition yields a k-cube whose entries are the integers from 0 to $n^k - 1$ so that the sums in all the rows, in all the coordinated directions, and in all the major diagonals are the same, $n(n^k - 1)/2$. We also consider the concept of *strongly magic* Latin k-cubes as magic cubes where the sums of the elements in the minor diagonals are equal to the row sums and the major diagonal sums. We define a k-cube as *completely Latin* if the elements in all diagonals are distinct Such completely Latin cubes are obviously strongly magic. The superposition of a system of k orthogonal strongly magic k-cubes with the interpretation of the entries as integers from 0 to $n^k - 1$ leads to a k-cube in which the sum in all rows and in all diagonals is $n(n^k - 1)/2$.

Many of the ideas in this paper occur in various forms in the mathematical literature. The construction of magic squares by the use of Latin squares can

Received January 24, 1975 and in revised form, February 24, 1976. The work of the third author was supported in part by NSF Grant No. 28696.

be found in [**2**]. Completely Latin k-cubes are discussed in [**4**] and systems of mutually orthogonal Latin k-cubes are considered in [**3**].

In §2 we discuss maximal systems of orthogonal magic Latin squares. In §3 we extend the construction of §2 and [**1**] to systems of orthogonal magic Latin k-cubes with $k > 2$. In §4 we consider some examples.

2. Orthogonal magic Latin squares. If n is a power of a prime then we can use finite fields to construct maximal systems of $n - 1$ mutually orthogonal Latin squares of order n. We now extend this to maximal systems of orthogonal magic squares of order n.

2.1 THEOREM. *If n is a power of an odd prime then there exists a system of $n - 1$ mutually orthogonal magic Latin squares of order n so that $n - 3$ of those squares are strongly Latin.*

Proof. Let $F = \{x_0, \ldots, x_{n-1}\}$ be the Galois field with n elements ordered so that $x_i = -x_{n-1-i}$ for $i = 0, 1, \ldots, n - 1$. We construct a system of $n - 1$ orthogonal Latin squares $A^{(t)} = (a_{ij}^{(t)})$, whose entries are the elements of F by setting $a_{ij}^{(t)} = x_i + tx_j$, where t ranges through F^*, the non-zero elements of F. The orthogonality of the system is immediate, since for any two distinct elements $s, t \in F^*$ and any pair $(y, z) \in F^2$ there is a unique solution x_i, x_j to the simultaneous equations $x_i + sx_j = y$, $x_i + tx_j = z$. For $t \neq \pm 1$ the diagonal elements are distinct and thus we get a system of $n - 3$ orthogonal strongly Latin squares of order n. For $t = \pm 1$ one of the diagonals has all its elements 0 while the other diagonal has distinct elements. We complete the construction by replacing the field elements by the integers $0, 1, \ldots, n - 1$: with 0 replaced by $(n - 1)/2$; so that the sums of all diagonals become $n(n - 1)/2$.

If n is even then $(n - 1)/2$ is not an integer and thus the above construction is not available. However there is a compensating feature in the fact that in fields of characteristic 2 we have $1 = -1$.

2.2. THEOREM. *If n is a power of 2 then there exists a system $n - 2$ orthogonal strongly Latin squares of order n.*

Proof. We use the same construction as in Theorem 2.1, this time setting $x_i = x_{n-1-i} + 1$. Then the Latin squares $A^{(t)} = (a_{ij}^{(t)})$ with $a_{ij}^{(t)} = x_i + tx_j$; $t \neq 0, 1$ have the desired property.

Kronecker products of strongly Latin squares are obviously strongly Latin. To show that we can always associate a magic Latin square of order mn with the Kronecker product of two magic Latin squares of order m and n respectively, write $A = (a_{ij})$; $i, j = 1, \ldots, m$; $B = (b_{kr})$; $k, r = 0, \ldots, n - 1$ and

$C = A \times B = ((a_{ij}, b_{kr})) \to (c_{in+k,jn+r})$ where $c_{in+k,jn+r} = a_{ij}n + b_{kr}$. Then for example

$$\sum_{s=1}^{mn} c_{ss} = n^2 \sum_{i=1}^{m} a_{ii} + m \sum_{k=1}^{n} b_{kk} = n^2 m(m-1)/2 + mn(n-1)/2$$
$$= mn(mn-1)/2.$$

