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FRACTION-DENSE ALGEBRAS AND SPACES

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ABSTRACT A fraction-dense (semi-prime) commutative ring A with 1 is one for which the classical quotient ring is rigid in its maximal quotient ring The fraction-dense f-rings are characterized as those for which the space of minimal prime ideals is compact and extremally disconnected For archimedean lattice-ordered groups with this property it is shown that the Dedekind and order completion coincide Fraction-dense spaces are defined as those for which $C(X)$ is fraction-dense If X is compact, then this notion is equivalent to the coincidence of the absolute of X and its quasi-F cover R-embeddings of Tychonoff spaces are re-introduced and examined in the context of fraction-density

Introduction. Fraction-dense algebras arise naturally in the consideration of quotient rings, and they give rise to an interesting class of topological spaces. For archimedean Riesz spaces Huijsmans and de Pagter have introduced a similar concept, in [HP], which they call *almost Dedekind completeness*. In fact, what we shall later call an *absolute l-group* coincides, in the context of uniformly complete Riesz spaces, with an almost Dedekind complete Riesz space. There is considerable overlap in the algebraic parts between this article and [HP]. We shall, however not attempt to reconcile their terminology with ours.

Unless further qualified, every ring in this exposition will be commutative, possess an identity, and also be semi-prime, in the sense that there are no non-zero nilpotent elements. An f-ring is a lattice-ordered ring in which $a \wedge b = 0$ implies that $a \wedge bc = 0$ for each $c \geq 0$. In ZFC this is equivalent to requiring that the lattice-ordered ring be a subdirect product of totally ordered rings. Likewise, all topological spaces are assumed to be Tychonoff, unless the contrary is expressly stated. Recall that a Hausdorff space is *Tychonoff* if the cozero-sets (of real-valued continuous functions) form a base for the topology.

All lattice-ordered groups in this article are abelian. Our standard references for this theory are [AF] and [BKW].

Suppose that A is an f-ring; then qA stands for its classical ring of quotients and QA for its maximal ring of quotients. qA should be familiar to the reader; however, let us recall some properties of QA. (For further reading we refer the reader to [La], [Ba] and [M].) First of all, the term "ring of quotients" should be interpreted as follows: assume that A is a subring of the ring B; we say that B is a *ring of quotients* of A if for each pair b_1, b_2, with $b_2 \neq 0$, there exists an $a \in A$ such that ab_1 and ab_2 both belong to A and $ab_2 \neq 0$. Each ring then has a (unique) maximal ring of quotients; in [La] the subject

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is treated in full generality, whereas Banaschewski gives a representational construction of the maximal ring of quotients of a *semi-prime* ring, see [Ba] That is the procedure followed in [M] for f-rings, and there it was shown that QA can be given a lattice-ordering so that it becomes an f-ring, and contains A as an f-subring Recall that qA also has a natural ordering making it an f-ring, $A \subseteq qA \subseteq QA$ QA is the A-injective hull of A and it is a von Neumann regular ring, that is, for each $a \in A$ there is an $x \in A$ such that $a^2x = a$

In considering the fractions in qA, first observe that in each fraction a/b one can assume without loss of generality that $b > 0$, then $a/b \geq 0$ precisely when $a \geq 0$ From this we conclude that A is *rigid* in qA (Recall that if H is a lattice-ordered group in which G is an l-subgroup, G is said to be rigid in H if for each $h \in H$ there is a $g \in G$ such that $h^{\perp\perp} = g^{\perp\perp}$, the symbol $\perp$ stands for "polar", and in any situation, such as this one, of inclusion, is understood in the only possible way, namely as denoting polars in the larger object)

Rigidity for lattice-ordered groups was first introduced in [CM2], it is shown there that if G is rigid in H then the contraction map $P \longrightarrow P \cap G$ is a homeomorphism from $\mathrm{Min}(H)$, the space of minimal prime l-ideals of H, as a topological space with the hull-kernel topology, onto $\mathrm{Min}(G)$ Recall the well-known fact that for semi-prime f rings, "minimal prime l-ideal" and "minimal prime (ring) ideal" mean the same thing

Now let us recall some of the main points from [M], about the quotient rings of A, qA and QA

The essence of Banaschewski's description of QA is to cast it as a direct limit of rings, each of which is naturally f-ordered The upshot is that one views each element of QA as a function defined on a dense open subset of $\mathrm{Min}(A)$, which locally is a quotient of elements of A, modulo the appropriate minimal primes The most concrete illustration of this construction—and see [FGL] for details—is that of $Q(X) \equiv Q\big(C(X)\big)$, the maximal ring of quotients of the ring of all continuous real-valued functions defined on a Tychonoff space X $Q(X)$ consists of the continuous real-valued functions defined on dense open subsets of X, with the obvious identifications on common domains (By contrast, $q(X) \equiv q\big(C(X)\big)$ is the ring of all continuous real-valued functions defined on cozero sets of X, with the same proviso for identification on common domains of definition)

Banaschewski's description of the maximal ring of quotients provides a natural link with Bleier's construction of the orthocompletion, see [Bl] (The lattice-ordered group G is said to be *orthocomplete* if it is laterally complete—that is, every subset of pairwise disjoint elements has a supremum—and projectable, (please refer to the note at the end of the next paragraph) The orthocompletion of a lattice-ordered group G, denoted by oG, is a lattice-ordered group containing G densely, which is orthocomplete, and such that no proper l-subgroup of oG contains G and is orthocomplete)

Indeed, for any semi-prime f-ring A, QA contains the orthocompletion oA, moreover, one of the main results of [M] (Theorem 1 5) asserts that $QA = q(oA)$ If A is projectable or archimedean then the order of the operators q and o can be reversed, see 1 4 and 1 8 1 in [M] Anderson and Conrad show in [AC] that for $A = C(X)$, $oA = QA$ (Note

A lattice-ordered group G is *projectable* if for each $g \in G$, $G = g^{\perp} + g^{\perp\perp}$. In [HP], projectable l-groups are called *normal*.)

In an off-hand manner it is also asserted in [AC] that for $A = C(X)$, $qA = QA$. Now this is not true, as one sees by taking the space $\beta\mathbb{N} \setminus \mathbb{N}$; it has no proper, dense cozero-sets—see Chapter 6 of [GJ]—and so $q(\beta\mathbb{N} \setminus \mathbb{N}) = C(\beta\mathbb{N} \setminus \mathbb{N})$, whereas the maximal ring of quotients is much bigger: since $\beta\mathbb{N} \setminus \mathbb{N}$ is not extremally disconnected, it has plenty of dense open subsets U which are not C^*-embedded; any function which belongs to $C^*(U)$ and cannot be extended continuously to $\beta\mathbb{N} \setminus \mathbb{N}$ is in $Q(\beta\mathbb{N} \setminus \mathbb{N})$.

And so the springboard for this article is the question: when is $qA = QA$? For reasons which we shall not motivate at this juncture, it is more interesting to ask the question: when is qA rigid in QA? As we shall presently demonstrate, the example just given is one in which qA is not rigid in QA. Let us now proceed to examine f-rings A for which QA contains qA rigidly.

