Canad. J. Math. Vol. **57** (6), 2005 pp. 1279–1290

A Semilinear Problem for the Heisenberg Laplacian on Unbounded Domains

Sara Maad

Abstract. We study the semilinear equation

$$-\Delta_{\mathbb{H}} u(\eta) + u(\eta) = f(\eta, u(\eta)), \quad u \in \mathring{S}_1^2(\Omega),$$

where Ω is an unbounded domain of the Heisenberg group $\mathbb{H}^N$, $N \geq 1$. The space $\mathring{S}_1^2(\Omega)$ is the Heisenberg analogue of the Sobolev space $W_0^{1,2}(\Omega)$. The function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is supposed to be odd in u, continuous and satisfy some (superlinear but subcritical) growth conditions. The operator $\Delta_{\mathbb{H}}$ is the subelliptic Laplacian on the Heisenberg group. We give a condition on Ω which implies the existence of infinitely many solutions of the above equation. In the proof we rewrite the equation as a variational problem, and show that the corresponding functional satisfies the Palais–Smale condition. This might be quite surprising since we deal with domains which are far from bounded. The technique we use rests on a compactness argument and the maximum principle.

1 Introduction

Let $\mathbb{H}^N$ be the space $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ equipped with the group operation

$$\eta \circ z = (\alpha, \beta, \tau) \circ (x, y, t) = (x + \alpha, y + \beta, t + \tau + 2(x \cdot \beta - y \cdot \alpha)).$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^N$. This operation endows $\mathbb{H}^N$ with the structure of a Lie group. We define vector fields $X_1, \ldots, X_N, Y_1, \ldots, Y_N$, and T by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \tag{1.1}$$

$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \tag{1.2}$$

$$T = \frac{\partial}{\partial t}. \tag{1.3}$$

It is easy to check that these vector fields form a basis for the left-invariant vector fields on $\mathbb{H}^N$. The commutation relations are

$$[X_i, Y_j] = -4\delta_{ij} T,$$
$$[X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0,$$

Received by the editors January 19, 2004.
AMS subject classification: 22E30, 22E27.
Keywords: Heisenberg group, concentration compactness, Heisenberg Laplacian.

and they imply that the Lie algebra of left-invariant vector fields is generated by X_1, $\dots$, X_N, Y_1, $\dots$, Y_N. This also means that these vector fields satisfy the Hörmander condition of order 1 (see [6]).

The Heisenberg Laplacian is by definition

$$\begin{aligned}\Delta_{\mathbb{H}} &= \sum_{j=1}^{N}(X_j^2 + Y_j^2)\\ &= \sum_{j=1}^{N}\Big(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j\frac{\partial^2}{\partial x_j\partial t} - 4x_j\frac{\partial^2}{\partial y_j\partial t} + 4(x_j^2 + y_j^2)\frac{\partial^2}{\partial t^2}\Big),\end{aligned}$$

and we use the notation $\nabla_{\mathbb{H}} u$ for the $2N$-vector

$$(X_1 u, \dots, X_N u, Y_1 u, \dots, Y_N u).$$

Let $\Omega \subset \mathbb{H}^N$ be a domain (bounded or unbounded). The space $\mathring{S}_1^2(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{S_1^2}$ given by

$$\|u\|_{S_1^2}^2 = \int_{\mathbb{R}^N} |u(\eta)|^2\, d\eta + \sum_{j=1}^{N}\int_{\mathbb{R}^N}\big(|X_j u(\eta)|^2 + |Y_j u(\eta)|^2\big)\, d\eta.$$

The left and right Haar measures on $\mathbb{H}^N$ are the Lebesgue measure, and the integral above is taken with respect to this measure. Let $Q = 2N + 2$ be the homogeneous dimension of $\mathbb{H}^N$ and let $2^* = 2Q/(Q-2) = 2 + 2/N$.

A model problem for those we are studying is

$$-\Delta_{\mathbb{H}} u + u = |u|^{p-2}u, \quad u \in \mathring{S}_1^2(\Omega), \tag{1.4}$$

where $p \in (2, 2^*)$ and Ω is a domain in $\mathbb{H}^N$. Here Ω can be bounded or unbounded, but for Ω bounded, our main result can also be obtained by using the compactness of the embedding of $\mathring{S}_1^2(\Omega)$ into $L^p(\Omega)$. Thus, we are mainly interested in unbounded Ω.

Equation (1.4) has been studied by, among others, Garofalo and Lanconelli [5] and Birindelli and Cutri [1].

