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ON THE SIZE OF A MAXIMUM TRANSVERSAL IN A STEINER TRIPLE SYSTEM

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Let $(X, \mathscr{B})$ be a Steiner triple system on $v = |X|$ points, and suppose that $\mathscr{F} \subset \mathscr{B}$ is a partial parallel class (transversal, clear set, set of pairwise disjoint blocks) of maximum size $t = |\mathscr{F}|$. We want to derive a bound on $r = |X \backslash \cup \mathscr{F}| = v - 3t$. (I conjecture that in fact r is bounded, e.g., $r \leqq 4 - 4$ is attained for the Fano plane, but all that has been proved so far (cf. [**1**], [**2**]) are bounds $r < C.v$ for some C. Here we shall prove $r < 5v^{2/3}$.)

Define a sequence of positive real numbers by $q_0 = Q \cdot r^2/v$, $q_1 = \frac{1}{2} q_0$, $\ldots q_i = \frac{1}{2} q_{i-1}, \ldots, q_l$, where l is determined by $q_l \geqq 6$, $\frac{1}{2} q_l < 6$, i.e.,

$$l = [\log (Qr^2/6v)/\log 2].$$

(The constant Q will be chosen later.) Define inductively sets A_i, K_i and collections $\mathscr{B}_i, \mathscr{F}_i$ as follows. Let

$$A_0 = X \backslash \cup \mathscr{F},$$

and for $0 \leqq i \leqq l$, let

$$\begin{aligned} \mathscr{B}_i &= \{T \in \mathscr{B} | \quad |T \cap A_i| \geqq 2\}, \\ K_i &= \{x \in X \backslash A_i | \quad \#\{T \in \mathscr{B}_i | x \in T\} \geqq q_i\}, \\ \mathscr{F}_i &= \{T \in \mathscr{F} | \quad |T \cap K_i| \geqq 1\}, \\ A_{i+1} &= A_0 \cup (\cup \mathscr{F}_i) \backslash K_i. \end{aligned}$$

One verifies immediately that each of these series is increasing: $A_i \subset A_{i+1}$, $K_i \subset K_{i+1}$ etc. Also that $A_i \cap K_j = \emptyset$ $(\forall i, j)$. It is convenient to set $\mathscr{F}_{-1} = \emptyset$. {The numbers q_i are chosen in such a way that an exchange process works. If B is an arbitrary block and we want to add it to $\mathscr{F}$, we must discard at most three members of $\mathscr{F}$ in order to maintain disjointness. But if the discarded triples are in $\mathscr{F}_i$ for some i then they are of the form $\{a, b, x\}$ with $x \in K_i$, and now that we no longer use x (supposing that $x \notin B$) we may add new triples $\{x, c, d\} \in \mathscr{B}_i$ to $\mathscr{F}$. In order to be able to add three pairwise disjoint triples $\{x_j, c_j, d_j\} \in \mathscr{B}_i$ $(j = 1, 2, 3)$ we must be sure that each x_j is incident with sufficiently many blocks in $\mathscr{B}_i$. (In fact it suffices if x_1 is incident with 1 block, x_2 with 3 blocks and x_3 with 5 blocks.) If $i = 0$ we are finished and have increased the size of our transversal. If $i > 0$ then we must continue, discard the at

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most six members of $\mathscr{F}_{i-1}$ containing the points c_j, d_j and add again members of $\mathscr{B}_{i-1}$ etc.}

Claim. (i) A_i does not contain a block $B \in \mathscr{B}$ $(0 \leqq i \leqq l+1)$.
(ii) No block $T \in \mathscr{F}$ intersects K_i in more than one point $(0 \leqq i \leqq l)$.

Proof. Ad (i): If $B \subset A_0$ for some block $B \in \mathscr{B}$ then $\mathscr{F} \cup \{B\}$ would be a larger partial parallel class, a contradiction. If $B \subset A_{i+1}$ then we can enlarge $\mathscr{F}$ by an exchange process:

Define $\mathscr{N}_j$, $\mathscr{R}_j$ by backward induction on j $(i+1 \geqq j \geqq 0)$:

$$\mathscr{R}_{i+1} = \emptyset, \mathscr{N}_{i+1} = \{B\},$$

$$\mathscr{R}_j = \left\{ T \in \mathscr{F}_j \backslash \mathscr{F}_{j-1} \,\middle|\, T \cap \bigcup_{k=j+1}^{i+1} \cup \mathscr{N}_k \neq \emptyset \right\}.$$

Choose for $\mathscr{N}_j$ some collection of $|\mathscr{R}_j|$ blocks from $\mathscr{B}_j$ such that each $T \in \mathscr{R}_j$ meets exactly one of them, and such that $\mathscr{N}_j \cup \mathscr{N}_{j+1} \cup \ldots \cup \mathscr{N}_{i+1}$ is a collection of pairwise disjoint blocks. That the latter is possible follows from

$$\left| \left(\bigcup_{k=j}^{i+1} \cup \mathscr{N}_k \right) \cap A_j \right| \leqq 3.2^{i-j}$$

and

$$q_j \geqq 6.2^{i-j} - 1.$$

Now

$$\mathscr{F}' = \left(\mathscr{F} \cup \bigcup_{j=0}^{i+1} \mathscr{N}_j \right) \Big\backslash \bigcup_{j=0}^{i} \mathscr{R}_j$$

is a larger partial parallel class, a contradiction.

Ad (ii): This is proved using an almost identical argument.

Let $a_i = |A_i|$, so that $r = a_0$, and let $k_i = |K_i|$. By (ii) it follows that

(1) $\quad a_{i+1} = 2k_i + r.$

From (i) it follows that

$$\binom{a_i}{2} \leqq k_i \cdot \frac{a_i}{2} + (v - k_i - a_i) \cdot q_i,$$

hence

(2) $\quad a_i < k_i + \dfrac{2q_i v}{a_i},$

and, using (1) and $a_j \geqq a_0$, $q_j \leqq q_0$,

(3) $\quad a_{i+1} > 2a_i + r(1 - 4Q).$

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Now $v \geqq a_{l+1} + k_l = r + 3k_l$ so that

$$\tfrac{1}{3} v > a_l - 2Qr > 2a_{l-1} + r(1 - 6Q) > 4a_{l-2} + r(3 - 14Q) > \dots$$
$$> 2^l a_0 + r(2^l - 1 - (2^{l+2} - 2)Q) = r(2^{l+1} - 1)(1 - 2Q)$$
$$> r\left(\frac{Qr^2}{6v} - 1\right)(1 - 2Q).$$

Take $Q = \frac{1}{4}$. Then we have for large r:

$$(16 + \epsilon)v^2 > r^3$$

and one verifies immediately that $r \geqq 5v^{2/3}$ leads to a contradiction for all r. In this proof we implicitly assume that $l \geqq 0$. But $l < 0$ means $Qr^2 < 6v$ so that again $Q = \frac{1}{4}$, $r \geqq 5v^{2/3}$ leads to a contradiction. Thus we proved:

THEOREM. *A maximum transversal of an* STS (v) *has size at least*

$$\frac{1}{3}v - \frac{5}{3}v^{2/3}.$$

It is easy to improve the constant 5 (a minor change in this proof gives 3, and further improvement is possible) but I am presently unable to improve on the exponent 2/3.

Note. An almost identical proof works for Steiner quadruple systems, and again gives $r = O(v^{2/3})$.

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