1. **Fraction-dense algebras.** We say that the semi-prime f-ring A is *fraction-dense* if qA is rigid in QA. If X is a Tychonoff space and $A = C(X)$ then we say that X is a *fraction-dense* space if A is fraction-dense. If $qA = QA$, A will be called *strongly fraction-dense*; likewise, X is a *strongly fraction-dense* space if $C(X)$ is strongly fraction-dense. The reader should notice at the outset that the class of fraction-dense spaces is quite extensive; since in a metric space every open set is a cozero-set, it follows that $Q(X) = q(X)$ for every metric space X. Indeed, every metric space is *strongly* fraction-dense.

Observe as well, that if X is an extremally disconnected space then every (dense) open subset is C^*-embedded—see 1H.6 in [GJ]—which implies that $q(X) = Q(X) = D(X)$, the f-algebra of all continuous functions f defined on X with values in the extended reals, and for which $\text{fin}(f) = \{x \in X : |f(x)| < \infty\}$ is a dense subset of X. Thus, every extremally disconnected space is strongly fraction-dense.

Theorem 1.1, which we shall state presently, gives a number of criteria for A to be a fraction-dense f-ring. Before proceeding to it however, let us recall some definitions. Now, we already recalled that of projectable lattice-ordered groups; we say that G is *strongly projectable* if $G = K^{\perp} + K^{\perp\perp}$, for each polar subgroup K. (Incidentally, we shall denote by $P(G)$ the boolean algebra of all polars of G, and by $\text{Pr}(G)$ the sublattice generated by the principal polars $g^{\perp\perp}$.)

The reader should bear in mind that strong projectability is indeed a more restrictive condition than projectability. There are many examples in the literature; for example, if X is a basically disconnected space, then $C(X)$ is projectable, although rarely strongly projectable.

We remind the reader of the definition in [CM1] of a *complemented* lattice-ordered group: G is said to be complemented if for each $g \in G$ there is an $h \in G$ such that $|g| \wedge |h| = 0$ and $|g| \vee |h|$ is a weak order unit. It is proved in [CM1] that G is complemented if and only if $\text{Min}(G)$ is compact. (Compare this with the result for rings in terms of the boolean algebra of annihilators in [HJ]. See as well Propositions 4.9 and 4.10 in [HP].)

From [M] we recall the following:

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PROPOSITION (0 2 IN [M]) *For a semi prime f-ring A, the following are equivalent*

(1) qA is von Neumann

(2) qA is projectable

(3) qA is complemented

Indeed, it is not hard to see that A is complemented precisely when qA is projectable Observe as well, that since QA is orthocomplete, it is both strongly projectable and, therefore, von Neumann (That a maximal ring of quotients of a semi-prime ring is von Neumann is well-known, see [La])

Before stating the theorem observe, finally, that G is a complemented lattice-ordered group if and only if $\mathrm{Pr}(G)$, the lattice of principal polars of G, is a subalgebra of $P(G)$ Moreover, in this event $\mathrm{Pr}(G)$ is the Stone dual of $\mathrm{Min}(G)$ Thus, $\mathrm{Pr}(G)$ is complete precisely when $\mathrm{Min}(G)$ is extremally disconnected, by the Stone-Nakano theorem, (see [Wl], p 47) (Recall that a space X is *extremally disconnected* if the closure of each open set is open) Appealing to Theorem 2 7 in [CM1], we see that $\mathrm{Min}(G)$ is compact and extremally disconnected if and only if every polar of G is principal

If A is an f-ring then $A(1)$ stands for the convex f-subring generated by 1, we shall refer to it as the *bounded subring* of A Note that $A(1)$ is rigid in A

THEOREM 1 1 *Suppose that A is a semi-prime f-ring Then the following are equivalent*

(1) A is fraction-dense

(2) $A(1)$ is fraction-dense

(3) A is rigid in QA

(4) qA is strongly projectable

(5) $\mathrm{Min}(A)$ is compact and extremally disconnected

(6) Every polar of A is principal

(7) qA and QA have the same idempotents

PROOF For any lattice-ordered groups G, H and K, such that G is an l-subgroup of H and H is an l-subgroup of K, G is rigid in K if and only if G is rigid in H and H is rigid in K (See [CM1])

With this in mind it is immediate that (1) and (3) are equivalent If qA is rigid in QA, then since QA is strongly projectable $P(QA) = \mathrm{Pr}(QA)$, and therefore the same is true for qA From this it follows that qA is projectable, whence strongly projectable, thus (3) implies (4)

Assuming (4), we have from earlier remarks that A is complemented, and from this that $\mathrm{Min}(A)$ is compact As in the previous paragraph, $P(qA) = \mathrm{Pr}(qA)$, and the same holds for A Also from comments made before, this means that $\mathrm{Min}(A)$ is extremally disconnected Thus, (4) implies (5)

From what has already been said it is clear that (5) and (6) are equivalent

Next we show that (5) implies (1) This follows from the rigidity of A in qA, along the lines of previous arguments $\mathrm{Min}(qA)$ is compact and extremally disconnected, whence qA is strongly projectable (So we have actually shown that (4) follows from (5)) Now,

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QA is the orthocompletion of qA—from Theorem 1 4 in [M]—and this means that the contraction map $P \longrightarrow P \cap qA$ is a boolean isomorphism from $P(QA)$ onto $P(qA)$, all of which means that each principal polar of QA contracts to a principal polar of qA This shows that qA is rigid in QA

As we now know that (1) is equivalent to statements (3) through (6), observe that they are all equivalent to (7) because, in a strongly projectable semi-prime f-ring, every polar is the principal polar of an idempotent

Finally, note that $A(1)$ is rigid in A, so that their spaces of minimal prime ideals are homeomorphic This implies that (2) is equivalent to the rest, and the proof of the theorem is complete ∎

From Theorem 1 1 we have, right away, the following corollary, as usual, βX denotes the Stone-Cech compactification of the space X

COROLLARY 1 1 1 *A Tychonoff space X is fraction-dense if and only if βX is fraction-dense*

Observe that, since $q(X) = q\big(C^*(X)\big)$ and $Q(X) = Q\big(C^*(X)\big)$, we also get that X is strongly fraction-dense if and only if βX is

For the next corollary, let us first recall the notion of the absolute of a space, as well as the concept of an irreducible map It will be sufficient for our purposes to present the situation for compact spaces

If f $X \longrightarrow Y$ is a continuous surjection (of compact spaces) then f is said to *irreducible* if Y cannot be obtained as the image under f of a proper closed subset of X Here are some basic observations about irreducible maps, the proofs may be found in [H] or [BH1] To begin, note that the continuous surjection f $X \longrightarrow Y$ is irreducible if and only if the functorially induced embedding $C(f)$ $C(Y) \longrightarrow C(X)$, by $C(f)(g) = g$ f, is an (order)-dense embedding Also, recall that if f is irreducible then the inverse image of a dense subset of Y is dense in X (2 7 (a) in [H])

Now, for a given compact space let $R(X)$ denote the boolean algebra of regular closed sets This is a complete algebra, its Stone dual, EX, is therefore an extremally disconnected space Viewing EX as the space of ultrafilters on $R(X)$, we have a natural map e_X $EX \longrightarrow X$ which assigns to the ultrafilter α the unique point of X common to all its members (For further details see [Wl] or [PW] EX is sometimes called the *Gleason space* of X, it was first studied by A Gleason in [Gl] In this study the extremally disconnected spaces are cast as the projective spaces, and EX as the "projective cover" of X) We shall refer to EX as the *absolute (space)* of X

