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WHEN THE SUM OF ALIQUOTS DIVIDES THE TOTIENT

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Abstract Let $\varphi(\cdot)$ be the Euler function and let $\sigma(\cdot)$ be the sum-of-divisors function. In this note, we bound the number of positive integers $n \leqslant x$ with the property that $s(n) = \sigma(n) - n$ divides $\varphi(n)$.

Keywords: sum of divisors; Euler function; divisibility

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1. Introduction

Let $\varphi(\cdot)$ be the *Euler function*:

$$\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^* = n\prod_{p|n}(1-1/p), \quad n \geqslant 1.$$

Let $\sigma(\cdot)$ be the *sum-of-divisors function*,

$$\sigma(n) = \sum_{d|n} d, \quad n \geqslant 1,$$

and let $s(n) = \sigma(n) - n$ be the sum of the *aliquot* divisors of $n \geqslant 1$. The function $s(n)$ and related arithmetic functions (such as $f(n) = n - \varphi(n)$) have been previously studied in the literature (see, for example, [**1**, **2**, **5**–**8**]).

In this note, we study the set of positive integers n with the property that $s(n)|\varphi(n)$. Note that if n is prime, then $s(n) = 1$, and therefore $s(n)|\varphi(n)$; hence, we restrict our attention to *composite* integers n with this property. Let

$$\mathcal{B} = \{n \text{ composite} : s(n)|\varphi(n)\}.$$

Our main result is an unconditional upper bound for the counting function $\#\mathcal{B}(x)$ of the set $\mathcal{B}$, where $\mathcal{B}(x) = \mathcal{B} \cap [1, x]$.

Theorem 1.1. *The following estimate holds as $x \to \infty$:*

$$\#\mathcal{B}(x) \leqslant x \exp(-(\tfrac{1}{3}(\log 8)^{1/3} + o(1))(\log x)^{1/3}(\log\log x)^{2/3}).$$

We cannot show that $\mathcal{B}$ is an infinite set. However, if m is a positive integer such that $p = 5m+1$ and $q = 20m+13$ are both primes, then for $n = pq$ we have $s(n) = 25m+15$ and $\varphi(n) = 4m(25m+15)$, and therefore $n \in \mathcal{B}$. We recall that Dickson's *prime k-tuplets conjecture* [**4**] asserts that there are infinitely many examples of such pairs of primes p and q. Thus, it is reasonable to expect that $\#\mathcal{B} = \infty$.

1.1. Notation

Throughout the paper, the letters p and q (with or without subscripts) are used to denote prime numbers, and the letter n is used to denote a positive integer. As usual, we denote by: $P(n)$ the largest prime factor of n; $\omega(n)$ the number of distinct prime factors of n; $\Omega(n)$ the number of prime factors of n, counted with multiplicity; $\tau(n)$ the number of positive integer divisors of n; $\mathrm{ord}_q(n)$ the order at which the prime q divides n ($\mathrm{ord}_q(n) = \alpha$ if and only if $q^\alpha \| n$).

For any set $\mathcal{A}$ of positive integers and a positive real number x, we put $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. We also put $\log x = \max\{\ln x, 1\}$, where $\ln(\cdot)$ is the natural logarithm, and we use $\log_k(\cdot)$ to denote the kth iterate of $\log(\cdot)$. Finally, we use the Vinogradov symbols $\gg$ and $\ll$, as well as the Landau symbols O and o, with their usual meanings.

2. The proof of Theorem 1.1

Let x be a large real number, let $y = y(x)$ be a function of x to be determined later, and put

$$u = u(x) = \frac{\log x}{\log y}. \tag{2.1}$$

In what follows, we assume that y and u tend to infinity with x.

Let us consider the following sets:

$$\mathcal{B}_1(x) = \{n \in \mathcal{B}(x) : P(n) \leqslant y\},$$

$$\mathcal{B}_2(x) = \{n \in \mathcal{B}(x) \setminus \mathcal{B}_1(x) : P(n)^2 | n\},$$

$$\mathcal{B}_3(x) = \left\{n \in \mathcal{B}(x) \Big\backslash \left(\bigcup_{j=1}^{2} \mathcal{B}_j(x)\right) : \omega(n) > u\right\},$$

$$\mathcal{B}_4(x) = \left\{n \in \mathcal{B}(x) \Big\backslash \left(\bigcup_{j=1}^{3} \mathcal{B}_j(x)\right) : \max_{q|n}\{\mathrm{ord}_q(n)\} > u(\log u)/\log 2\right\},$$