The corresponding functional is given by

$$\varphi(u) = \frac{1}{2}\int_\Omega |u(\eta)|^2 + \frac{1}{2}\sum_{j=1}^{N}\int_\Omega \big(|X_j u(\eta)|^2 + |Y_j u(\eta)|^2\big)\, d\eta - \frac{1}{p}\int_\Omega |u(\eta)|^p\, d\eta,$$

and the critical points of φ are exactly the weak solutions of (1.4).

In [8], the author studied a similar problem for the classical Laplacian on $\mathbb{R}^N$, $N \geq 3$, of which the following is a special case: Let $\Omega \subset \mathbb{R}^N$ be a domain, and consider the problem

$$-\Delta u + u = |u|^{q-2}u \quad u \in H_0^1(\Omega),$$

where $q \in (2, 2N/(N-2))$. Then the notion of strongly asymptotically contractive domains was introduced, and it was shown that for such domains the preceding equation has infinitely many solutions. In $\mathbb{R}^N$, the main example of a strongly asymptotically contractive domain is a tubelike domain like, for example,

$$\Omega = \{(x, y) \in \mathbb{R}^l \times \mathbb{R}^{N-l} ; |y| < g(x)\},$$

where $1 \le l \le N-1$ and $g \in C(\mathbb{R}^l, \mathbb{R})$ is such that the limit $g_\infty = \lim_{|x| \to \infty} g(x)$ exists and $g(x) > g_\infty$ for all $x \in \mathbb{R}^l$.

In the present paper, we use similar methods to prove the existence of infinitely many solutions of (1.4). We define strongly asymptotically contractive domains in $\mathbb{H}^N$, and prove that for such domains there are infinitely many solutions of (1.4), and the corresponding functional φ has infinitely many critical values.

There is another paper which deals with more or less the same class of domains. In a paper by Tintarev [11], there is a theorem similar to Theorem 2.5. The two theorems are not independent; the proofs of both are based on an idea (in the case of $\mathbb{R}^N$) from the author's licentiate thesis [7], written under the supervision of Tintarev. An improved version of this thesis is published as [8]. Tintarev does not make any assumption that $sf(s, \eta) \ge 0$ (see condition (f_2) in Section 2), and without this condition, the solutions may change sign, and the maximum principle cannot be applied. Because of this, Tintarev's proof only works in the case of the ground state solution, which is known to be of constant sign. In this paper, we show the theorem for all values of φ.

It is interesting to see that there are many more domains which are strongly asymptotically contractive in $\mathbb{H}^N$ than in $\mathbb{R}^{2N+1}$. This fact depends on how the group operation of $\mathbb{H}^N$ comes into the definition of strongly asymptotically contractive domains. This will be seen in Section 2, where we also state the problem in full generality.

In Section 3, we prove that φ satisfies the (P-S)$_c$ condition for every $c > 0$, and in Section 4, we use a known minimax theorem to obtain an infinite sequence of critical values of φ.

2 A Semilinear Elliptic Equation

We study a more general problem than equation (1.4). Consider

$$-\Delta_{\mathbb{H}} u + u = f(\eta, u), \quad u \in \mathring{S}_1^2(\Omega) \tag{2.1}$$

where $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions

(f_1) there are constants $2 < p \le q < 2^*$ and $C > 0$ such that for any $\eta \in \Omega$ and $s \in \mathbb{R}$,

$$|f(\eta, s)| \le C(|s|^{p-1} + |s|^{q-1}),$$

(f_2) there are constants $\mu > 2$, $\nu > 2$ and $D > 0$ such that for any $\eta \in \Omega$ and $s \in \mathbb{R}^N \setminus \{0\}$,

$$sf(\eta, s) \ge \mu F(\eta, s) \equiv \mu \int_0^s f(\eta, \sigma)\, d\sigma > 0$$

and

$$\liminf_{s\to 0}\frac{F(\eta,s)}{|s|^{\nu}}\geq D.$$

(f_3) f is odd in s, *i.e.*, for any $\eta\in\mathbb{H}^N$ and $s\in\mathbb{R}$,

$$f(\eta,s)=-f(\eta,-s).$$

(f_4) there exists a continuous function $f_\infty\colon\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{R\to\infty}\sup_{\substack{x\in\Omega\setminus B_R(0)\\ s\in\mathbb{R}}}|f(x,s)-f_\infty(s)|=0.$$

The functional on $\mathring{S}_1^2(\Omega)$ corresponding to equation (2.1) is given by

$$\varphi(u)=\frac{1}{2}\|u\|_{S_1^2}^2-\int_\Omega F(\eta,u(\eta))\,d\eta, \tag{2.2}$$

and the critical points of φ are exactly the solutions of (2.1).

