J. Austral. Math. Soc. **23** (Series A) (1977), 9–27.

ON ARITHMETIC FUNCTIONS AND DIVISORS OF HIGHER ORDER

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(Received 15 January 1975)

Abstract

We discuss properties of arithmetic functions of higher order defined through the introduction of a new concept of divisor of higher order. We shall construct an infinite sequence of Euler-like functions and the well known Euler function will be the first member of this sequence. Asymptotic estimates of such functions are given and a study of error functions associated with the Euler-like sequence is made. We would like to mention that the familiar number theoretic functions become only the first members of an infinite sequence of functions of similar behaviour.

1. Introduction

If d and n are two positive integers and if $d \mid n$ we say d is a first order divisor of n and change the notation to $d \mid_1 n$. When a and b are two positive integers, (a, b) rewritten as $(a, b)_1$ shall denote the largest divisor of a dividing b. When $(a, b)_1 = 1$ we say a is prime to b first order.

If d and n are two positive integers then d is said to be a divisor of n of second order, denoted by $d \mid_2 n$ if

$$\left(\frac{n}{d}, d\right)_1 = 1. \tag{1.1}$$

(This is the definition of unitary divisor). The symbol $(a, b)_2$ represents the largest divisor c of a satisfying $c \mid_2 b$. If $(a, b)_2 = 1$ we say a is prime to b order 2. Here comes the departure: A divisor d of n is a divisor of third order (notation: $d \mid_3 n$) if

$$\left(\frac{n}{d}, d\right)_2 = 1. \tag{1.2}$$

The symbol $(a, b)_3$ stands for the largest divisor c of a that satisfies $c \mid_3 b$. If $(a, b)_3 = 1$ we say a is prime to b order 3. We generalise by saying that $d \mid_r n$ (read d is rth-order divisor of n) if

(1.3) $$\left(\frac{n}{d}, d\right)_{r-1} = 1$$

and

(1.4) $$(a, b)_r = \max\{c \mid_1 a : c \mid_r b\}.$$

If $(a, b)_r = 1$ then a is prime to b order r.

NOTE. The definition $d \mid_3 n$ given by us differs from the two well known extensions of the concept of unitary divisor given by Chidambaraswamy (1967) and Suryanarayana (1971) respectively. The former defines d to be a semi-unitary divisor of n if $(d, n/d)_2 = 1$ as opposed to our $d \mid_3 n$ where $(n/d, d)_2 = 1$. The latter defines d to be a bi-unitary divisor of n if $(d, n/d)^{**} = 1$ where $(a, b)^{**}$ represents the largest common unitary divisor of a and b. However in both these papers the concept of unitary divisor is just extended one step beyond.

Our definition of higher order divisors is given in such a way that the higher order divisors share many properties in common so that it is possible to discuss together the properties of arithmetic functions of rth order, as we shall see in the theorems that follow. In fact to discuss the entire system as a unit, it becomes necessary to place n/d on the left side. Moreover the familiar number theoretic results follow as corollaries if we set $r = 1$, and some of the results of Cohen (1960) can be deduced if we set $r = 2$.

We now define rth order analogues to some well known arithmetic functions. However as $(a, b)_r \neq (b, a)_r$ in general these functions have interesting dual functions. Denote by

(1.5) $$\varphi_r(n, x) = \sum_{\substack{0 < a \leqq x \\ (a, n)_r = 1}} 1; \quad \varphi_r(n, n) = \varphi_r(n)$$

and its dual

(1.6) $$\varphi_r^*(n, x) = \sum_{\substack{0 < a \leqq x \\ (n, a)_r = 1}} 1; \quad \varphi_r^*(n, n) = \varphi_r^*(n)$$

for $r \geqq 1$. We define $\varphi_0(n, x) = \varphi_0^*(n, x) = [x]$ where $[x]$ denotes the largest integer $\leqq x$ and $(a, n)_0 = (n, a)_0 = 1$ for all a and n. Note that $\varphi_1 = \varphi_1^* = \varphi$ (Euler). We define the divisor functions

(1.7) $$\sigma_{r,k}(n) = \sum_{d \mid_r n} d^k \quad \text{and} \quad \sigma_{r,k}^*(n) = \sum_{d \mid_r n} \left(\frac{n}{d}\right)^k$$

and

(1.8) $$\sigma_{r,0}(n) = \sigma_{r,0}^*(n) = \tau_r(n).$$

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Before we take up the study of these functions we need to define some more functions. Let $\{F_r\}_{r=0}^{\infty}$ denote the sequence given by

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}, \quad n \geqq 2. \tag{1.9}$$