We have thus proved the following:

2.3. COROLLARY. *If $n = p_1^{d_1} p_2^{d_2} \ldots p_m^{d_m}$ then there exists a system of q mutually orthogonal magic Latin squares of order n of which s are strongly Latin. Here*

$$q = \min_{i=1,\ldots,m} \{p_i^{d_i} - 1\}, \quad s = q - 2 \quad \textit{when} \quad 2 \neq p_1 < \ldots < p_m.$$

$$q = \min \{2^{d_1} - 2, p_2^{d_2} - 1, \ldots, p_m^{d_m} - 1\},$$

$$s = \min \{2^{d_1} - 2, p_2^{d_2} - 3, \ldots p_m^{d_m} - 3\} \quad \textit{when} \quad 2 = p_1 < \ldots < p_m.$$

3. Orthogonal magic k-cubes. For $k > 2$ we get an improvement on the results stated in [**1**] and we can even insist on obtaining magic cubes.

3.1 THEOREM. *If n is a power of an odd prime and $n \geqq k > 2$ then there exists a system of $n + 1$ orthogonal magic Latin k-cubes of order n of which at least $n - (k-1)2^{k-1}$ are strongly Latin.*

Proof. We first note that for any k-tuple $\mathbf{C} = (c_1, \ldots, c_k)$ of nonzero elements of the finite field $F = \{x_0, x_1, \ldots, x_{n-1}\}$, the k-cube $A_{\mathbf{C}} = (a_{i_1 \ldots i_k})$ with

$$a_{i_1 \ldots i_k} = c_1 x_{i_1} + \ldots + c_k x_{i_k}$$

is a Latin k-cube of order n.

If $\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(k)}$ are linearly independent vectors with components in F^* then the cubes $A_k^{(i)} = A_{\mathbf{C}^{(i)}}$, $i = 1, \ldots, k$ form an orthogonal k-tuple. More generally, if any k of the vectors $\mathbf{C}^{(1)}, \ldots, \mathbf{C}^{(r)}$, $r \geqq k$ with components in F^* are linearly independent, then the cubes $A^{(1)}, \ldots, A^{(r)}$ form an orthogonal system.

In order to construct the system $\mathbf{C}^{(i)}$; $i = 1, \ldots, n+1$, we use the same ordering of F that we used in the proof of Theorem 2.1. Next we find a polynomial $f(x) = x^{k-1} + a_1 x^{k-2} + \ldots + a_{k-1} \in F[x]$ with nonzero coefficients $a_1, \ldots, a_{k-1}$ and no zeros in F. To construct such a polynomial we can start for example with $g(x) = x^{k-1} + x^{k-2} + \ldots + x^2$. Since $k - 1 < n$ there must be some $x_i \in F^*$ for which $g(x_i) \neq 0$. Now pick $a_{k-2} = -g(x_i)/x_i$ so that the polynomial $h(x) = g(x) + a_{k-2}x$ has two zeros $x = 0, x_i$ in F. Thus there is some value $-a_{k-1} \in F^*$ which is not attained by $h(x_j)$ for any $x_j \in F$ and $f(x) = h(x) + a_{k-1}$ has the desired property.

Now pick any k distinct values $y_1, \ldots, y_k \in F^*$ and let $f_i(x) = y_i^{-k} f(y_i x)$; $i = 1, \ldots, k$. The $n + 1$ vectors $\mathbf{C}^{(t)} = (f_1(t), \ldots, f_k(t))$; $t \in F$ and $\mathbf{C}^{(\infty)} = (1, \ldots, 1)$ have the property that any k are linearly independent.

We need to show that every k $\times$ k submatrix of

$$\begin{bmatrix} f_1(x_1) \dots f_1(x_n) & 1 \\ \cdot \quad \cdot & \cdot \\ \cdot \quad \cdot & \cdot \\ \cdot \quad \cdot & \cdot \\ f_k(x_1) \dots f_k(x_n) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{k-1}y_1^{-k+1} & a_{k-2}y_1^{-k+2} \dots a_1 y_1^{-1} & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{k-1}y_k^{-k+1} & a_{k-2}y_k^{-k+2} \dots a_1 y_k^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dots 1 & 0 \\ x_1 & x_n & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ x_1^{k-1} & \dots x_n^{k-1} & 1 \end{bmatrix}$$

is of the rank k. This follows from the fact that the first matrix on the right is a regular $k \times k$ matrix, essentially a Vandermonde matrix, while any $k \times k$ submatrix of the second matrix is a Vandermonde.