Definition 2.1 We will say that a domain $\Omega\subset\mathbb{H}^N$ is *strongly asymptotically contractive* if $\Omega\neq\mathbb{H}^N$ and for any sequence $\eta_j\in\mathbb{H}^N$ such that $|\eta_j|\to\infty$, there exists a subsequence η_{j_l} such that either

(i) $|\bigcup_{n=1}^{\infty}\bigcap_{l=n}^{\infty}(\eta_{j_l}\circ\Omega)|=0$, or
(ii) there exists a point $\eta_0\in\mathbb{H}^N$ such that for any $R>0$ there exists an open set $M_R\Subset\eta_0\circ\Omega$, a closed set Z of measure zero and an integer $l_R>0$ such that

$$(\eta_{j_l}\circ\Omega)\cap B_R(0)\subset M_R\cup Z$$

for any $l\geq l_R$.

The following two examples show that many domains satisfy this condition. The condition of Example 2.2 was introduced by del Pino and Felmer [3], who studied a semilinear problem on unbounded domains in $\mathbb{R}^N$.

Example 2.2 Let $\Omega\subset\mathbb{H}^N$ be a domain. For $t\in\mathbb{R}$, let

$$\Omega_t=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\,;\,(x,y,t)\in\Omega\}.$$

Suppose that

(i) there exist compact sets $E,F\subset\mathbb{R}^N\times\mathbb{R}^N$ such that $E\supset\Omega_t\supset F$ for every $t\in\mathbb{R}$;
(ii) for any $\epsilon>0$ there exists $R>0$ such that if $|t|>R$, then $\Omega_t\subset F+B_\epsilon(0)$.

Then Ω is strongly asymptotically contractive.

Proof Let $\eta_j = (\alpha_j, \beta_j, \tau_j)$ be a sequence in $\mathbb{H}^N$ such that $|\eta_j| \to \infty$. If there exists a subsequence η_{j_l} such that $|\alpha_{j_l}|^2 + |\beta_{j_l}|^2 \to \infty$ as $l \to \infty$, then it is easy to see that for $R > 0$ fixed, there exists $l_R \in \mathbb{N}$ such that

$$(\eta_{j_l} \circ \Omega) \cap B_R(0) = \varnothing$$

for any $l \geq l_R$.

Thus, we may assume that α_j and β_j are bounded. By restricting ourselves to a subsequence, we may assume that $\alpha_j \to \alpha_0$, $\beta_j \to \beta_0$ and $|\tau_j| \to \infty$ as $j \to \infty$. Let $\eta_0 = (\alpha_0, \beta_0, 0)$, and let $\epsilon > 0$ and $R > 0$ be given. Then for j large, the set $(\eta_j \circ \Omega) \cap B_R(0) = \{(x + \alpha_j, y + \beta_j, t + \tau_j + 2(x \cdot \beta_j - y \cdot \alpha_j)\,;\, (x, y, t) \in \Omega\} \cap B_R(0)$ is contained in an $\epsilon/2$-neighborhood of

$$\{(x, y, t + \tau_j) \in \mathbb{H}^N \,;\, (x, y, t) \in (\eta_0 \circ \Omega)\} \cap B_R(0),$$

so that for j large,

$$(\eta_j \circ \Omega) \cap B_R(0) \subset \Big(\big((\alpha_0, \beta_0) + F + B_\epsilon(0) \big) \times \mathbb{R} \Big) \cap B_R(0).$$

Thus, if $\epsilon > 0$ is small enough, we can put

$$M_R = \Big(\big((\alpha_0, \beta_0) + F + B_\epsilon(0) \big) \times \mathbb{R} \Big) \cap B_R(0),$$

and we are done. ∎

Example 2.3 Let $\Omega \subset \mathbb{H}^N$ be a domain such that there exist constants $C > 0$ and $r < 1$ with

$$\Omega \subset \{(x, y, t) \in \mathbb{H}^N \,;\, |t| \leq C(1 + |x|^r + |y|^r)\}.$$

Then Ω is strongly asymptotically contractive.