Finally, note that the stipulations on f $E \longrightarrow X$ that f be irreducible and E be extremally disconnected, characterize the absolute of X, in the following sense If f and E have the stated properties, then there is a homeomorphism g $EX \longrightarrow E$ such that $fg = e_X$ Any map f $E \longrightarrow X$ which is irreducible, out of an extremally disconnected space E is said to *realize* the absolute of X

Now to the second corollary of Theorem 1 1 Let $J(A)$ denote the Jacobson radical of A It will also be most convenient to suppose that A satisfies the *bounded inversion property* $a > 1$ implies that $a-1$ exists, this property is satisfied by $C(X)$, for any space X

It ıs well-known that a semı-prıme f-rıng A satısfies the bounded ınversıon property pre cısely when each maxımal ıdeal of A ıs an l-ıdeal It then follows that the space Max(A) of maxımal ıdeals of A, relatıve to ıts hull-kernel topology, ıs Hausdorff Max(A) ıs always compact, we refer the reader to Lemma 0 0 ın [M])

Now let δ_A stand for the (contınuous) map whıch assıgns to each mınımal prıme ıdeal P the maxımal ıdeal $\delta_A(P)$ containıng ıt, thıs map ıs well-defined because the prıme l ıdeals of A form a root-system—see [BKW]—whıch ıs to say that no two ıncomparable prımes contaın a thırd prıme δ_A ıs a contınuous surjectıon of Mın(A) on Max(A)

COROLLARY 1 1 2 *Suppose that A ıs a semı-prıme f-rıng satısfyıng the bounded ınversıon property, and that* $J(A) = 0$ *Then A ıs fractıon-dense ıf and only ıf* Mın(A) *and* δ_A *realıze the absolute of* Max(A)

PROOF The suffıcıency ıs obvıous from the theorem As to the necessıty, all that must be verıfied ıs that δ_A ıs ırreducıble Every closed subset of Mın(A) ıs of the form $V(C) = \{P \in \mathrm{Min}(A) \quad P \subseteq C\}$, where C ıs an ıntersectıon of mınımal prımes If $\delta_A[V(C)] = \mathrm{Max}(A)$, then each maxımal ıdeal of A contaıns C, sınce $J(A) = 0$ thıs ımplıes that $C = 0$, and so $V(C) = \mathrm{Min}(A)$ ∎

Note that every archımedean f-rıng wıth bounded ınversıon has trıvıal Jacobson radıcal (See [M], the dıscussıon precedıng 3 9) The converse ıs false, as evıdenced by any non-archımedean ordered field

Before proceedıng to examıne fractıon-dense spaces more closely, a comment ıs ın order on the heels of Corollary 1 1 2 Let us assume that A stands for a semı-prıme f-rıng wıth the bounded ınversıon property

Sınce QA ıs von Neumann, we have that Max(QA) = Mın(QA) On the other hand, ın QA every polar ıs prıncıpal, and the algebra ıs orthocomplete, whıch means that Max(QA) ıs extremally dısconnected

Next, ıf B ıs any f-subrıng of A (also wıth the bounded ınversıon,) consıder the map θ Max(A) $\longrightarrow$ Max(B) whıch assıgns to a maxımal ıdeal M of A the unıque maxımal ıdeal of B whıch contaıns the contractıon $M \cap B$ As shown by Scott Woodward, and soon to appear ın hıs Unıversıty of Florıda dıssertatıon, thıs ıs a contınuous surjectıon of Max(A) on Max(B) So let us consıder thıs map θ_A Max(QA) $\longrightarrow$ Max(A)

PROPOSITION 1 2 *Let A be an f-rıng satısfyıng the bounded ınversıon, for whıch* $J(A) = 0$ *Then the map* θ_A Max(QA) $\longrightarrow$ Max(A) *realızes the absolute of* Max(A)

PROOF Once agaın, the only matter left to settle ıs the ırreducıbılıty of θ_A So let us suppose that $\theta_A[K] = \mathrm{Max}(A)$, for some closed set K Thus, every maxımal ıdeal N of A contaıns a contractıon $M \cap A$, for some $M \in K$ Sınce $J(A) = 0$ ıt follows that $\bigcap\{M \cap A \quad M \in K\} = 0$, puttıng ıt dıfferently, $(\bigcap K) \cap A = 0$ Sınce QA ıs an A essentıal extensıon of A—after all, QA ıs the A-ınjectıve hull of A—thıs ımplıes that $\bigcap K = 0$, thıs, together wıth the assumptıon that K ıs closed, gıves the conclusıon that $K = \mathrm{Max}(QA)$ ∎

Proposıtıon 1 2 has the followıng appealıng corollary There are some detaıls whıch need checkıng, we leave thıs to the reader as an exercıse

COROLLARY 1.2.1. *Suppose that A is an f-ring with bounded inversion and $J(A) = 0$. Then the following are equivalent.*

(1) $\mathrm{Max}(A)$ *is extremally disconnected.*

(2) A is strongly projectable.

(3) A and QA have the same idempotents.

(4) A is fraction-dense and δ_A is a homeomorphism from $\mathrm{Min}(A)$ *onto* $\mathrm{Max}(A)$.

We shall proceed now to a more detailed study of fraction-dense spaces. Recall that the two principal classes of fraction-dense spaces we have already mentioned, namely, metric spaces and extremally disconnected spaces, are, in fact, strongly fraction-dense. As far as we know every fraction-dense space is strongly fraction-dense; although it is unlikely that this implication should hold in general, we have yet to discover a counter-example.

By contrast, observe that if A is the f-algebra of all real sequences with finite range, then $qA = A$, while QA is the algebra of all real sequences. A is rigid in QA, so that A is fraction-dense, but not strongly fraction-dense.

2. **Fraction-dense spaces.** We shall begin this section with an introduction to the notion of covers of topological spaces. We have already mentioned irreducible maps. Now recall that a continuous map is *perfect* if it is a closed mapping and the inverse image of every singleton set is compact; evidently, if the spaces in question are compact, then every continuous map between them is perfect. Now, if $f: Y \longrightarrow X$ is a perfect, irreducible surjection, we say that the pair (Y,f) is a *cover* of X. For a comprehensive discussion of the theory of covers we refer the reader to [H]. Much of the deep work on this subject has been done by Vermeer; [V1] and [V2] have much to recommend them.

Let us consider the lattice $\mathrm{COV}(X)$ of all covers of X. First, recall that two covering maps $f: Y \longrightarrow X$ and $g: Z \longrightarrow X$ are said to be *equivalent*, if there is a homeomorphism $h: Y \longrightarrow Z$ such that $gh = f$. Such a map is unique. Modulo this equivalence relation one can then order $\mathrm{COV}(X)$, the collection of all covers of X, as follows: with the same designations for f and g as before, we say that $(Y,f) \geq (Z,g)$ if there is a continuous map $h: Y \longrightarrow Z$ such that $gh = f$. We note that $(Y,f) \geq (Z,g)$ and $(Z,g) \geq (Y,f)$ together imply that f and g are equivalent. (See [H] for details; if $(Y,f) \geq (Z,g)$ then the map h is perfect and irreducible and unique with respect to making $gh = f$.) Under this partial ordering, $\mathrm{COV}(X)$ is a complete lattice, in which the least element is $(X, 1_X)$, while (EX, e_X) is the largest element. The suprema in $\mathrm{COV}(X)$ can be described by means of pullbacks; see [H].