The elements in the main diagonals of $A^{(t)}$ are

$$\{(f_1(t) \pm f_2(t) \pm \dots \pm f_k(t))\, x_i | i = 0, \dots, n-1\} \quad \text{for } t \in F \text{ and}$$

$$\{(1 \pm 1 \pm 1 \dots \pm 1)x_i | i = 1, \dots, n\} \quad \text{if } t = \infty .$$

In either case the elements are either all distinct or they are all 0. Thus all k-cubes will become magic if we replace the field elements by the numbers $0, 1, \dots, n-1$ where the field element 0 is replaced by the number $(n-1)/2$.

If t is chosen so that none of the 2^{k-1} polynomials $f_1(t) \pm f_2(t) \pm \dots \pm f_k(t)$ vanishes then $A^{(t)}$ is strongly Latin. Since none of these polynomials vanishes identically, none has more than $k-1$ zeros in F there must be at least $n - (k-1)2^{k-1}$ orthogonal strongly Latin k-cubes of order n.

The case in which n is a power of 2 leads to an interesting ramification.

3.2. THEOREM. *Let $n \geqq 4$ be a power of 2. Then there exists a system of $n + 2$ orthogonal Latin cubes (3-cubes) of which at least n are strongly Latin.*

Proof. Let F be the field of n elements ordered as in the proof Theorem 2.2. Let $f(x) = x^2 + ax + b$, $ab \neq 0$, be an irreducible polynomial in $F[x]$ and pick three distinct elements $y_1, y_2, y_3, \in F^*$. Define $f_i(x) = y_i^{-2} f(y_i x)$; $i = 1, 2, 3$ and construct the set of $n + 2$ Latin cubes of order n with entries from F corresponding to the vectors $\mathbf{C}^{(t)} = (f_1(t), f_2(t), f_3(t))$; $t \in$ F, $\mathbf{C}^{(\infty)} = (1, 1, 1)$, $C' = (y_1^{-1}, y_2^{-1}, y_3^{-1})$.

To prove that these $n + 2$ cubes form an orthogonal system it again suffices

https://doi.org/10.4153/CJM-1976-113-0 Published online by Cambridge University Press

to show that all 3×3 submatrices of the matrix

$$\begin{bmatrix} f_1(x_1) \dots f_1(x_n) & 1 & y_1^{-1} \\ f_2(x_1) \dots f_2(x_n) & 1 & y_2^{-1} \\ f_3(x_1) \dots f_3(x_n) & 1 & y_3^{-1} \end{bmatrix} = \begin{bmatrix} by_1^{-2} & ay_1^{-1} & 1 \\ by_2^{-2} & ay_2^{-1} & 1 \\ by_3^{-2} & ay_3^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \dots 1 & 0 & 0 \\ x_1 & \dots x_n & 0 & a^{-1} \\ x_1^2 & \dots x_n^2 & 1 & 0 \end{bmatrix}$$

are regular. This is easily seen since $ab \neq 0$ and all the 3×3 submatrices on the ring are Vandermonde. Perhaps we should justify the inclusion of the $(n + 2) - nd$ cube A' by showing explicitly that

$$\begin{vmatrix} 1 & 1 & 0 \\ x_i & x_j & a^{-1} \\ x_i^2 & x_j^2 & 0 \end{vmatrix} = a^{-1} \begin{vmatrix} 1 & 1 \\ x_i^2 & x_j^2 \end{vmatrix} = a^{-1}(x_i + x_j)^2 \neq 0$$

for all $1 \leqq i < j \leqq n$.

The elements in the main diagonals of $A^{(t)}$ are

$$\{(f_1(t) + f_2(t) + f_3(t))x_i + e_2 f_2(t) + e_3 f_3(t) | i = 1, \dots, n\}$$

where e_2, e_3 are 0 or 1. Thus $A^{(t)}$ is strongly Latin provided

$$f_1(t) + f_2(t) + f_3(t) \neq 0.$$

If $A^{(t)}$ is not strongly Latin then each main diagonal of $A^{(t)}$ consists of n equal elements.