Proof Let $\eta_j = (\alpha_j, \beta_j, \tau_j)$ be a sequence in $\mathbb{H}^N$ such that $|\eta_j| \to \infty$. Note that

$$\begin{aligned}(\eta_j \circ \Omega) &= \{(x + \alpha_j, y + \beta_j, t + \tau_j + 2(x \cdot \beta_j - y \cdot \alpha_j))\,;\, (x, y, t) \in \Omega\} \\ &= \{(x, y, t + \tau_j + 2(x \cdot \beta_j - y \cdot \alpha_j)\,;\, (x - \alpha_j, y - \beta_j, t) \in \Omega\} \\ &\subset \{(x, y, t + \tau_j + 2(x \cdot \beta_j - y \cdot \alpha_j)\,;\, |t| < C(1 + |x - \alpha_j|^r + |y - \beta_j|^r)\}.\end{aligned}$$

Hence, if $(x, y, t) \in (\eta_j \circ \Omega) \cap B_R(0)$, then $|x| < R$, $|y| < R$, and

$$|t + \tau_j + 2(x \cdot \beta_j - y \cdot \alpha_j)| < R.$$

If $|\alpha_j| + |\beta_j| \neq 0$, we let

$$\gamma_j = \frac{\tau_j}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}}.$$

We get two different situations depending on whether the sequence γ_j is bounded or unbounded.

We consider first the case when γ_j is unbounded. Let $R > 0$ be arbitrary. Then

$$\frac{|\tau_j + 2(x\cdot\beta_j - y\cdot\alpha_j)|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}} \le \frac{R + C(1 + (R + |\alpha_j|)^r + (R + |\beta_j|)^r)}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}}. \tag{2.3}$$

Since the right-hand side of this inequality is bounded whereas the left-hand side is unbounded, we have a contradiction. Consequently, there is a subsequence η_{j_l} such that

$$(\eta_{j_l} \circ \Omega) \cap B_R(0) = \varnothing,$$

and so Definition 2.1(ii) is satisfied for this sequence.

The case when $|\alpha_j|^2 + |\beta_j|^2 \equiv 0$ on a subsequence is similar. We just have to replace equation (2.3) by

$$|\tau_j + 2(x\cdot\beta_j - y\cdot\alpha_j)| \le R + C(1 + (R + |\alpha_j|)^r + (R + |\beta_j|)^r),$$

and argue in the same way as before.

If γ_j is bounded, we may extract a subsequence and assume that $\gamma_j \to \gamma_0$. We put $\alpha_j = (\alpha_j^1, \alpha_j^2, \dots, \alpha_j^N)$ and $\beta_j = (\beta_j^1, \beta_j^2, \dots, \beta_j^N)$. After a rotation of the x and y coordinates we may also assume that $\alpha_j^1 \to +\infty$ while $\lim_{j\to\infty} \alpha_j^2 = \cdots = \lim_{j\to\infty} \alpha_j^N = 0$, and that $\lim_{j\to\infty} \beta_j^1 = \cdots = \lim_{j\to\infty} \beta_j^N = 0$.

Let $R > 0$, and let $\epsilon > 0$ be given. For j large, the domain $(\eta_j \circ \Omega) \cap B_R(0)$ lies between the two hypersurfaces

$$\eta_j \circ \mathcal{H}_+ = \{(x, y, \tau_j + (C+\epsilon)(1 + |x - \alpha_j|^r + |y|^r) - 2\alpha_j^1 y_1) \,;\, x, y \in \mathbb{R}^N\}$$

and

$$\eta_j \circ \mathcal{H}_- = \{(x, y, \tau_j - (C+\epsilon)(1 + |x - \alpha_j|^r + |y|^r) - 2\alpha_j^1 y_1) \,;\, x, y \in \mathbb{R}^N\}.$$

Let $(x, y, t) \in (\eta_j \circ \Omega) \cap B_R(0)$. Then the distance of (x, y, t) to the hyperplane $y_1 = \gamma_0/2$ can be estimated by

$$\begin{aligned} |y_1 - \gamma_0/2| &\le \left| \frac{C+\epsilon}{2\alpha_j^1}(1 + |x - \alpha_j|^r + |y|^r) + \frac{R}{2\alpha_j^1} \right| \\ &\le \frac{(C+\epsilon)(1 + |R + \alpha_j|^r + R^r) + R}{2\alpha_j^1} \to 0 \end{aligned}$$

as $j \to \infty$.

This shows that for each $n \ge 1$,

$$\bigcap_{j=n}^{\infty} (\eta_j \circ \Omega) \subset \{(x, y, t) \in \mathbb{H}^N \,;\, y_1 = 0\},$$

which leads to

$$\Big|\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty}(\eta_j\circ\Omega)\Big| = 0,$$

so that Definition 2.1(ii) is fulfilled for this sequence. ■

With the guidance of these examples, it is easy to find other examples of strongly asymptotically contractive domains. We could, for example, consider unions of intersecting domains of the above types. We conclude with an example of a domain which is not strongly asymptotically contractive. This shows that we cannot improve Example 2.3 by allowing $r = 1$.