$$\mathcal{B}_5(x) = \left\{n \in \mathcal{B}(x) \Big\backslash \left(\bigcup_{j=1}^{4} \mathcal{B}_j(x)\right) : \omega(p-1) > u \text{ for some prime } p|n\right\},$$

$$\mathcal{B}_6(x) = \left\{ n \in \mathcal{B}(x) \Big\backslash \left(\bigcup_{j=1}^{5} \mathcal{B}_j(x) \right) : \max_{p|n,\ q|p-1} \{\mathrm{ord}_q(p-1)\} > u(\log u)/\log 2 \right\},$$

$$\mathcal{B}_7(x) = \mathcal{B}(x) \Big\backslash \left(\bigcup_{j=1}^{6} \mathcal{B}_j(x) \right).$$

Since $\mathcal{B}(x)$ is the union of the sets $\mathcal{B}_j(x)$, $j = 1, \ldots, 7$, it suffices to find an appropriate bound on the cardinality of each set $\mathcal{B}_j(x)$.

We begin with the following well-known estimate of Canfield *et al.* [**3**] for the number of *y-smooth* numbers $n \leqslant x$ (see also [**9**]):

$$\Psi(x, y) = \#\{n \leqslant x : P(n) \leqslant y\} = xu^{-u+o(u)}, \quad u \to \infty,$$

which holds uniformly in the range

$$(\log x)^{1+\varepsilon} \leqslant y \leqslant x^{1/3} \tag{2.2}$$

for every fixed $\varepsilon > 0$. From now on, we assume that (2.2) holds, and thus we have the bound

$$\#\mathcal{B}_1(x) \leqslant x \exp(-(1 + o(1))u \log u). \tag{2.3}$$

Next, let $n \in \mathcal{B}_2(x)$, and write $n = p^2 m$, where $p = P(n) > y$ and $P(m) \leqslant p$. For each prime p that arises in this way, the number of such integers $n \in \mathcal{B}_2(x)$ does not exceed $\lfloor x/p^2 \rfloor$; therefore,

$$\#\mathcal{B}_2(x) \leqslant \sum_{p>y} \frac{x}{p^2} \leqslant x \sum_{k>y} \frac{1}{k^2} \ll \frac{x}{y}. \tag{2.4}$$

Put $K = \lfloor u \rfloor$. For every $n \in \mathcal{B}_3(x)$, there exist primes $p_1 < \cdots < p_K$ such that $p_1 \cdots p_K | n$. For every sequence of primes $p_1 < \cdots < p_K$ arising in this way, the number of such integers $n \in \mathcal{B}_3(x)$ does not exceed $\lfloor x/(p_1 \cdots p_K) \rfloor$; consequently,

$$\#\mathcal{B}_3(x) \leqslant \sum_{\substack{p_1 \cdots p_K \leqslant x, \\ p_1 < \cdots < p_K}} \frac{x}{p_1 \cdots p_K} \leqslant \frac{x}{K!} \left(\sum_{p \leqslant x} \frac{1}{p} \right)^K.$$

Using *Stirling's formula* together with *Mertens's estimate*

$$\sum_{p \leqslant x} \frac{1}{p} = \log_2 x + O(1), \tag{2.5}$$

we see that the bound

$$\#\mathcal{B}_3(x) \leqslant x \left(\frac{e \log_2 x + O(1)}{\lfloor u \rfloor} \right)^{\lfloor u \rfloor} = x \exp(-(1 + o(1))u \log u) \tag{2.6}$$

holds, provided that

$$\log_3 x = o(\log u). \tag{2.7}$$

From now on, we assume that (2.7) is satisfied.

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Let $L = \lfloor u(\log u)/\log 2 \rfloor$. For every $n \in \mathcal{B}_4(x)$, there exists a prime q such that $q^L | n$. Since the number of $n \in \mathcal{B}_4(x)$ divisible by q^L does not exceed $\lfloor x/q^L \rfloor$, we have

$$\#\mathcal{B}_4(x) \leqslant \sum_{q \leqslant x} \frac{x}{q^L} \leqslant x \sum_{k \geqslant 2} \frac{1}{k^L}.$$