Let $l(y)$ and $l^*(y)$ denote respectively the least integer $>$ and $\geqq y$. Further define

$$f_r(x) = l\left(\frac{F_{r-1}}{F_r}x\right) \quad \text{when} \quad r \equiv 1 \pmod 2 \tag{1.10}$$

$$f_r(x) \equiv l^*\left(\frac{F_{r-1}}{F_r}x\right) \quad \text{when} \quad r \equiv 0 \pmod 2). \tag{1.11}$$

Let $f_r^{-1}(x)$ denote the largest integer y with $f_r(y) = x$. And if $n = \Pi_{i=1}^s p_i^{\alpha_i}$ be the canonical decomposition of n then let

$$\beta_r(n) = \prod_{i=1}^{s} p_i^{f_r^{-1}(\alpha_i)+1}\ r > 1, \quad \beta_1(n) = n. \tag{1.12}$$

2. Properties of higher order divisors and arithmetic functions

We will now show

THEOREM 1. *If $n = \Pi_{i=1}^s p_i^{\alpha_i}$ be the canonical representation of n as a product of distinct primes, and if $d \mid_1 n$ then $d \mid_r n$ if and only if $d = \Pi_{i=1}^s p_i^{\beta_i}$ where*

$$\beta_i = 0 \quad or \quad f_r(\alpha_i) \leqq \beta_i \leqq \alpha_i \quad i = 1, 2, \cdots, s.$$

PROOF. For $r = 1$ by (1.9) and (1.10) $f_r(\alpha_i) = 1$ and so the theorem holds trivially. For $r = 2$, $f_r(\alpha_i) = \alpha_i$ (again by (1.9) and (1.11)) and $\beta_i = 0$ or $\beta_i = \alpha_i$ for a unitary divisor and the theorem is true.

Let $r = 3$ and $d = \Pi_{i=1}^s p_i^{\beta_i}$ satisfy $d \mid_3 n$. Clearly $d \mid_1 n$ and so $\alpha_i \geqq \beta_i$ trivially holds. Now

$$\frac{n}{d} = \prod_{i=1}^{s} p_i^{\alpha_i - \beta_i}.$$

If $d \mid_3 n$ then $(n/d, d)_2 = 1$. Thus there is no divisor of n/d except 1 which is a divisor of d of second order. This is possible if and only if

$$\alpha_i - \beta_i < \beta_i \quad \text{or} \quad \beta_i = 0.$$

For if $\alpha_i - \beta_i \geqq \beta_i$ then $p_i^{\beta_i} \mid_1 n/d$ and $p_i^{\beta_i} \mid_2 d$ and this is a contradiction. Thus $\alpha_i - \beta_i < \beta_i$. If $\alpha_i - \beta_i < \beta_i$, and $p^{\nu_i} \mid n/d$ then $0 \leqq \nu_i \leqq \alpha_i - \beta_i < \beta_i$ so that $p_i^{\nu_i} \mid_2 d$. Hence $(n/d, d)_2 = 1$. Thus

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$$\alpha_i - \beta_i < \beta_i \Leftrightarrow \beta_i > \frac{\alpha_i}{2} = \frac{F_2}{F_3}\alpha_i.$$

Moreover β_i is an integer and so $\beta_i \geqq f_3(\alpha_i)$ proving theorem for $r = 3$.

In general let the theorem hold for $1, 2, \cdots, r$, r even. Now $d\,|_{r+1}\, n$ if and only if $(n/d, d)_r = 1$ where n and d are represented as above. Now $(n/d, d)_r = 1$ says that there is no divisor of n/d save 1 that is a divisor of d order r. This is possible if and only if

$$\alpha_i - \beta_i < \frac{F_{r-1}}{F_r}\beta_i \quad \text{or} \quad \beta_i = 0. \tag{2.1}$$

For otherwise if $\alpha_i - \beta_i \geqq F_{r-1}F_r/\beta_i$ then one can find a ν_i satisfying

$$\alpha_i - \beta_i \geqq \nu_i \geqq F_{r-1}\beta_i/F_r$$

so that $p^{\nu_i}|_1(n/d)$ and $p_i^{\nu_i}|_r d$, a contradiction. Thus (2.1) holds. The sufficiency of (2.1) is clear. We rewrite (2.1) as

$$\beta_i > \frac{F_r}{F_{r+1}}\alpha_i \tag{2.2}$$

and β_i is an integer. Thus $\beta_i \geqq f_{r+1}(\alpha_i)$ proving lemma for $r+1$ odd. The proof for the case $r+1$ even is similar, only that $\beta_i \geqq F_r\alpha_i/F_{r+1}$ will replace (2.2) for (2.1) will be replaced by a weak inequality.