Example 2.4 The domain given by

$$\Omega = \{(x, y, t) \in \mathbb{H}^N ; |t| \leq 1 + |x| + |y|\}$$

is not strongly asymptotically contractive.

Proof Let $\eta_j = (\alpha_j, 0, 0)$, where $\alpha_j = (\alpha_j^1, 0, \dots, 0)$ and $|\alpha_j| \to \infty$. A similar calculation as in Example 2.3 shows that

$$\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty}(\eta_j\circ\Omega) \supset \{(x, y, t) \in \mathbb{H}^N ; |y_1| < 1/2\}.$$

It is also easy to see that Definition 2.1(ii) cannot hold for this η_j. ■

Now we are ready to state our main result.

Theorem 2.5 *Suppose that $\Omega \subset \mathbb{H}^N$ is a strongly asymptotically contractive domain and let $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy (f_1), (f_2), (f_3) and (f_4). Then equation* (2.1) *has infinitely many distinct solutions $u \in \mathring{S}_1^2(\Omega)$.*

3 The P-S$_c$ Condition

The proof of the following lemma is the same as the proof of [8, Lemma 8], and so we merely state the result here.

Lemma 3.1 *Let φ be given by* (2.2)*, where $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies condition (f_2). Let u_j be a* P-S$_c$*-sequence for φ,* i.e., *a sequence such that*

(i) $\varphi(u_j) \to c$,
(ii) $\varphi'(u_j) \to 0$.

Then $c \geq 0$. Moreover, $\|u_j\|_{S_1^2}$ is bounded, and

$$\limsup_{j\to\infty} \|u_j\|_{S_1^2}^2 \leq \frac{c}{\frac{1}{2} - \frac{1}{\mu}}.$$

We also need the following lemma from Schindler and Tintarev [9, 12]:

Lemma 3.2 *Let u_j be a bounded sequence in $S_1^2(\mathbb{H}^N)$. Then there exist $w^{(0)}$, $w^{(1)}$, $w^{(2)}$, $\dots, \in S_1^2(\mathbb{H}^N)$, $r_j \in S_1^2(\mathbb{H}^N)$ and $\eta_j^{(n)} \in \mathbb{H}^N$, $j, n \in \mathbb{N}$ such that on a renumbered subsequence,*

(i) $\tau_{\eta_j^{(n)}} u_j \rightharpoonup w^{(n)}$,

(ii) $u_j = w^{(0)} + \sum_1^\infty \tau_{(\eta_j^{(n)})^{-1}} w^{(n)} + r_j$, *where* $r_j \to 0$ *in* $L^p(\mathbb{H}^N)$, $p \in (2, 2^*)$,

(iii) $d(0, \eta_j^{(n)}) \to \infty$ *if* $n \geq 1$,

(iv) $\sum_0^\infty \|w^{(n)}\|_{S_1^2}^2 \leq \lim_{j\to\infty} \|u_j\|_{S_1^2}^2$

(v) $\sum_0^\infty \|w^{(n)}\|_{L^p}^p = \lim_{j\to\infty} \|u_j\|_{L^p}^p$, *for all* $p \in (2, 2^*)$.

Lemma 3.3 *Let $\Omega \subset \mathbb{H}^N$ be a strongly asymptotically contractive domain, and let φ be given by* (2.2), *where $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies conditions* (f_1), (f_2), (f_3) *and* (f_4). *Then φ satisfies the* P-S$_c$ *condition for any $c > 0$.*

Proof Let $c > 0$, and let $u_j \in \mathring{S}_1^2(\Omega)$ be a P-S$_c$ sequence for φ. By Lemma 3.1, the sequence u_j is bounded. Hence Lemma 3.2 is applicable, and so

$$u_j = w^{(0)} + \sum_{n=1}^\infty \tau_{(\eta_j^{(n)})^{-1}} w^{(n)} + r_j,$$

where $r_j \in L^p(\mathbb{H}^N)$, $\eta_j^{(n)} \in \mathbb{H}^N$, $d(\eta_j^{(n)}, 0) \to \infty$ and $w^{(n)} \in S_1^2(\mathbb{H}^N)$ are as in Lemma 3.2.

Since Ω is strongly asymptotically contractive, one of the two conditions of Definition 2.1 is satisfied. We will show that in either case, $w^{(n)} = 0$ for all $n \geq 1$.