Now suppose that $\mathcal{K}$ is a class of topological spaces. (One need not assume that the spaces are Tychonoff, but we shall continue to do so.) We say that $\mathcal{K}$ is a *covering class* if for each space X there is a least element (Y,f)—where $f: Y \longrightarrow X$ is perfect and irreducible—with $Y \in \mathcal{K}$. If such a minimum cover exists we speak of the $\mathcal{K}$*-cover* of a space X, and denote it $\mathcal{K}X$.

Thus, if E is the class of extremally disconnected spaces, then every space X has a E-cover, namely EX, the absolute space. Let QF denote the class of *quasi-F* spaces; (X is quasi-F provided every dense cozero-set is C^* embedded) Various contributions to the literature have discussed the quasi-F cover [DHH], [HVW1], [HP] and [ZK]

Recall that any irreducible map $f\ Y \longrightarrow X$ induces, by way of the assignment $A \longrightarrow f[A]$, a boolean isomorphism between the respective algebras of regular closed sets, (see [PW], 6 5(d)(3)) Also, $P\big(C(X)\big)$ is isomorphic to $R(\beta X)$ βX is, after all, homeomorphic to $\mathrm{Max}\big(C(X)\big)$, and then the (boolean) isomorphism in question is defined by assigning to a polar P of $C(X)$ the set of maximal ideals containing P Under this assignment the principal polars are associated with the closures of cozero-sets

We are now ready for the first result of this section

PROPOSITION 2 1 *A space X is fraction-dense if and only if every regular closed subset of X is the closure of a cozero-set of X Also, X is fraction-dense if and only if every regular open set of X densely contains a cozero-set*

PROOF Since X is fraction-dense if and only if every polar of $C(X)$ is principal, it follows that X is fraction-dense if and only if every regular closed set of βX is the closure of a cozero-set of βX However, it is easily seen that the latter condition holds for βX precisely when it does for X ■

Following [BH1], we shall say that the perfect, irreducible map $f\ Y \longrightarrow X$ is *sequentially irreducible* if for each cozero-set $W \subseteq Y$ there is a cozero-set $V \subseteq X$ such that W and $f-1(V)$ have the same closure This notion is called $Z^{\#}$-*irreducible* in [HVW1] and ω_1-*irreducible* elsewhere As is demonstrated in [HVW1], Theorem 2 13, the quasi-F cover (Y, ϕ) of a space X is characterized by Y being a quasi-F space and the covering map $\phi\ Y \longrightarrow X$ being sequentially irreducible (The unpublished result is originally due to F Dashiell)

From Theorem 2 4 in [BH1] we conclude the following

PROPOSITION 2 2 *Suppose that $f\ Y \longrightarrow X$ is a perfect, irreducible map Then f is sequentially irreducible if and only if the induced embedding $C(f)\ C(X) \longrightarrow C(Y)$ is a rigid embedding*

PROOF According to 2 4 of [BH1], f is sequentially irreducible if and only if the contraction map $P \longrightarrow P \cap C(X)$ sends countably generated polars to countably generated polars However, in $C(X)$ every countably generated polar is principal if $P = \{f_1, f_2 \quad\}^{\perp\perp}$, then we may, without loss of generality, suppose that the f_n are positive and bounded Letting $f = \sum_{n\in N} f_n/2^n$, we obtain that $P = f^{\perp\perp}$ ■

As an immediate corollary of this proposition and Theorem 1 1 one gets

COROLLARY 2 2 1 *If $f\ Y \longrightarrow X$ is a perfect, sequentially irreducible map and X is fraction dense then so is Y*

Next, let us recall the following definition a space X is *cozero-complemented* if for each cozero-set U there is a cozero-set V which is disjoint from U and such that the union is dense in X Then observe that, since any fraction-dense f-ring is complemented, every fraction-dense space is cozero-complemented

Now let us recall the following notion from [HVW2]: X is a *cloz-space* if every complemented cozero set has an open closure. The authors show that every quasi-F space is a cloz-space, and that the converse is true for strongly zero-dimensional spaces, but not in general. Moreover, the class of cloz-spaces is a covering class. We now get, immediately from Proposition 2.1 and Corollary 2.2.1:

PROPOSITION 2.3. *Each fraction-dense cloz-space X is extremally disconnected. Moreover, if Y is fraction-dense then its cloz-cover, quasi-F cover and absolute all coincide.*

NOTE. For every fraction-dense space X, $QFX = EX$. The converse is false: let X be the space of all ordinals *less* than the first uncountable ordinal, endowed with the order topology. By 3.15 in [HVW1], $EX = QFX$; however, X is not fraction-dense: βX is its one-point compactification, by adjoining the first uncountable ordinal, which is a P-point in βX. As we shall see presently, a fraction-dense compact space (in a universe without measurable cardinals) has no non-isolated P-points.

For compact spaces the converse is true:

PROPOSITION 2.4. *If X is a compact space then X is fraction-dense if and only if $QFX = EX$.*

PROOF. Again note that every countably generated polar of $C(X)$ is principal. Also, all we are required to prove is the sufficiency: if $EX = QFX$, then by Theorem 2.16 in [HVW], $\mathrm{Min}\big(C(X)\big)$ is compact. By Lemma 3.20 in [HVW], $\mathrm{Min}\big(C(X)\big)$ is extremally disconnected. It now follows from Theorem 1.1 that $C(X)$ is fraction-dense. ∎

For the remainder of this section we shall assume that all spaces are compact. We shall also assume that, henceforth, all f-rings satisfy the bounded inversion property.

Let us now turn to some more "intimate" properties of fraction-dense spaces. We need a few definitions before proceeding. Recall that a point $p \in X$ is an *almost P-point* if every zero set containing p has interior. We say that p is an *F-point* if $O_p = \{f \in C(X) : Z(f)$ is a neighborhood of $p\}$ is a prime ideal. For aesthetic reasons we call $p \in X$ a *quasi-F point* if, under the covering map $\phi_X\colon QFX \longrightarrow X$, $\phi^{-1}\{p\}$ is a singleton. It is easy to see that X is a quasi-F space if and only if each of its points is a quasi-F point.

Let us first record the following lemma. Note that if A is an f-ring and $M \in \mathrm{Max}(A)$ then $O(M)$ stands for the intersection of all the minimal prime ideals of A which are contained in M. If B is an f-subring of A then $\theta\colon \mathrm{Max}(A) \longrightarrow \mathrm{Max}(B)$ is the canonical surjection of spaces of maximal ideals.