The elements in the main diagonals of $A^{(\infty)}$ are $\{x_i | i = 1, \dots, n\}$ and $\{x_i + 1 | i = 1, \dots, n\}$ so that $A^{(\infty)}$ is strongly Latin. Finally the elements in the main diagonals of A' are

$$\{(y_1^{-1} + y_2^{-1} + y_3^{-1})x_i + e_2 y_2^{-1} + e_3 y_3^{-1} | i = 1, \dots, n\}$$

where $e_2, e_3 = 0$ or 1. Thus either A' is strongly Latin or all its main diagonals consist of n equal elements. Thus, if there exist two cubes which are not strongly Latin, then by the orthogonality of the system the other n cubes must be strongly Latin.

3.3. THEOREM. *Let $n \geqq k$ be a power of 2. Then there exists an orthogonal system of $n + 1$ Latin k-cubes of order n of which at least $n + 2 - k$ are strongly Latin.*

Proof. We make a construction which is completely analogous to that made in the proof of the preceding theorem except that we no longer get an analog to A'. The resulting k-cubes are either strongly Latin or have all the main diagonals consisting of n equal elements. If there are $k - 1$ of the cubes which are not strongly Latin, then by the orthogonality of the system the remaining cubes are all strongly Latin.

Using Kronecker products we get results for n which are not powers of primes.

3.4. COROLLARY. *Let $n = p_1^{d_1} \dots p_m^{d_m}$. Then there exists an orthogonal system of q magic Latin k-cubes of order n of which r are strongly Latin. Here*

$$q = \min_{i=1,\dots,m} \{p_i^{d_i} + 1\}, \quad r = q - 1 - (k-1)2^{k-1}$$

if $2 < p_1 < \dots < p_m$;

$$q = \min \{2^{d_1}, p_2^{d_2} + 1, \dots, p_m^{d_m} + 1\}$$
$$r = \min \{2^{d_1}, p_2^{d_2} - 8, \dots, p_m^{d_m} - 8\}$$

if $2 = p_1 < \dots < p_m$; $k = 3$;

$$q = \min \{2^{d_1} + 2 - k, p_2^{d_2} + 1, \dots, p_m^{d_m} + 1\}$$
$$r = \min \{2^{d_1} + 2 - k, p_2^{d_2} - (k-1)2^{k-1}, \dots, p_m^{d_m} - (k-1)2^{k-1}\}$$

if $2 = p_1 < \dots < p_m$, $k > 3$.

Since the polynomials chosen in the proofs of Theorems 3.1, 3.2, and 3.3 are linearly independent, it follows that for any given k and any sufficiently large power of a prime n we get a system of orthogonal completely Latin k-cubes of order n. The superposition of any k of these cubes leads to a large number of completely magic k-cubes in the sense that the integers from 0 to $n^k - 1$ are placed in the cubes so that the sums in all straight lines which pass through n entries are the same number $n(n^k - 1)/2$.

3.5. THEOREM. *If n is a power of an odd prime and*

$$n \geqq g(k) + k = \tfrac{1}{2}(3^k - 1)(k-1) - k(k-2)$$

then there exists a system of $n - g(k)$ orthogonal completely Latin k-cubes of order n.

Proof. We use the same constructions as in the proofs of Theorems 3.1 and 3.3 but, for odd n, we have to exclude all values of t for which any of the sums

$$(3.6) \quad f_{i_1}(t) \pm f_{i_2}(t) \pm \dots f_{i_s}(t) = 0$$

where

$$1 \leqq i_1 < i_2 < \dots < i_s \leqq k \quad (1 < s \leqq k).$$

The number of such choices is $(3^k - 1 - 2k)/2$ since for each f_i we either fail to include it or include f_i or $-f_i$ in the sum (3.6). This would give 3^k choices. However we must include at least two f_i so this decreases the number of choices by 1 (choice of none) $+ 2k$(choice of one). Finally we pick the sign of f_{i_1} to be $+$ and thus divide the number of terms by 2. No polynomial in (3.6) has more than $k - 1$ zeros in F and thus the number of k-cubes $A^{(t)}$ in Theorem 3.1 which are completely magic is at least $n - (k-1)(3^k - 1 - 2k)/2 = n - g(k)$.