Let $n \geq 1$. If $|\bigcup_{n=1}^\infty \bigcap_{j=n}^\infty (\eta_j^{(n)} \circ \Omega)| = 0$, then since weak convergence implies a.e. convergence on a subsequence, $w^{(n)} = 0$ almost everywhere.

If the second case holds, for any $n \geq 1$ there exists a subsequence $\eta_{j_l}^{(n)}$ and a point $\eta_0^{(n)} \in \mathbb{H}^N$ such that for any $R > 0$ there exists an open set $M_R^{(n)} \Subset \eta_0^{(n)} \circ \Omega$, a closed set $Z^{(n)}$ of measure zero and an integer $l_R > 0$ such that for any $R > 0$ and any $l > l_R$,

$$(\eta_{j_l}^{(n)} \circ \Omega) \cap B_R(0) \subset M_R^{(n)} \cup Z^{(n)}.$$

By extracting a subsequence, we can assume that this relation holds for any $j \geq j_R$. We have

$$\text{supp}\, w^{(n)} \cap B_R(0) \subset \overline{\bigcup_{j=j_R}^\infty (\eta_j^{(n)} \circ \Omega)} \cap B_R(0) \subset \overline{M}_R^{(n)} \cup Z^{(n)} \Subset (\eta_0^{(n)} \circ \Omega) \cup Z^{(n)}, \tag{3.1}$$

and so

$$\text{supp}\, w^{(n)} \subset \eta_0^{(n)} \circ \Omega$$

modulo a set of measure zero.

Let $\tilde{w}^{(n)} = \tau_{\eta_0^{(n)}} w^{(n)}$ and let $\tilde{\eta}_j^{(n)} = \eta_j^{(n)} \circ \eta_0^{(n)}$.

Note that by (3.1), there exists an open set $U \subset \Omega$ such that $\tilde{w}^{(n)} \equiv 0$ in U. This is important since we will use the strong maximum principle to conclude that $\tilde{w}^{(n)} \equiv 0$ in Ω.

Let $v \in C_0^\infty(\Omega)$ be arbitrary. Then

$$\begin{aligned}
\int_\Omega \big(\nabla_{\mathbb{H}} \tilde{w}^{(n)}(z) \cdot \nabla_{\mathbb{H}} v(z) + \tilde{w}^{(n)}(z) v(z) \big)\, dz \\
&= \lim_{j\to\infty} \int_\Omega \big(\nabla_{\mathbb{H}} (\tau_{\tilde{\eta}_j^{(n)}} u_j)(z) \cdot \nabla_{\mathbb{H}} v(z) + \tau_{\tilde{\eta}_j^{(n)}} u_j(z) v(z) \big)\, dz \\
&= \lim_{j\to\infty} \int_{\tilde{\eta}_j^{(n)} \circ \Omega} f\big((\tilde{\eta}_j^{(n)})^{-1} \circ z, \tau_{\tilde{\eta}_j^{(n)}} u_j(z) \big)\, v(z)\, dz \\
&= \lim_{j\to\infty} \int_{\tilde{\eta}_j^{(n)} \circ \Omega} f\big((\tilde{\eta}_j^{(n)})^{-1} \circ z, \tilde{w}^{(n)}(z) \big)\, v(z)\, dz \\
&= \int_\Omega f_\infty(\tilde{w}^{(n)}(z)) v(z)\, dz.
\end{aligned}$$

Hence $\tilde{w}^{(n)}$ is a solution of

$$-\Delta_{\mathbb{H}} u + u = f_\infty(u), \qquad u \in \mathring{S}_1^2(\Omega).$$

Note that f_∞ satisfies conditions (f_1) and (f_2). By the L^p estimates (see [4, Theorem 9.4]) $\tilde{w}^{(n)}$ is continuous in Ω. Let Ω_+, Ω_- and Ω_0 be subsets of Ω defined by

$$\Omega_+ = \{\eta \in \Omega \,;\, \tilde{w}^{(n)}(\eta) > 0\},$$

$$\Omega_- = \{\eta \in \Omega \,;\, \tilde{w}^{(n)}(\eta) < 0\},$$

$$\Omega_0 = \{\eta \in \Omega \,;\, \tilde{w}^{(n)}(\eta) = 0\}.$$

By our previous discussion, Ω_0 has a nonempty interior.