The higher order divisors share in common the property

THEOREM 2. (a) *If a and n are integers, then for any nonnegative integer λ*

$$(a, n)_r = (\lambda n + a, n)_r = (\lambda n - a, n)_r. \tag{2.3}$$

(b) *We have $(n, a)_r = 1$ if and only if*

$$(n, a)_r = (n, \lambda\beta_r(n) + a)_r = (n, \lambda\beta_r(n) - a)_r = 1 \tag{2.4}$$

where $\beta_r(n)$ is as defined in (1.12).

We omit the proofs of Theorem 2 as they are direct consequences of the definitions in section 1. We shall need Theorem 2 in the discussion of error functions.

We now take up the study of the functions defined in (1.5) to (1.8). We shall always represent n in the canonical form $n = \Pi_{i=1}^{s} p_i^{\alpha_i}$.

THEOREM 3. $\varphi_r(n) = n\Pi_{i=1}^{s}\left(1 - \dfrac{1}{p_i^{f_r(\alpha_i)}}\right)$ *for* $r \geqq 1$.

PROOF. We know from (1.5) that

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$$\varphi_r(n,x)=\sum_{\substack{0<a\leqq x\\(a,n)_r=1}}1=[x]-\sum_{\substack{0<a\leqq x\\(a,n)_r>1}}1.$$

Now $(a,n)_r>1$ if there exists $d\,|_r n$, $d>1$ with $d\,|_1\, a$. We know from Theorem 1 that $d\,|_r n$ if and only if $\beta_i=0$ or $f_r(\alpha_i)\leqq\beta_i\leqq\alpha_i$. This implies that if $p_i\,|\,a$ and $p_i\,|_r n$ then $p_i^{f_r(\alpha_i)}|_1\, a$. Thus a simple combinatorial argument leads to

$$\varphi_r(n,x)=[x]-\sum_{0<i\leqq s}\left[\frac{x}{p_i^{f_r(\alpha_i)}}\right]+\sum_{0<i<j\leqq s}\left[\frac{x}{p_i^{f_r(\alpha_i)}p_j^{f_r(\alpha_j)}}\right]$$

(2.5)
$$-\cdots+(-1)^s\sum\left[\frac{x}{p_1^{f_r(\alpha_1)}\cdots p_s^{f_r(\alpha_s)}}\right].$$

If we put $x=n$ in (2.5) we get Theorem 3.

Now (2.5) also indicates that

COROLLARY 1. *If* $e_r(n,x)=x\varphi_r(n)/n-\varphi_r(n,x)$ *then*

$$e_r(n,x)=0(\tau_1(n))$$

PROOF. We can rewrite (2.5) as

$$\varphi_r(n,x)=x-\sum\frac{x}{p_i^{f_r(\alpha_i)}}+\cdots+0\Big(1+\sum_{p_i|n}1+\sum_{p_ip_j|n}+\cdots\Big)$$
$$=\frac{x\varphi_r(n)}{n}+0(\tau_1(n)).$$

THEOREM 4. *With* $\beta_r(n)$ *as in* (1.12) *we have*

$$\varphi_r^*(n,\beta_r(n))=\varphi_1(n,\beta_r(n))\prod_{i=1}^{s}\left(1+\frac{1}{p_i^{f_r^{-1}(\alpha_i)+1}\left(1-\frac{1}{p_i}\right)}\right)$$

PROOF. Going back to the definition of $\varphi_r^*(n,x)$ in (1.6) we find that $(n,a)_r=1$ can arise out of two cases. If $(n,a)_1=1$ then $(n,a)_r=(a,n)_r=1$. Or $(n,a)_1>1$ in which case there is a $p_i\,|_1\, n$ and $p_i\,|_1\, a$. As $(n,a)_r=1$ even if $d_1\,|\,a$, $d\nmid_r n$ for all $d\,|_1\, n$. Thus $p_i^{f_r^{-1}(\alpha_i)+1}|_1\, a$. Again a simple combinatorial argument leads to

$$\varphi_r^*(n,x)=\varphi_1(n,x)+\sum_{0<i\leqq s}\varphi_1\left(\frac{n}{p_i^{\alpha_i}},\frac{x}{p_i^{f_r^{-1}(\alpha_i)+1}}\right)$$

(2.6)
$$+\sum_{0<i<j\leqq s}\varphi_1\left(\frac{n}{p_i^{\alpha_i}p_j^{\alpha_j}},\frac{x}{p_i^{f_r^{-1}(\alpha_i)+1}p_j^{f_r^{-1}(\alpha_j)+1}}\right)+\cdots$$

If we put $x=\beta_r(n)$ in (2.6) and use Theorem 2 which for $r=1$ gives $\varphi_1(n,\lambda n+\mu)=\lambda\varphi_1(n)+\varphi_1(n,\mu)$ we get Theorem 4.