If n is a power of 2 then we need only exclude those values of t for which

$$f_{i_1}(t) + \dots + f_{i_s}(t) = 0 \quad 1 \leqq i_1 < \dots < i_s \leqq k, 1 < s \leqq k.$$

https://doi.org/10.4153/CJM-1976-113-0 Published online by Cambridge University Press

This leads to the exclusion of at most $(k - 1)(2^k - 1 - k)$ values of t and thus the number of k-cubes $A^{(t)}$ in Theorem 3.3 which are completely magic is at least $n - h(k)$.

For sufficiently large prime powers n it is possible to select polynomials $f_i(t)$ with care so that we get systems (with $n + 1$ or $n + 2$ elements) of orthogonal completely Latin k-cubes of order n. We illustrate this here for the case $k = 3$, $n = 2^m$.

3.7. LEMMA. *Let F be a finite field with 2^m elements considered as an m dimensional vector space over the prime field $F_0 = \{0, 1\}$. Then for each quadratic polynomial $g(t) = at^2 + bt \in F[t]$ with $ab \neq 0$ the values attained by $g(t)$, $t \in F$ form an $(m - 1)$-dimensional hyperplane H_g over F_0.*

The hyperplane H_g is uniquely determined by the ratio $a/b^2 = c$ and

$$(3.8) \quad H_g = H_c = \{ct^2 + t | t \in F\}$$

is the set of solutions of the equation

$$(3.9) \quad \mathrm{Tr}(cx) = cx + (cx)^2 + \ldots + (cx)^{2^{m-1}} = 0.$$

Since there are $2^m - 1$ distinct equations (3.9) it follows that every $(m - 1)$-dimensional subspace of F has the form H_g for a suitable g.

Proof. Since $(t_1 + t_2)^2 = t_1^2 + t_2^2$ for $t_1, t_2 \in F$ we have $g(t_1 + t_2) = g(t_1) + g(t_2)$ so that H_g is a linear manifold over F_0. Since $g(t_1) = g(t_2)$ if and only if $t_1 = t_2$ or $t_1 = t_2 + b/a$ it follows that H_g has 2^{m-1} elements and is a hyperplane. For any $s \in F^*$ the polynomial $h(t) = g(st)$ attains the same values over F as the polynomial $g(t)$. Thus $H_h = H_g$. The choice $s = 1/b$ yields

$$H_g = H_{ct^2+t} = H_c.$$

For $x = ct^2 + t$ we have

$$cx = (ct)^2 + ct \quad \text{and} \quad \mathrm{Tr}(cx) = \mathrm{Tr}(ct) + \mathrm{Tr}((ct)^2) = 2\mathrm{Tr}(ct) = 0.$$

The last statement of the lemma follows from the fact that there are $2^m - 1$ $(m - 1)$-dimensional subspaces of F.

3.8. COLLARY. *The intersection of k hyperplanes $H_{c_1}, \ldots, H_{c_k}$ defined in Lemma 3.7 is the set of elements $x \in F$ which satisfy $\mathrm{Tr}(cx) = 0$ for all c in the linear subspace of F spanned by $c_1, \ldots, c_k$ over F_0.*

3.9. THEOREM. *If $m \geqq 11$ then there exists a system of $n + 2$ orthogonal completely Latin cubes of order $n = 2^m$.*

Proof. In the field of order n there is an element e of order $m \geqq 11$ we now consider the 7 polynomials $g_1 = e^2t^2 + t$, $g_2 = et^2 + et$, $g_3 = t^2 + e^2t$, $g_4 = g_1 + g_2$, $g_5 = g_1 + g_3$, $g_6 = g_2 + g_3$, $g_7 = g_1 + g_2 + g_3$ and the corresponding 7 hyperplanes $H_i = H_{u_i}$ where $u_1 = e^2$, $u_2 = 1/e$, $u_3 = 1/e^4$, $u_4 = e/(1 + e)$, $u_5 = 1/(1 + e)^2$, $u_6 = 1/(e + e^2)^2$, and $u_7 = 1/(1 + e + e^2)$. These 7 values

https://doi.org/10.4153/CJM-1976-113-0 Published online by Cambridge University Press

u are linearly independent over F_0 and the 7 hyperplanes H_i are therefore in general position.