We claim that $\tilde{w}^{(n)} \equiv 0$ in Ω. We argue by contradiction, and assume that either Ω_+ or Ω_- is nonempty. If Ω_+ is nonempty, then it has a component $\tilde{\Omega}_+$ such that $\partial\Omega_+ \cap \partial\Omega_0^\circ \neq \varnothing$. By the strong maximum principle (see [2, Corollary 3.1]), $w^{(n)} \equiv 0$ in $(\tilde{\Omega}_+ \cup \Omega_0)^\circ$. Hence $\tilde{\Omega}_+ = \varnothing$. In a similar manner we see that Ω_- cannot have a component whose boundary intersect the boundary of $\partial\Omega_0$. Thus $\Omega_+ = \Omega_- = \varnothing$ and $\tilde{w}^{(n)} \equiv 0$ in Ω.

Let p and q be as in condition (f_1). The above argument shows that $u_j \to w^{(0)}$ in $L^p(\mathbb{H}^N) \cap L^q(\mathbb{H}^N)$, and since $u_j \in \mathring{S}_1^2(\Omega) \subset L^p(\Omega) \cap L^q(\Omega)$, $u_j \to w^{(0)}$ in $L^p(\Omega) \cap L^q(\Omega)$. Hence $f(\eta, u_j) \to f(\eta, w^{(0)})$ in $L^{p/(p-1)}(\Omega) + L^{q/(q-1)}(\Omega)$. Let $f = f_1 + f_2$, where $f_1 \in L^{p/(p-1)}(\Omega)$ and $f_2 \in L^{q/(q-1)}(\Omega)$. Observe that

$$\begin{aligned}
\|u_j - w^{(0)}\|_{S_1^2}^2 = \langle \varphi'(u_j) - \varphi'(w^{(0)}), u_j - w^{(0)} \rangle \\
+ \int_\Omega \big(f(z, u_j(z)) - f(z, w^{(0)}(z)) \big)\, (u_j(z) - w^{(0)}(z))\, dz.
\end{aligned}$$

https://doi.org/10.4153/CJM-2005-051-4 Published online by Cambridge University Press

Obviously

$$\langle \varphi'(u_j) - \varphi'(w^{(0)}), u_j - w^{(0)} \rangle \to 0,$$

and by the Hölder inequality,

$$\begin{aligned}
&\left| \int_\Omega \left(f(z,u_j) - f(z,w^{(0)}) \right) (u_j - w^{(0)})\, dz \right| \\
&\quad \le \left| \int_\Omega \left(f_1(z,u_j) - f_1(z,w^{(0)}) \right) (u_j - w^{(0)})\, dz \right| \\
&\qquad + \left| \int_\Omega \left(f_2(z,u_j) - f_2(z,w^{(0)}) \right) (u_j - w^{(0)})\, dz \right| \\
&\quad \le \left(\int_\Omega \left| f_1(z,u_j) - f_1(z,w^{(0)}) \right|^{p/(p-1)} dz \right)^{(p-1)/p} \left(\int_\Omega \left| u_j - w^{(0)} \right|^p dz \right)^{1/p} \\
&\qquad + \left(\int_\Omega \left| f_2(z,u_j) - f_2(z,w^{(0)}) \right|^{q/(q-1)} dz \right)^{(q-1)/q} \left(\int_\Omega \left| u_j - w^{(0)} \right|^q dz \right)^{1/q}.
\end{aligned}$$

Thus $u_j \to w^{(0)}$ in $\mathring{S}_1^2(\Omega)$, and so the P-S$_c$ condition holds. ■

4 Infinitely Many Solutions

To obtain an infinite sequence of critical values of φ, we use the following result, see Struwe [10] for a proof.

Theorem 4.1 *Suppose that V is an infinite dimensional Banach space and suppose that $\varphi \in C^1(V, \mathbb{R})$ satisfies P-S$_c$ for every $c > 0$, $\varphi(u) = \varphi(-u)$ for all u, and assume the following conditions:*

(i) *There exist $\alpha > 0$ and $\rho > 0$ such that if $\|u\| = \rho$ and $u \in V$, then $\varphi(u) \ge \alpha$.*

(ii) *For any finite dimensional subspace $W \subset V$, there exists $R = R(W)$ such that $\varphi(u) \le 0$ for $u \in W$, $\|u\| \ge R$.*

Then φ possesses an unbounded sequence of critical values.

Proof of Theorem 2.5 We apply Theorem 4.1 with $V = \mathring{S}_1^2(\Omega)$. By Lemma 3.3, φ satisfies the P-S$_c$ condition for every $c > 0$. Thus the assertion follows if we show that conditions (i) and (ii) of Theorem 4.1 are satisfied.