COROLLARY 2. *If* $e^*_r(n,x) = x\varphi^*_r(n,\beta_r(n))/\beta_r(n) - \varphi^*_r(n,x)$ *then* $e^*_r(n,x) = 0(\tau_1(n))$.

We omit details of the proof which is similar to Corollary 1.

Similarly formulae can be found for $\sigma_{r,k}(n)$, $\sigma^*_{r,k}(n)$, and $\tau_r(n)$. These are given below

$$\tau_r(n) = \prod_{i=1}^{s} (\alpha_i - f_r(\alpha_i) + 2) \tag{2.7}$$

and

$$\sigma_{r,k}(n) = \prod_{i=1}^{s} (1 + p_i^{f_r(\alpha_i)} + \cdots p_i^{\alpha_i}). \tag{2.8}$$

We are now in a position to prove the following which is somewhat interesting since it is novel.

THEOREM 5. *For any pair of integers n and* $k \geqq 0$ *we have*

(a) $\varphi_1(n) \leqq \varphi_3(n) \leqq \varphi_5(n) \leqq \cdots \leqq \varphi_6(n) \leqq \varphi_4(n) \leqq \varphi_2(n) \leqq \varphi_0(n)$

(b) $\sigma_{2,k}(n) \leqq \sigma_{4,k}(n) \leqq \sigma_{6,k}(n) \leqq \cdots \leqq \sigma_{5,k}(n) \leqq \sigma_{3,k}(n) \leqq \sigma_{1,k}(n)$

(c) $\sigma^*_{2,k}(n) \leqq \sigma^*_{4,k}(n) \leqq \sigma^*_{6,k}(n) \leqq \cdots \leqq \sigma^*_{5,k}(n) \leqq \sigma^*_{3,k}(n) \leqq \sigma^*_{1,k}(n)$

(d)
$$\frac{\varphi^*_1(n,\beta_1(n))}{\beta_1(n)} \leqq \frac{\varphi^*_3(n,\beta_3(n))}{\beta_3(n)} \leqq \frac{\varphi^*_5(n,\beta_5(n))}{\beta_5(n)}$$
$$\leqq \cdots \frac{\varphi^*_6(n,\beta_6(n))}{\beta_6(n)} \leqq \frac{\varphi^*_4(n,\beta_4(n))}{\beta_4(n)} \leqq \frac{\varphi^*_2(n,\beta_2(n))}{\beta_2(n)}.$$

PROOF. We shall prove (a) and (b). The proofs of (c) and (d) are similar. First we observe that F_{2k}/F_{2k+1} form an increasing sequence and F_{2k-1}/F_{2k} form a decreasing sequence both sequences converging to $(\sqrt{5}-1)/2$. Further if $x < y$ then

$$l(x) \leqq l(y) \qquad l^*(x) \leqq l^*(y) \quad \text{and} \quad l(x) \leqq l^*(y). \tag{2.9}$$

These follow from the definitions of l and l^*. Now (2.9) implies that for any integer m we have

$$f_1(m) \leqq f_3(m) \leqq f_5(m) \leqq \cdots \leqq f_6(m) \leqq f_4(m) \leqq f_2(m). \tag{2.10}$$

If we use (2.10) and Theorem 3 we get (a). Now (2.9) and (2.10) will give on similar lines of reasoning the reverse inequalities for $f_r^{-1}(m)$. Then if we use Theorem 4 we get (d).

To prove (b) and (c) it is enough to observe that (2.10) implies that

$$d\,|_{2m}\, n \Rightarrow d\,|_{2m+2}\, n;\ d\,|_{2m+1}\, n \Rightarrow d\,|_{2m-1}\, n;\ d\,|_{2m}\, n \Rightarrow d\,|_{2m'+1}\, n$$

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for any pair of integers m and m'. That is if $D_m(n)$ denotes the set of mth order divisors of n then

$$(2.11)\qquad D_2(n) \subset D_4(n) \subset D_6(n) \subset \cdots \subset D_5(n) \subset D_3(n) \subset D_1(n).$$

Clearly (2.11) gives (b) and (c). This proves the theorem.