LEMMA 2.5. *Suppose that B is an f-subring of A. Then if B is rigid in A, $O(\theta M) = \bigcap\{O(N)\cap B : \theta N = \theta M\}$, and if $\theta N_i = \theta M$ $(i = 1,2)$ with $N_1 \parallel N_2$, then $O(N_1)\cap B \parallel O(N_2)\cap B$.*

PROOF. Clearly $O(\theta M) \subseteq O(N)\cap B$, for each $N \in \mathrm{Max}(A)$ such that $\theta N = \theta M$. Conversely, if $P \in \mathrm{Min}(B)$ and $O(\theta M) \subseteq P \subseteq \theta M$, then for some $Q \in \mathrm{Min}(A)$, $Q\cap B = P$ (by rigidity), and such a Q must lie between N and $O(N)$, for some N so that $\theta N = \theta M$. We leave the incomparability argument to the reader. ∎

In the final part of the following proposition, as elsewhere, (NMC) designates that it is assumed that no measurable cardinals exist For the relevance of this to P-points in an extremally disconnected space, we refer the reader to Chapter 12 of [GJ]

PROPOSITION 2 6 *Suppose that X is a (compact) space Then*

(a) Every almost P-point is quasi-F

(b) Every F-point is quasi-F

(c) If X is fraction-dense then every quasi-F point is an F-point

(d) (NMC) *Every almost P-point in a fraction dense space is isolated*

PROOF (a) From 3 11 in [HVW1]

(b) Once again, ϕ_X $QFX \longrightarrow X$ denotes the covering map It is sequentially irreducible, so that $C(\phi)$ $C(X) \longrightarrow C(QFX)$ is rigid Now apply Lemma 2 5

(c) If $p \in X$ is a quasi-F point then $\phi\ {}^{1}\{p\} = \{q\}$ By Lemma 2 5, $O_p = O_q \cap C(X)$ Now, since O_q is prime in $C(EX)$, it follows that O_p is likewise prime in $C(X)$ Hence, p is an F-point

(d) It is well-known that in an extremally disconnected space of non-measurable cardinality every P-point is isolated The same proof—see [GJ] or [PW], for example—can be used to show that every almost P-point is isolated (This has already been observed in [Lv])

Now, by 3 11 in [HVW1], if X is fraction-dense and $p \in X$ is almost P, then $\phi\ {}^{1}\{p\}$ is a singleton, which is also almost P But this inverse-image is then isolated, making p isolated as well ∎

NOTE If one does not assume anything about cardinalities, then Proposition 2 6 (d) can be put as follows, (recall that $p \in X$ is a C^* *point* if $X\{p\}$ is C^*-embedded in X) In a compact fraction-dense space X every non-isolated almost P-point is a C^*-point We shall obtain this as Corollary 4 2 3

The preceding proposition raises the following question If every almost P-point of X is isolated, does it follow that X is fraction-dense? The answer is no, but the example prompts a second, stickier question, which we shall state presently

Once again, let X stand for the space of all ordinal numbers less than the first un countable ordinal All the points of X are isolated, except the limit ordinals, which are G_δ-points, and hence not almost P Of course, X is not compact, so we can—and do—rephrase the question (notice, by the way, that X is cozero-complemented, and recall that any fraction-dense space has this property)

QUESTION 2 6 1 If X is compact, cozero-complemented and every almost P-point is isolated, is X fraction-dense?

In perfect analogy to the fraction-dense case one can establish the result which fol lows We state a version for spaces first, followed by the generalization to f-rings It should be noted that Theorem 2 16 in [HVW1] shows that if X is cozero complemented then its basically disconnected cover and its quasi-F cover coincide, our result shows that the canonical map θ_X $\mathrm{Min}\big(C(X)\big) \longrightarrow \mathrm{Max}\big(C(X)\big) = X$ realizes this identity

The proofs of the next two results are very similar to those of Corollary 1.1.2, and are therefore left to the reader.

PROPOSITION 2.6.2. *Suppose that X is compact and cozero-complemented. Then*
(1) The canonical map $\theta_X\colon \mathrm{Min}\big(C(X)\big) \longrightarrow \mathrm{Max}\big(C(X)\big) = X$ *realizes the basically disconnected cover of X.*
(2) θ_X *is sequentially irreducible, and hence* $QFX = BDX = \mathrm{Min}\big(C(X)\big)$.

PROPOSITION 2.6.3. *Suppose that A is a semi-prime f-ring with bounded inversion, which in addition satisfies: (a)* $J(A) = 0$, *(b) A is complemented and (c)* $\mathrm{Pr}(A)$ *is* σ*-complete.*

Then $\mathrm{Min}(A)$ *is basically disconnected and the canonical map* $\theta_A\colon \mathrm{Min}(A) \longrightarrow \mathrm{Max}(A)$ *realizes* $\mathrm{Min}(A)$ *as both the basically disconnected cover as well as the quasi-F cover of* $\mathrm{Max}(A)$.

In connection with the question in Question 2.6.1 and our earlier work on fraction-dense spaces, it should be clear that if X is compact, cozero-complemented and every almost P-point is isolated then its basically disconnected cover has this last property as well. Moreover, X is fraction-dense if and only if $BDX = QFX$ is extremally disconnected. Therefore Question 2.6.1 can be reduced to:

QUESTION 2.6.4. Is there a compact, basically disconnected space in which every almost P-point is isolated, which is not extremally disconnected?

The question can be reduced even more. Observe, the following, which can easily be derived from results in [HJ].

PROPOSITION 2.6.5. *Suppose that A is a complemented f-ring. Then every non-minimal prime ideal of A contains a regular element.*

COROLLARY 2.6.6. *If X is a compact cozero-complemented space, then every almost P-point is a P-point.*

PROOF. If $p \in X$ is almost P then the maximal ideal $M_p = \{f \in C(X) : f(p) = 0\}$ contains no regular elements. By Proposition 2.6.5, it must be minimal; hence $M_p = O_p$, and by $4L$ in [GJ], p is a P-point. ∎

In particular, in any basically disconnected space an almost P-point is already a P-point. So Question 2.6.4 is equivalent to:

QUESTION 2.6.7. Is there a compact, basically disconnected space in which every P-point is isolated, which is not extremally disconnected?

The answer is yes, and here is an example: of a basically disconnected space which is not extremally disconnected and has no P-points. Going back to Question 2.6.1, where the succession of questions got started, we see that it is answered in the negative.

Consider the boolean algebra $2^{\mathbb{R}}$ of all subsets of the real numbers, modulo the σ-ideal M of all meager subsets. It is shown in [Si], 21G and 21H, that $2^{\mathbb{R}}/M$ is σ-complete but not complete. Its Stone dual is, therefore, basically disconnected, but not extremally disconnected. We proceed to show that it has no P-points.