We now complete the choice of polynomial $f_1 = g_1 + c_1$, $f_2 = g_2 + c_2$, $f_3 = g_3 + c_3$ where the c_i are determined successively so as to make the 7 polynomials $f_1, f_2, f_3\ f_1 + f_2, f_1 + f_3, f_2 + f_3, f_1 + f_2 + f_3$ irreducible over F and so that

$$\begin{vmatrix} e^2 & e & 1 \\ 1 & e & e^2 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

We first pick $c_1 \notin H_{g_1}$. This gives us $n/2$ possible choices for c_1. Once we have chosen c_1 we pick c_2 so that $c_2 \notin H_{g_2}$, $c_2 \notin c_1 + H_{g_4}$. This gives us $n/4$ possible choices for c_2. Having chosen c_2 we pick c_3 so that $c_3 \notin H_{g_3}$, $c_3 \notin c_1 + H_{g_5}$, $c_3 \notin c_2 + H_{g_6}$, $c_3 \notin c_1 + c_2 + H_{g_7}$, and

$$\begin{vmatrix} e^2 & e & 1 \\ 1 & e & e^2 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0.$$

This gives us at least $n/16 - 1$ choices for c_3. Once these choices have been made we get $n + 2$ orthogonal completely Latin cubes

$$A_{ijk}{}^{(t)} = f_1(t)x_i + f_2(t)x_j + f_3(t)x_k, \quad t \notin F;$$
$$A_{ijk}{}^{(\infty)} = e^2x_i + ex_j + x_k;$$
$$A_{ijk}{}' = x_i + ex_j + e^2x_k;$$

where $F = \{x_0, \ldots, x_{n-1}\}$ is arranged as in the proof of Theorem 3.2.

4. Examples. We can use the results of Section 3 to construct strongly magic cubes of every prime power order $q \geqq 7$ and hence, by Kronecker products, for every order n whose least primary divisor is no less than 7.

We need only show that there exist triples of linearly independent vectors $\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \mathbf{C}^{(3)}$ over finite fields of order $q \geqq 7$ with the properties $c_j{}^{(i)} \neq 0$, $c_j{}^{(i)} \neq Ic_k{}^{(i)}$ for $j \neq k$. For q a power of 2 we also need the property $c_1{}^{(i)} + c_2{}^{(i)} + c_3{}^{(i)} \neq 0$. The vectors $(1, t, t^2)$ with $t \neq 0, \pm 1$ have the desired properties for odd q and so for odd $q \geqq 7$ there are at least 4 Latin cubes so that the superposition of any 3 yields a strongly magic cube. For q a power of 4 we also have to rule out the two values of t for which $t^2 + t + 1 = 0$. Thus for even q there are at least 5 Latin cubes so that the superposition of any 3 yields a strongly magic cube.

Choosing the values $t = \pm 2, 3$ for $q = 7$ we get the 3 cubes $A^{(t)}$ whose entries are

$$a_{ijk}{}^{(t)} \equiv i + tj + t^2k - 3(t + t^2) \pmod 7$$

$i, j, k = 0, \ldots, 6$. Their superposition yields a strongly magic cube with entries expressed in base 7. For 4-dimensional cubes our construction yields strongly magical yields strongly magical cubes of every primary order $q \geqq 17$.

https://doi.org/10.4153/CJM-1976-113-0 Published online by Cambridge University Press

References

1. Joseph Arkin and E. G. Straus, *Latin k-cubes*, Fibonacci Quarterly, *12* (1974), 288–292.
2. W. W. R. Ball, *Mathematical Recreations and Essays* (New York 1962).
3. Denes and Keedwell, *Latin Squares* (London 1974).
4. Walter Taylor, *On the coloration of cubes*, Discrete Math. *2* (1972), 187–190.

197 Old Nyack Turnpike,
Spring Valley, New York;
San Jose State University,
San Jose, California;
University of California,
Los Angeles, California

https://doi.org/10.4153/CJM-1976-113-0 Published online by Cambridge University Press