Since

$$\sum_{k \geqslant 2} \frac{1}{k^L} \ll \int_2^\infty \frac{\mathrm{d}t}{t^L} \ll \frac{1}{2^L} = \exp(-(1 + o(1))u \log u), \tag{2.8}$$

we derive the bound

$$\#\mathcal{B}_4(x) \leqslant x \exp(-(1 + o(1))u \log u). \tag{2.9}$$

Let $K = \lfloor u \rfloor$ as before. For every $n \in \mathcal{B}_5(x)$, there exists a sequence of primes $p_1 < \cdots < p_K$ and a prime $p | n$ such that $p_1 \cdots p_K | p - 1$. Write $p - 1 = p_1 \cdots p_K \ell$, where $\ell \geqslant 1$; then the number of $n \in \mathcal{B}_5(x)$ divisible by p is at most

$$\left\lfloor \frac{x}{p} \right\rfloor \leqslant \frac{x}{p} < \frac{x}{p - 1} = \frac{x}{p_1 \cdots p_K \ell}.$$

Summing over the possible choices of the primes $p_1 < \cdots < p_K$ and the integer ℓ, and applying Stirling's formula together with (2.5), we obtain

$$\begin{aligned} \#\mathcal{B}_5(x) &\leqslant \sum_{\substack{p_1 \cdots p_K \ell \leqslant x, \\ p_1 < \cdots < p_K}} \frac{x}{p_1 \cdots p_K \ell} \\ &\leqslant x \biggl(\sum_{\substack{p_1 \cdots p_K \leqslant x, \\ p_1 < \cdots < p_K}} \frac{1}{p_1 \cdots p_K} \biggr) \biggl(\sum_{\ell \leqslant x} \frac{1}{\ell} \biggr) \\ &\ll \frac{x \log x}{K!} \biggl(\sum_{p \leqslant x} \frac{1}{p} \biggr)^K \\ &\leqslant x \log x \biggl(\frac{e \log_2 x + O(1)}{\lfloor u \rfloor} \biggr)^{\lfloor u \rfloor} \\ &= x \exp(-(1 + o(1))u \log u), \end{aligned} \tag{2.10}$$

where we have used (2.7) in the last step.

Let $L = \lfloor u(\log u)/\log 2 \rfloor$ as before. For every $n \in \mathcal{B}_6(x)$, there exists a prime $p | n$ and a prime q such that $q^L | p - 1$. Write $p - 1 = q^L \ell$, where $\ell \geqslant 1$; then the number of $n \in \mathcal{B}_6(x)$ divisible by p is at most

$$\left\lfloor \frac{x}{p} \right\rfloor \leqslant \frac{x}{p} < \frac{x}{p - 1} = \frac{x}{q^L \ell}.$$

Summing over the possible choices of the prime q and the integer ℓ, we derive that

$$\#\mathcal{B}_6(x) \leqslant \sum_{q^L \ell \leqslant x} \frac{x}{q^L \ell} \leqslant x \biggl(\sum_{q^L \leqslant x} \frac{1}{q^L} \biggr) \biggl(\sum_{\ell \leqslant x} \frac{1}{\ell} \biggr) \ll x \log x \sum_{k \geqslant 2} \frac{1}{k^L}.$$

Using (2.8) together with (2.7), we derive the bound

$$\#\mathcal{B}_6(x) \leqslant x \exp(-(1+o(1))u \log u). \tag{2.11}$$

Finally, we come to the set $\mathcal{B}_7(x)$. Every integer $n \in \mathcal{B}_7(x)$ can be uniquely expressed in the form $n = pm$, where $p = P(n)$, $y < p \leqslant x/m$, and $P(m) < y$. Thus,

$$\#\mathcal{B}_7(x) = \sum_{m<x/y} \#\mathcal{P}_m, \tag{2.12}$$

where

$$\mathcal{P}_m = \{p : pm \in \mathcal{B}_7(x) \text{ and } p > P(m)\}, \quad m < x/y.$$

Let $m < x/y$ be fixed in what follows, and suppose that $p \in \mathcal{P}_m$. We have

$$s(pm) = \sigma(pm) - pm = p(\sigma(m) - m) + \sigma(m). \tag{2.13}$$

Since pm is not prime (as it is an element of $\mathcal{B}$), $m \neq 1$, and hence $\sigma(m) - m > 0$; therefore, the number $s(pm)$ determines p uniquely. As $s(pm)$ is a divisor of $\varphi(pm) = (p-1)\varphi(m)$, we can write $s(pm) = d_1 d_2$ for some divisors $d_1 | p-1$ and $d_2 | \varphi(m)$. Reducing the identity (2.13) modulo d_1, it follows that $2\sigma(m) - m \equiv 0 \pmod{d_1}$ and, consequently,