Let Ston(B) denote the Stone dual of the boolean algebra B We think of Ston(B) as the space of ultrafilters of B, the clopen set of all ultrafilters which contain $b \in B$ is denoted by b' Now observe the following points about Stone duality

(a) The boolean epimorphism of B onto B/J, where J is an ideal of the boolean algebra B, induces an embedding of Ston(B/J) in Ston(B) which identifies the former with the complement in Ston(B) of $\bigcup\{x' \quad x \in J\}$ (See [Si], p 29)

(b) The epimorphism $B \longrightarrow B/J$ preserves countable suprema and infima of and only if J is a σ-ideal Moreover, this occurs precisely when the dual map Ston(B/J) $\longrightarrow$ Ston(B) has the property that if a subset E of Ston(B) which is nowhere dense as well as an intersection of countably many clopen sets, then its inverse image is a nowhere dense set (See [Si], 22 3)

Now, let's apply (a) and (b) to the canonical map $2^{\mathbb{R}} \longrightarrow 2^{\mathbb{R}}/M$ First, Ston($2^{\mathbb{R}}$) $= \beta\mathbb{R}_d$, where $\mathbb{R}_d$ denotes the discrete space of reals Then, the dual map Ston($2^{\mathbb{R}}/M$) $\longrightarrow \beta\mathbb{R}_d$ maps onto a subspace of $\beta\mathbb{R}_d \setminus \mathbb{R}_d$ Since M is a σ-ideal this embedding also satisfies the property highlighted in the second sentence of (b) Since the cardinality of $\mathbb{R}_d$ is non-measurable the space is real-compact (See [GJ], 12 2) This means that if $p \in$ Ston($2^{\mathbb{R}}/M$) then, viewing it in $\beta\mathbb{R}_d \setminus \mathbb{R}_d$, there is a zero-set Z of $\beta\mathbb{R}_d$ which contains p and lies in $\beta\mathbb{R}_d \setminus \mathbb{R}_d$ The trace of Z upon Ston($2^{\mathbb{R}}/M$) is then a nowhere dense G_δ-set containing p Thus, p cannot be a P-point

Some familiar classes are not covering classes, for example, the class of all F-spaces and the class of all zero-dimensional spaces (See [H], 9 3 and 9 4 Recall that X is an F-space if every cozero-set is C^*-embedded) To conclude this section we show that the class of fraction-dense spaces also is not a covering class Moreover, every compact space X is the infimum in COV(X) of fraction-dense covers

Let D stand for a discrete space of cardinality ω_1, and assume that it is well-ordered $D = \{d_1, d_2, \quad \}$ For each countable ordinal α, let $D(\alpha) = \{d_\mu \quad \mu < \alpha\}$ and $D = D(\alpha) \cup D'(\alpha)$ be the obvious partition of D Clearly βD is the disjoint union of $\beta D(\alpha)$ and $\beta D'(\alpha)$

Now let $L(\alpha)$ be the space obtained from βD by identifying all the points in $\beta D(\alpha) \setminus D(\alpha)$ Let $D = D \cup \{2\}$, where all the points of D remain isolated, and the basic neighborhoods of λ are the subsets with countable complement in D It is well known that λD is a P-space, so that $\beta\lambda D$ is basically (but not extremally) disconnected Then observe

(a) In COV($\beta\lambda D$) the spaces $L(\alpha)$ form a chain

$$\beta D \geq \qquad \geq L(\alpha_1) \geq \qquad \geq L(\alpha_2) \geq \qquad , \quad (\text{with } \alpha_1 \leq \alpha_2)$$

(b) The infimum of the $L(\alpha)$ is $\beta\lambda D$, and each $L(\alpha)$ is fraction-dense (Note each $L(\alpha)$ is homeomorphic to the disjoint union $\alpha N \cup \beta D$ Any disjoint union of fraction-dense spaces is fraction-dense)

(c) The $L(\alpha)$ are in fact strongly fraction-dense, thus, strong fraction-density is not a covering property either

The example just exhibited shows in fact that fraction-density is not preserved under an infimum of a chain We have not been able to decide whether the infimum of two fraction-dense spaces in COV(X) is fraction-dense. This question amounts to asking, by Proposition 2.2, whether the following is true: if $C(Y_1)$ and $C(Y_2)$ are both rigid f-subalgebras of $C(Y)$, then is the intersection rigid in $C(Y)$? Even though we cannot answer this question, it is easy to find examples of a pair of rigid f-subrings the intersection of which is not rigid. In $C(N)$ let A_1 be the subalgebra of sequences with finite range, and $A_2 = C(\alpha N)$, the convergent real sequences. Both are rigid in $C(N)$, while $A_1 \cap A_2$, the algebra of eventually constant sequences, is not. The next proposition establishes the stronger assertion made earlier, concerning the non-covering property of fraction-density.

PROPOSITION 2.7. (NMC) *Every compact space X is the infimum in* COV(X) *of strongly fraction-dense covers.*

PROOF. Suppose that X is compact and EX is its absolute. Let $E(p,q)$ stand for the space obtained from EX by identifying two of its points, p and q. Let $e_X\colon EX \longrightarrow X$ be the canonical irreducible surjection, and consider all pairs of points for which $e_X(p) = e_X(q)$. It should be clear that all such $E(p,q)$ are covers of X and that X is the infimum of them.

Now, if $p \neq q$ yet $e_X(p) = e_X(q)$ then neither one can be isolated, and so not a P-point either, since we are assuming that there are no measurable cardinals. On the other hand, if neither p nor q is a P-point, then it is not difficult to verify that $E(p,q)$ is strongly fraction-dense; indeed, every continuous real-valued function defined on EX can be obtained as a quotient of functions which vanish at both p and q, but not on a neighborhood of either of them. This says that $q(EX) = q\big(E(p,q)\big)$, which is sufficient to show that $E(p,q)$ is strongly fraction-dense. ∎

In [HW] the authors consider spaces X for which $C(X)/P$ is a valuation ring, for each prime ideal P of $C(X)$. In view of the fact that this condition generalizes the F-space condition, and that fraction-dense F-spaces are extremally disconnected, it is reasonable to ask whether spaces with the above valuative condition which are also fraction-dense are necessarily extremally disconnected.

The answer is no; using the same argument as in the second paragraph of the preceding proof one can show that the space X obtained by identifying two non-isolated points of βN is fraction-dense; it is not an F-space, and hence not extremally disconnected. On the other hand, it is not hard to verify that $C(X)/P$ is a valuation ring, for every prime ideal P.

3. **Coincidence of completions.** f-Rings with bounded inversion are divisible, as lattice-ordered groups. In addition, conditions (5) and (6) of Theorem 1.1 suggest how to define "fraction-density" for (abelian) lattice-ordered groups; let us suppose that G is an arbitrary abelian lattice-ordered group, and say that it is *absolute* if Min(G) is compact and extremally disconnected. This is the concept, which in the context of archimedean Riesz spaces, is called *almost Dedekind complete* in [HP]. The reader should note the overlap between Theorem 4.7(ii) in [HP] and the equivalence of (5) and (6) in our Theorem 1.1.