Integrating (f_1), there is a constant C_1 such that for all $\eta \in \Omega$ and $s \in \mathbb{R}$,

$$|F(\eta, s)| \le C_1(|s|^p + |s|^q).$$

By the Folland–Stein embedding theorem, we then have the estimate

$$\begin{aligned}
\varphi(u) &\ge \frac{1}{2}\|u\|_{S_1^2}^2 - C_1 \int_\Omega (|u(\eta)|^p + |u(\eta)|^q)\, d\eta \\
&\ge \frac{1}{2}\|u\|_{S_1^2}^2 - C_2\|u\|_{S_1^2}^p - C_2\|u\|_{S_1^2}^q.
\end{aligned}$$

https://doi.org/10.4153/CJM-2005-051-4 Published online by Cambridge University Press

Let $\|u\| = \rho$, where $\rho > 0$ is free for the moment. Then

$$\varphi(u) \geq \frac{1}{2}\rho^2 - C_2\rho^p - C_2\rho^q.$$

Now we choose $\rho > 0$ such that it satisfies the equation

$$\rho - C_2 p\rho^{p-1} - C_2 q\rho^{q-1} = 0.$$

Since the left-hand side of this equation is positive for small values of ρ and negative for large ρ, by the intermediate value theorem there is a solution. Then

$$C_2\rho^p = \frac{1}{p}\rho^2 - C_2\frac{q}{p}\rho^q,$$

and so for $\|u\| = \rho$,

$$\varphi(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right)\rho^2 + C_2\left(\frac{q}{p} - 1\right)\rho^q \geq \left(\frac{1}{2} - \frac{1}{p}\right)\rho^2.$$

This shows that condition (i) is satisfied with $\alpha = (1/2 - 1/p)\rho^2$.

By (f_2), there is a constant $C_3 > 0$ such that for every $x \in \Omega$ and $s \in \mathbb{R}$, $|F(\eta, s)| \geq C_3|s|^{\mu_1}$, where $\mu_1 = \min(\mu, \nu)$. Indeed, let $\epsilon > 0$ be given. By integration of the first identity of (f_2), we have for $|s| > \epsilon$ and $\eta \in \Omega$,

$$F(\eta, s) \geq \frac{F(\eta, \epsilon)}{\epsilon^\mu}|s|^\mu.$$

Letting $\epsilon \to 0$ and using the second identity of (f_2), the claim follows. Let W be a finite dimensional subspace of $\mathring{S}_1^2(\Omega)$. Since all norms are equivalent on W and since

$$\varphi(u) \leq \frac{1}{2}\|u\|^2_{S_1^2} - C_3\|u\|^{\mu_1}_{L^{\mu_1}},$$

(ii) follows. ■

Acknowledgements The author would like to thank Svante Janson for discussions and helpful comments.

References

[1] I. Birindelli and A. Cutrì, *A semi-linear problem for the Heisenberg Laplacian.* Rend. Sem. Mat. Univ. Padova **94**(1995), 137–153.

[2] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés.* Ann. Inst. Fourier (Grenoble) **19**(1969), 277–304.

[3] M. A. del Pino and P. L. Felmer, *Least energy solutions for elliptic equations in unbounded domains.* Proc. Roy. Soc. Edinburgh Sect. A **126**(1996), 195–208.

[4] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group.* Comm. Pure Appl. Math. **27**(1974), 429–522.

[5] N. Garofalo and E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group.* Indiana Univ. Math. J. **41**(1992), 71–98.
[6] L. Hörmander, *Hypoelliptic second order differential equations.* Acta math. **119**(1967), 147–171.
[7] S. Maad, *Multiplicity of solutions of nonlinear elliptic equations with $\mathbb{Z}/2$-symmetry.* U.U.D.M. Report 2001:12, ISSN 1101-3591, http://www.math.uu.se/research/pub/Maad1.pdf
[8] ———, *Infinitely many solutions of a symmetric semilinear equation on an unbounded domain.* Ark. Mat. **41**(2003), 105–114.
[9] I. Schindler and K. Tintarev, *An abstract version of the concentration compactness principle.* Rev. Mat. Complut. **15**(2002), 417–436.
[10] M. Struwe, *Variational methods.* Second Edition, Springer-Verlag, Berlin, 1996.
[11] K. Tintarev, *Semilinear elliptic problems on unbounded subsets of the Heisenberg group.* Electron. J. Differential Equations **2001**, 18.
[12] ———, *Semilinear subelliptic problems without compactness on Lie groups.* NoDEA Nonlinear Differential Equations Appl. **11**(2004), 299–309.

Department of Mathematicsand Statistics
School of Electronic and Physical Sciences
University of Surrey
Guildford, Surrey
GUZ 7XH
U.K.
e-mail: sara_maad@hotmail.com