3. Asymptotic estimates

We saw in the last section properties of $d|_r n$. Theorem 5 gave for instance relations between φ_r, $\sigma_{r,k}$, φ_r^* and $\sigma_{r,k}^*$ for $r = 1, 2, \cdots$ separately. When we take up asymptotic estimates of these functions, we find that the φ's and the σ's are related. For example the average order of $\sigma_{r,k}$ and $\sigma_{r,k}^*$ involve φ_{r-1}^* and φ_{r-1} respectively. We begin by proving

THEOREM 6. *There exists a constant c_r so that*

$$\sum_{1 \leqq n \leqq x} \varphi_r(n) = c_r x^2 + 0(x^{3/2+\varepsilon}) \forall \varepsilon > 0.$$

PROOF. We note that Theorem 3 implies that φ is multiplicative. Also if n is square free then $\varphi_r(n) = \varphi_1(n)$.

Decompose every number n as $n = Nn'$ where n' is square free and $(n', N) = 1$. The number N has the property that if $p \mid N$, p-prime then $p^2 \mid N$. (N is called a powerful number.) We call N the powerful part of n. Keep N fixed and ask for those $n \leqq x$ for which N is the powerful part. Sum over all such N. Thus

$$(3.1)\qquad \sum_{1 \leqq n \leqq x} \varphi_r(n) = \sum_{\substack{N \leqq x \\ N\text{-powerful}}} \sum_{\substack{Nn' \leqq x \\ n' \text{ square free} \\ (n', N) = 1}} \varphi_r(Nn') = \sum_{\substack{N \leqq x \\ N\text{-powerful}}} \varphi_r(N) \sum_{\substack{n' \leqq x/N \\ n' \text{square free} \\ (n', N) = 1}} \varphi_1(n').$$

So we need to know the sum

$$(3.2)\qquad \sum_{\substack{m \leqq x \\ m \text{ square free} \\ (m, N) = 1}} \varphi_1(m) = \sum_{\substack{m \leqq x \\ (m, N) = 1}} \varphi_1(m) |\mu(m)| = \sum_{\substack{m \leqq x \\ (m, N) = 1}} |\mu(m)| \sum_{d \mid m} \mu(d) \frac{m}{d}$$

$$= \sum_{\substack{1 \leqq d \leqq \sqrt{x} \\ (d, N) = 1}} \mu(d) \sum_{\substack{0 \leqq d' \leqq x/d \\ (d', N) = 1}} d' |\mu(dd')| + \sum_{\substack{1 \leqq d \leqq \sqrt{x} \\ (d, N) = 1}} d \sum_{\substack{\sqrt{x} \leqq d' \leqq x/d \\ (d', N) = 1}} \mu(d) |\mu(dd')|$$

$$= \sum_{\substack{1 \leqq d \leqq \sqrt{x} \\ (d, N) = 1}} \mu(d) \sum_{\substack{0 \leqq d' \leqq x/d \\ (d', N) = 1}} d' |\mu(dd')| + 0(x^{3/2})$$

$$= \sum_{\substack{1 \leqq d \leqq \sqrt{x} \\ (d, N) = 1}} \mu(d) \sum_{\substack{d' \leqq x/d \\ d' \text{ square free} \\ (d', Nd) = 1}} d' + 0(x^{3/2}).$$

The function μ used above is the Möbius function. It is not difficult to show that the number of integers $\leqq x$ that are square free and prime to Nd is

$$c_{Nd}x + 0(\sqrt{x})$$

where

$$c_{Nd} = \prod_{p \mid Nd}\left(1 - \frac{1}{p}\right) \cdot \prod_{q \nmid Nd}\left(1 - \frac{1}{q^2}\right) = \frac{6}{\pi^2}\prod_{p \mid Nd}\left(1 + \frac{1}{p}\right)^{-1} < \frac{6}{\pi^2}.$$

So the sum in (3.2) is of the form

$$\frac{x^2}{2}\sum_{\substack{1 \leqq d \leqq \sqrt{x} \\ (d, N)=1}} \frac{\mu(d)}{d^2} c_{Nd} + 0(x^{3/2}) = k_N \frac{x^2}{2} + 0(x^{3/2}) \tag{3.3}$$

where k_N is a constant depending on N and bounded for all N-powerful. Substituting this in (3.1) we get

$$\sum_{1 \leqq n \leqq x} \varphi_r(n) = x^2 \sum_{\substack{N \leqq x \\ N \text{ powerful}}} k_N \frac{\varphi_r(N)}{2N^2} + 0(x^{3/2}).$$

Now every powerful number N is of the form SS' where S is a perfect square, S' square free and $S' \mid S$. This decomposition is unique. So the number of powerful numbers $\leqq x$ is

$$0\left(\sum_{\substack{S \leqq x \\ S \text{ square}}} \tau_1(S)\right) = 0(x^{1/2+\varepsilon}) \forall \varepsilon > 0.$$

This shows that the series

$$\sum_{\substack{N=1 \\ N\text{-powerful}}}^{\infty} \frac{\varphi_r(N) k_N}{2N^2} < \infty$$

so that if c_r is the sum of this series we have

$$\sum_{1 \leqq n \leqq x} \varphi_r(n) = c_r x^2 + 0(x^{3/2+\varepsilon}) \forall \varepsilon > 0$$

which proves the theorem.