We choose to proceed directly from the archimedean l-group, via the Yosida embedding to achieve Theorem 3 3, which is essentially Theorem 4 11 in [HP] The only extra assumption made in the sequel is that G is divisible

If G has an order unit u, let $\mathrm{Yos}(G,u)$—or $\mathrm{Yos}(G)$, if the unit is fixed or otherwise understood—denote the set of *values* of u, that is, the set of all prime l-ideals of G which are maximal with respect to excluding u $\mathrm{Yos}(G)$ is a compact, Hausdorff space relative to its hull-kernel topology, as is customary—see [BH1] or [BH2], for example—we shall refer to this space as the *Yosida space* of G Let us now examine the relationship between the absoluteness of G and covers of $\mathrm{Yos}(G)$

Let us begin by recalling the notion of an (order) essential extension of a lattice-ordered group Suppose that G is an l-subgroup of the lattice-ordered group H, then H is an *essential extension* of G if each non-trivial l-ideal of H has a non-trivial intersec tion non-trivial intersection with G If G is archimedean then it has a (unique) maximal, archimedean essential extension, denoted eG, see [C] for details Here we add but one more observation about eG it is l-isomorphic to $D(X)$, where X is the Stone-dual of $P(G)$)

On the other hand, let's observe the following, assume that G is a complemented lattice-ordered group For each $g \in G$ the basic open set $u(g)$, consisting of all the minimal primes of G which exclude g, is compact-open, and $u(g)$ is homeomorphic to $\mathrm{Min}(g^{\perp\perp}) = \mathrm{Min}\big(G(g)\big)$, where $G(g)$ denotes the l-ideal generated by g Therefore, if G is absolute, then so is (g), and $u(g)$ is the absolute of $\mathrm{Yos}\big(G(g), g\big)$, while the map $\theta_g\ u(g) = \mathrm{Min}\big(G(g)\big) \longrightarrow \mathrm{Yos}\big(G(g), g\big)$, assigning the minimal prime l-ideal $P \in u(g)$ to the value of g in $G(g)$ containing $P \cap G(g)$, realizes the absoluteness of $u(g)$ and is sequentially irreducible (The proofs mimic the ones for f-rings completely)

Thus

PROPOSITION 3 1 *Suppose that G is an absolute abelian l-group Then for each* $g \in G$, $\mathrm{Yos}\big(G(g), g\big)$ *is a fraction dense space*

For the remainder of this article, let us suppose that G is a divisible archimedean lattice-ordered group We recall the notion of o-convergence and the associated o completion, (see [DHH], [HP] or [Pa] for details) We say that a sequence (g_n) in G *o converges to g* if there is a decreasing sequence (v_n) of positive elements such that $\inf_n v_n = 0$ and $|g_n - g| \leq v_n$ The *o-Cauchy* condition is defined analogously, and we say that G is *o complete* if every o-Cauchy sequence converges

We shall not recall the precise definition of *o-completion* here, save to recollect that if X is any compact space then $C(QFX)$ is the o-completion of $C(X)$, (Theorem 3 9b in [DHH]) Recall as well that $C(EX)$ and $C(BDX)$ are, respectively, the Dedekind and Dedekind σ-completions of $C(X)$ Clearly, X is fraction-dense if and only if all three of these completions coincide Now let us proceed to generalize this

LEMMA 3 2 *Suppose that G is an archimedean lattice-ordered group If G is absolute and o complete then it is Dedekind complete*

PROOF. Clearly G is Dedekind complete if and only if each $G(g)$ is Dedekind complete. Since the absoluteness and the o-completeness of G imply the same for each $G(g)$, we may assume without loss of generality that G has a strong order unit u.

The Johnson-Kist-Yosida embedding—see Chapter 7 of [LZ]—then puts G in $C\big(\mathrm{Yos}(G)\big)$; from Proposition 3.1 we have that Yos(G) is fraction-dense.

Now it is well-known that an o-complete l-group is uniformly complete. This observation, together with the one that G is uniformly dense in $C\big(\mathrm{Yos}(G)\big)$—because G is divisible—implies that $G = C\big(\mathrm{Yos}(G)\big)$. Since G is o-complete Yos(G) must be a quasi-F space—Theorem 3.7 in [DHH]—which implies that Yos(G) is extremally disconnected, by Proposition 2.3, and hence G is Dedekind complete. ∎

We now have the following theorem; compare with Theorem 4.11 of [HP].

THEOREM 3.3. *Suppose that G is a (divisible) archimedean lattice-ordered group with order unit. If G is absolute then its o-completion and Dedekind completion coincide. The converse is true provided* Pr(G) *is σ-complete.*

PROOF. Consider H, the o-completion of the absolute lattice-ordered group G. Since H lies between G and eG and G is rigid in eG, it follows that G is rigid in H, whence H is absolute. By Lemma 3.2, H is Dedekind complete, and obviously the Dedekind completion of G.

Conversely, if the o-completion and the Dedekind completion of G coincide and Pr(G) is σ-complete, then, in the language of [BH1], this common completion K lies in aG, the largest *akd*-extension of G, and because Pr(G) is σ-complete, and countably generated polars correspond under the contraction map $P \longrightarrow P \cap G$, it follows that G is rigid in K. However, if G is rigid in its Dedekind completion it should be obvious that Min(G) is extremally disconnected. In addition, as G has an order unit K does too, which makes Min(G) compact, proving that G is an absolute lattice-ordered group. ∎

One might be able to relax the σ-completeness of Pr(G) in Theorem 3.3; however, it cannot be discarded altogether. If A is the f-ring of all eventually constant sequences of real numbers, then Pr(A) is not σ-complete and A is not fraction-dense—not absolute, as a lattice-ordered group. However, the o-completion of A coincides with its Dedekind completion; namely, $C(\beta N)$.

4. ***R*-embeddings.** Proposition 2.2 suggests the following definition: suppose that Y is a dense subspace of the Tychonoff space X. The embedding induces an inclusion of f-algebras $C(X) \subseteq C(Y)$, by restriction of functions. We say that Y is *R-embedded* in X if this inclusion if rigid. Note that this concept is precisely what is called *$Z^{\#}$-embedded* in [HVW1]: for each cozero-set V in Y there is a cozero-set W in X such that V and $W \cap Y$ have the same closure in Y. As observed in [HVW1], if Y is a dense subspace of X, then if Y is C^*-embedded in X then it is also R-embedded in X. The concepts of *z-embedding*: every zero-set of Y is the contraction of a zero-set of X, stands in between C^*-embedding and R-embedding. Let us collect the basic facts about R-embeddings in the following proposition, which is quite easy to prove from what has gone before.

PROPOSITION 4 1 *Suppose that Y is a dense subspace of X Then the following are equivalent*

(a) *Y is R-embedded in X*

(b) *The induced inclusion of C(X) in C(Y) is rigid*

(c) *The induced inclusion of $C^*(X)$ in $C^*(Y)$ is rigid*

(d) *The Stone extension of the embedding, $\beta Y \longrightarrow \beta X$, is sequentially irreducible*

A space Y is said to be *absolutely R-embedded* if it is R-embedded in any space containing it densely In 3 7(b) of [HVW1] it is shown that every weakly Lindelof space is absolutely R-embedded, the proof employs the fact that a cozero-set of a weakly Lindelof space is weakly Lindelof, which is shown in [CHN] This proves half of Theorem 4 2, coming up shortly (X is *weakly Lindelof* if every open cover of X has a countable subfamily, the union of which is dense in X) Recall, from 6J of [GJ] that a space X is absolutely C^*-embedded—that is, X is C^*-embedded in any subspace containing it—precisely when X is *almost compact*, which is to say that $|\beta X \setminus X| \leq 1$

As to absolute z-embeddedness, we refer the reader to [B], [BlH] and [HaJ] X is absolutely z-embedded if and only if X is almost compact or Lindelof

For R-embeddedness we have the following theorem

THEOREM 4 2 *A Tychonoff space X is absolutely R-embedded if and only if X is almost compact or weakly Lindelof*

By the remarks already offered, the sufficiency follows Before proving the necessity in the theorem we need a couple of lemmas, including a special case of the theorem

LEMMA 4 2(A) *Suppose that X is locally compact Then X is R embedded in αX, the one-point compactification of X, if and only if X is almost compact or weakly Lindelof*