$$s(pm) | (2\sigma(m) - m)\varphi(m).$$

In particular, for each $p \in \mathcal{P}_m$, all of the prime factors of the number $s(pm)$ lie in the set

$$\mathcal{Q}_m = \{q : q | (2\sigma(m) - m)\varphi(m)\}.$$

Using standard estimates, we have

$$\#\mathcal{Q}_m = \omega((2\sigma(m) - m)\varphi(m)) \leqslant \lfloor \log x \rfloor$$

if x is sufficiently large (in fact, $\#\mathcal{Q}_m \ll (\log x)/\log_2 x$). On the other hand, for each $p \in \mathcal{P}_m$,

$$\begin{aligned} \omega(s(pm)) \leqslant \omega((p-1)\varphi(m)) &\leqslant \omega(p-1) + \omega(m) + \sum_{q|m} \omega(q-1) \\ &\leqslant 2K + \sum_{q|m} L \leqslant 2K + KL, \end{aligned}$$

where $K = \lfloor u \rfloor$ and $L = \lfloor u(\log u)/\log 2 \rfloor$ as before. Let $\gamma(k) = \prod_{q|k} q$ denote the square-free kernel of the integer k. Then the preceding argument shows that, as p varies over the set $\mathcal{P}_m$, the number $\gamma(s(pm))$ takes at most

$$\binom{\lfloor \log x \rfloor}{2K + KL} \leqslant \left(\frac{\log x}{2K + KL}\right)^{2K+KL}$$

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possible values. Since the order at which any prime q divides $s(pm)$ is bounded by

$$\begin{aligned}\operatorname{ord}_q(s(pm)) &\leqslant \operatorname{ord}_q((p-1)\varphi(m))\\ &\leqslant \operatorname{ord}_q(p-1) + \operatorname{ord}_q(m) + \sum_{q'\mid m} \operatorname{ord}_q(q'-1)\\ &\leqslant 2L + \sum_{q'\mid m} L \leqslant 2L + KL,\end{aligned}$$

we see that the total number of possibilities for the number $s(pm)$, as p varies over the set $\mathcal{P}_m$, does not exceed

$$\left(\frac{\log x}{2K+KL}\right)^{2K+KL} (2L+KL)^{2K+KL}.$$

Since the number $s(pm)$ determines p uniquely, we therefore obtain the bound

$$\begin{aligned}\#\mathcal{P}_m &\leqslant \left(\frac{\log x}{2K+KL}\right)^{2K+KL} (2L+KL)^{2K+KL}\\ &= \exp((2K+KL)(\log_2 x - \log(2K+KL) + \log(2L+KL)))\\ &= \exp\left((1+o(1))\frac{u^2 \log u}{\log 2}\log_2 x\right),\end{aligned}$$

where we have substituted the predefined values of K and L and used the fact that

$$\log(2L+KL) - \log(2K+KL) = \log\left(\frac{2L+KL}{2K+KL}\right) = \log(1+o(1)) = o(1)$$

as $x \to \infty$. Using the previous estimate in (2.12), we deduce that

$$\#\mathcal{B}_7(x) = \frac{x}{y}\exp\left((1+o(1))\frac{u^2\log u}{\log 2}\log_2 x\right). \tag{2.14}$$

Combining the estimates (2.3), (2.4), (2.6), (2.9)–(2.11) and (2.14), we have

$$\#\mathcal{B}(x) \leqslant x\exp(-(1+o(1))u\log u) + \frac{x}{y}\exp\left((1+o(1))\frac{u^2\log u}{\log 2}\log_2 x\right).$$

To optimize this estimate, we balance the two expressions by choosing the value of u for which

$$\frac{u^2\log u}{\log 2}\log_2 x - \log y = -u\log u.$$

In view of (2.1), this is equivalent to

$$\frac{u^3\log u}{\log 2}\log_2 x + u^2\log u = \log x.$$

For such u, we have

$$u = (1+o(1))\left((\log 8)\frac{\log x}{\log_2 x}\right)^{1/3}$$

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and

$$y = \exp(((\log 8)^{-1/3} + o(1))(\log x)^{2/3}(\log_2 x)^{1/3});$$

thus, y and u tend to infinity with x, and the required conditions (2.2) and (2.7) are clearly satisfied. With the above choice of u, we have

$$\#\mathcal{B}(x) \leqslant x \exp(-(1 + o(1))u \log u),$$

and the theorem follows.

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