It is interesting to observe

THEOREM 7. *We have*

$$\sum_{1 \leqq n \leqq x} \frac{\varphi_r(n)}{n} \sim \sum_{1 \leqq n \leqq x} \frac{\varphi_r^*(n, \beta_r(n))}{\beta_r(n)} \sim 2c_r x.$$

PROOF. We begin by stating Abel's summation formula as in LeVeque (1960).

"Suppose that $\lambda_1, \lambda_2, \cdots$ is a non-decreasing sequence with limit infinity, that $a_1, a_2, \cdots$ is an arbitrary sequence of real or complex numbers, and f a function with a continuous derivative for $x \geqq \lambda_1$. Put

$$A(x) = \sum_{\lambda_n \leqq x} a_n.$$

Then for $x \geqq \lambda_1$

$$\sum_{\lambda_n \leqq x} a_n f(\lambda_n) = A(x)f(x) - \int_{\lambda_1}^{x} A(t)f'(t)dt."$$

Clearly, by Theorem 6 and Abel's summation formula with $\lambda_n = n$, $f(x) = 1/x$ and $a_n = \varphi_r(n)$ we infer that

$$\sum_{1 \leqq n \leqq x} \frac{\varphi_r(n)}{n} = 2c_r x + 0(x^{\frac{1}{2}+\varepsilon}) \forall \varepsilon > 0$$

which proves half the assertion. The second half is more interesting. We have

$$\sum_{n=1}^{m} \varphi_r(n, m) = \sum_{n=1}^{m} \sum_{\substack{1 \leqq a \leqq m \\ (a,n)_r = 1}} 1 = \sum_{1 \leqq a \leqq m} \sum_{\substack{n=1 \\ (a,n)_r = 1}}^{m} 1$$

(3.4)

$$= \sum_{a=1}^{m} \varphi_r^*(a, m) = \sum_{n=1}^{m} \varphi_r^*(n, m).$$

But

(3.5) $$\sum_{n=1}^{m} \varphi_r(n, m) = m \sum_{n=1}^{m} \frac{\varphi_r(n)}{n} + 0(m \log m)$$

by virtue of Corollary 1 and

(3.6) $$\sum_{n=1}^{m} \varphi_r^*(n, m) = m \sum_{n=1}^{m} \frac{\varphi_r^*(n, \beta_r(n))}{\beta_r(n)} + 0(m \log m)$$

by Corollary 2 for

$$\sum_{n \leqq m} \tau_1(n) = 0(m \log m).$$

Replacing m by x, dividing by x we see that (3.4), (3.5) and (3.6) give Theorem 7.

We can now prove

THEOREM 8. $\Sigma_{1 \leqq n \leqq m} \tau_r(n) = 2c_{r-1} m \log m + 0(m)$.

PROOF. Let us first give an interpretation to the sum

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$$\sum_{1 \leqq n \leqq m} \tau_r(n) = \sum_{1 \leqq n \leqq m} \sum_{d |_r n} 1 = \sum_{1 \leqq n \leqq m} \sum_{(n/d, d)_{r-1} = 1} 1. \tag{3.7}$$

So (3.7) represents the number of lattice points (x_0, y_0) under the graph of the hyperbola $xy = m, x > 0, y > 0$ with $(y_0, x_0)_{r-1} = 1$. We first count the lattice points with $x_0 \leqq \sqrt{m}$. They are given by

$$\begin{aligned} \sum_{1 \leqq d \leqq \sqrt{m}} \sum_{\substack{d' \leqq m/d \\ (d', d)_{r-1} = 1}} 1 &= \sum_{1 \leqq d \leqq \sqrt{m}} \varphi_{r-1}(d, m/d) \\ &= \sum_{d \leqq \sqrt{m}} \left\{ m \frac{\varphi_{r-1}(d)}{d^2} + 0(\tau_1(d)) \right\} \\ &= m \sum_{d \leqq \sqrt{m}} \frac{\varphi_{r-1}(d)}{d^2} + 0(\sqrt{m} \log m). \end{aligned} \tag{3.8}$$