PROOF As the sufficiency has already been proved, we move on to the necessity

Recall that X is open in αX, and observe that $V = V' \cap X$ where V' is a cozero-set of αX if and only if $X \setminus V$ is a compact G_δ-set or else V is an open σ-compact set, that is to say, a countable union of compact sets We leave the verification of this remark to the reader

Now we show that the cozero set V of X and the contraction $V' \cap X$, where V' is a cozero-set of αX, have the same closure in X precisely when either $X \setminus V$ is compact or else V densely contains an open σ-compact set The sufficiency is clear from our preceding remark As to the necessity, suppose that $\mathrm{cl}_X V = \mathrm{cl}_X(V' \cap X)$, where V and V' are cozero-sets in X and αX respectively Then $\mathrm{cl}_X V$ contains the open σ-compact set $V' \cap X$ Now $V = U_n F_n$, where the F_n are closed in X, and $V' \cap X = U_n K_n$, with each K_n compact Since each $F_n \cap K_m$ is compact and $V \supseteq U_{n\,m} F_n \cap K_m = V \cap (V' \cap X)$, which is dense in V since V is open Thus, if $X \setminus V$ is not compact, V densely contains an open σ-compact set

If X is not almost compact we have, by 6J (1) in [GJ], that X possesses two disjoint zero-sets Z_1 and Z_2 which are non-compact By our remarks in the foregoing, each of these densely contains a σ-compact set, say K_1 and K_2 respectively Now, $X = (X \setminus Z_1) \cup$

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$(X \setminus Z_2)$ and densely contains $K_1 \cup K_2$ (which is a σ-compact set); since σ-compact sets are Lindelöf, it follows that X is weakly Lindelöf, and the proof of the lemma is done. ∎

LEMMA 4.2(B). *For a space X the following are equivalent.*

(1) X is weakly Lindelöf.

(2) For each open subspace Y, such that $X \subseteq Y \subseteq \beta X$, there is a subspace $W \subseteq Y$, dense in βX, which is an open σ-compact set.

(3) For each open subspace Y, such that $X \subseteq Y \subseteq \beta X$, there is a subspace $W \subseteq Y$, dense in βX, which is a cozero-set in βX.

(4) Each open subspace Y, such that $X \subseteq Y \subseteq \beta X$, is weakly Lindelöf.

We leave the proof to the reader. In addition, Lemma 4.2(b) should be compared to the result of Smirnov [Sm] which gives that X is Lindelöf if and only if condition (3) of the lemma holds with the added stipulation that $X \subseteq W$. Finally, observe that in the lemma, as in Smirnov's result, βX can be replaced by any compactification of X.

Now let us conclude the proof of Theorem 4.2. That is, we show that if X is absolutely R-embedded but not almost compact, then it is weakly Lindelöf.

Suppose that Y is an open subspace of βX, with $X \subseteq Y \subseteq \beta X$. We may suppose that Y too is not almost compact; furthermore, Y is absolutely R-embedded as well, because rigid containment is transitive. As Y is locally compact, we have by Lemma 4.2(a) that it is weakly Lindelöf. Now apply Lemma 4.2(b). ∎

We record a summary of the special situation for locally compact spaces in a corollary. Observe that that the stipulation that X not be almost compact is necessary: the ordered space of all ordinals less than the first uncountable ordinal is almost compact, yet its point at infinity is a P-point.

COROLLARY 4.2.1. *Suppose that X is locally compact but not almost compact. Then the following are equivalent.*

(1) X is weakly Lindelöf.

(2) X densely contains an open σ-compact set.

(3) The point at infinity in αX is not an almost P-point.

(4) X is absolutely R-embedded but not absolutely C^*-embedded.

A space which satisfies the *countable chain condition* (ccc) obviously is weakly Lindelöf—see 3.6(b) in [HVW1]—and so if X satisfies ccc it follows that X is absolutely R-embedded.

According to 3P(3) in [PW], a space satisfies the ccc precisely when every open subspace is weakly Lindelöf. (Compare with the proof of Proposition 3.6(a) of [HVW1], which shows that X is weakly Lindelöf precisely when every cozero-set of X is weakly Lindelöf.) Recall as well Proposition 2.1: a space X is fraction-dense precisely when every regular open set densely contains a cozero-set. From this it should be reasonably clear that

COROLLARY 4 2 2 *A compact space X ıs fractıon-dense ıf and only ıf every regular open set of X ıs weakly Lındelof*

It mıght be worthwhıle to put Corollary 4 2 2 another way, for emphasıs If X ıs compact then $EX = QFX$ ıf and only ıf every regular open subset ıs weakly Lındelof

Let us recall here the remark made followıng Proposıtıon 2 6, and record the corollary promısed then

COROLLARY 4 2 3 *If X ıs compact and fractıon-dense and $p \in X$ ıs a non-ısolated almost-P poınt, then p ıs a C^*-poınt*

We also derıve, wıth lıttle addıtıonal effort

PROPOSITION 4 3 *A space whıch satısfies the* ccc *ıs strongly fractıon-dense*

PROOF If every open subset of X ıs weakly Lındelof then every dense open subset contaıns a dense cozero-set Hence $q(X) = Q(X)$, provıng that X ıs strongly fractıon-dense ■

Note, as well, that a Tychonoff space X ıs fractıon-dense ıf and only ıf every dense open subspace ıs R-embedded ın X (Recall that $Q(X)$ ıs the dırect lımıt of $C(V)$, where V ranges over all dense open subsets of X, so ıf every dense open subset ıs R-embedded, then $C(X)$ ıs rıgıd ın $Q(X)$, and X ıs fractıon-dense by Theorem 1 1 The converse ıs trıvıal)

We should also mentıon the followıng as we remarked after Corollary 1 2 1, we do not know of any examples of fractıon-dense spaces whıch are not strongly fractıon-dense By Proposıtıon 4 3, such a space must faıl the ccc

Our final proposıtıon (for locally compact spaces) ıs a summary of many of the results ın thıs sectıon We leave the detaıls of the verıficatıon to the reader

PROPOSITION 4 4 *For a locally compact space X the followıng are equıvalent*

(a) X ıs absolutely R-embedded and fractıon-dense

(b) αX ıs fractıon-dense

(c) Every compactıficatıon of X ıs fractıon-dense

Moreover, ıf there are no measurable cardınals, then the above are equıvalent to

(d) X ıs weakly Lındelof and fractıon-dense

(e) X contaıns a dense, open σ-compact set, whıch ıs fractıon-dense and R-embedded ın αX

(f) X contaıns a dense, R-embedded σ-compact, fractıon-dense space

(Recall that a σ *compact* space ıs a countable unıon of compact spaces)

PROOF The equıvalence of (a), (b) and (c) should be clear from Lemma 4 2(a) and Theorem 4 2, and the observatıon that ıf βX and αX are fractıon-dense then so ıs each compactıficatıon of X As to the final condıtıons, recall Proposıtıon 2 6(d) wıthout measurable cardınals every almost P-poınt ıs ısolated ın a fractıon-dense space Thus ıf X ıs fractıon-dense and almost compact (but not compact), then the poınt at ınfinıty ın αX cannot be almost P, whence X ıs weakly Lındelof, accordıng to Lemma 4 2(b) ■

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