Now again Abel's summation formula and Theorem 7 tell us that the summation in (3.8) is

$$2c_{r-1} m \log \sqrt{m} + 0(\sqrt{m} \log m).$$

Now the number of lattice points with $y_0 \leqq \sqrt{m}$ is

$$\sum_{1 \leqq d \leqq \sqrt{m}} \sum_{\substack{d' \leqq m/d \\ (d, d')_{r-1} = 1}} 1 = \sum_{1 \leqq d \leqq \sqrt{m}} \varphi^*_{r-1}(d, m/d)$$

which by use of Corollary 2 is

$$m \sum_{d \leqq \sqrt{m}} \frac{\varphi^*_{r-1}(d, \beta_{r-1}(d))}{d \beta_{r-1}(d)} + 0(\sqrt{m} \log m).$$

Again the use of Abel's summation formula and Theorem (7) gives that this sum is

$$2c_{r-1} m \log \sqrt{m} + 0(\sqrt{m} \log m).$$

The overlap in these two processes of counting is the points in the square $1 \leqq x_0 \leqq \sqrt{m}$, $1 \leqq y_0 \leqq \sqrt{m}$, which is $0(m)$. Thus

$$\begin{aligned} \sum_{n \leqq m} \tau_r(n) &= 2c_{r-1} m \log \sqrt{m} + 2c_{r-1} m \log \sqrt{m} + 0(\sqrt{m} \log m) + 0(m) \\ &= 2c_{r-1} m \log m + 0(m). \end{aligned}$$

COROLLARY 3. *If $\tau_1(n)$ is the divisor function then*

$$\sum_{n \leqq m} \tau_1(n) \sim m \log m.$$

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PROOF. $c_0 = 1/2$. Conclusion is clear.

COROLLARY 4. *If $\tau_2(n)$ is the number of unitary divisors of n then*

$$\sum_{n \leq m} \tau_2(n) \sim \frac{6}{\pi^2} m \log m.$$

PROOF. By Theorem 6, the constant $c_1 = 3/\pi^2$. So $2c_1 = 6/\pi^2$ and corollary holds.

REMARK. The average order of $\varphi_r(n)$ is $2c_r n$, while the average order of $\tau_r(n)$ is $2c_{r-1} \log n$.

We now take up the asymptotic estimates of $\sigma_{r,k}$ and $\sigma^*_{r,k}$ for $k > 0$. For $k > 0$ define two constants

$$\alpha_{r,k} = \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{\varphi_{r-1}(n)}{n^{k+2}} \tag{3.9}$$

and

$$\alpha^*_{r,k} = \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{\varphi^*_{r-1}(n, \beta_{r-1}(n))}{\beta_{r-1}(n) \cdot n^{k+1}}. \tag{3.10}$$

Our theorem is

THEOREM 9.

$$\text{(a)} \quad \sum_{n=1}^{m} \sigma_{r,k}(n) = \alpha^*_{r,k} m^{k+1} + 0(m^{k+\frac{1}{2}});$$

$$\text{(b)} \quad \sum_{n=1}^{m} \sigma^*_{r,k}(n) = \alpha_{r,k} m^{k+1} + 0(m^{k+\frac{1}{2}}).$$

PROOF. We shall prove the second part of the theorem. Part (a) will follow on similar reasoning. We shall first need an estimate of

$$\sum_{0 < a \leq x, (a,n)_r = 1} a^k. \tag{3.11}$$

Let $A(n, r, s)$ denote the sth number 'a' such that $(a, n)_r = 1$. It is obvious that

$$\varphi_r(n, A(n, r, s)) = s.$$

But from Corollary 1 we infer that

$$\varphi_r(n, A(n, r, s)) = \frac{A(n, r, s)}{n} \varphi_r(n) + 0(n^{\epsilon}) = s$$

so that

(3.12) $$A(n,r,s)=\frac{ns}{\varphi_r(n)}+\frac{n}{\varphi_r(n)}\cdot 0(n^{\varepsilon})\forall\varepsilon>0.$$

We deduce from Theorem 5 that for $r\geqq 0$ $\varphi_r(n)\geqq\varphi_1(n)=\varphi(n)$. Since it is known that $n/\varphi_1(n)=0(\log\log n)$ we infer that

$$\frac{n}{\varphi_r(n)}=0(\log\log n)$$

so that (3:12) is rewritten as

(3.13) $$A(n,r,s)=\frac{ns}{\varphi_r(n)}+0(n^{\varepsilon})\forall>0.$$

Thus

$$\sum_{\substack{0<a\leqq x\\(a,n)_r=1}}a^k=\sum_{0<s\leqq\varphi_r(n,x)}A(n,r,s)^k=\sum_{0<s\leqq\varphi_r(n,x)}\left(\frac{ns}{\varphi_r(n)}+0(n^{\varepsilon})\right)^k\forall\varepsilon>0$$

$$=\left\{\sum_{0<s\leqq\varphi_r(n,x)}s^k\right\}\frac{n^k}{\varphi_r(n)^k}+\frac{n^{k-1}}{\varphi_r(n)^{k-1}}\sum_{0<s\leqq\varphi_r(n,x)}0(s^{k-1}n^{\varepsilon})\forall\varepsilon>0$$

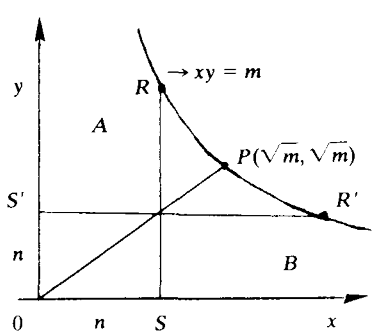
(3.14) $$=\frac{n^k}{\varphi_r(n)^k}\left(\frac{\varphi_r(n,x)^{k+1}}{k+1}+0(\varphi_r(n,x)^k)\right)+0\left(\frac{n^{k-1+\varepsilon}}{\varphi_r(n)^{k-1}}\varphi_r(n,x)^k\right)$$

$$=\frac{n^k}{\varphi_r(n)^k}\left(\frac{x^{k+1}\varphi_r(n)^{k+1}}{(k+1)n^{k+1}}+0(x^{k+\varepsilon})\right)+0(n^{\varepsilon}\varphi_r(n,x)^k)$$

$$=\frac{x^{k+1}\varphi_r(n)}{(k+1)n}+0(x^{k+\varepsilon})\forall\varepsilon>0$$

where x is taken as $\geqq n$.

We shall return to (3.14) after making a geometric interpretation of $\Sigma\,\sigma^*_{r,k}$. Consider the lattice points discussed in Theorem 8. Call these lattice points 'good' and let G denote the set of good lattice points. Divide the region under the curve into three non-intersecting regions A, OP, and B. Clearly



$$\sum_{n=1}^{m} \sigma^*_{r,k}(n) = \sum_{(x_0, y_0)\in G} y_0^k$$

which can be split up as

$$\sum_{n=1}^{m} \sigma^*_{r,k}(n) = \sum_{(x_0, y_0)\in G\cap A} y_0^k + \sum_{(x_0, y_0)\in G\cap B} y_0^k + \sum_{(x_0, y_0)\in G\cap(OP)} y_0^k$$

(3.15)

$$= S_1 + S_2 + S_3 \text{ say.}$$

Clearly

$$S_3 = 0\,(m^{(k+1)/2}).$$

To estimate S_2, pick a point S' on OY at a distance n from 0 with $n \leqq \sqrt{m}$. The sum of $y_0^k = n^k$ over $R'S'$ through S' is

$$\sum_{n<x_0\leqq m/n;\,(n,x_0)_{r-1}=1} n^k = n^k \varphi^*_{r-1}(n, n, m/n)$$

where

$$\varphi^*_{r-1}(n, c, d) = \sum_{c<a\leqq d,\,(n,a)_{r-1}=1} 1.$$

Thus we have

$$S_2 = \sum_{n=1}^{[\sqrt{m}]} n^k \varphi^*_{r-1}(n, n, m/n) = \sum_{n=1}^{[\sqrt{m}]} n^k 0(m/n) = 0(m^{(k+2)/2})$$

$$= 0(m^{k+1/2}) \quad \text{for} \quad k \geqq 1.$$

To estimate S_1 pick an S on OX and a distance n from 0, $n \leqq \sqrt{m}$. Draw RS through it. The sum of y_0^k over y_0 lying on RS is

$$\sum_{\substack{n<y_0\leqq m/n\\(y_0,n)_{r-1}=1}} y_0^k = \frac{m^{k+1}\varphi_{r-1}(n)}{(k+1)n^{k+2}} + 0\left(\frac{m^{k+\varepsilon}}{n^{k+\varepsilon}}\right) - \frac{n^k\varphi_{r-1}(n)}{k+1} + 0(n^{k+\varepsilon})$$

using (3.14) where x takes values n and $m/n \geqq n$. If we sum from 1 to $\sqrt{m}$ we get S_1 which is

$$S_1 = \frac{m^{k+1}}{k+1}\sum_{n=1}^{[\sqrt{m}]} \frac{\varphi_{r-1}(n)}{n^{k+2}} + 0(m^{k+\varepsilon}) + 0(m^{(k+2)/2})$$

$$= \frac{m^{k+1}}{k+1}\left\{\sum_{n=1}^{\infty} \frac{\varphi_{r-1}(n)}{n^{k+2}} - \sum_{n=[\sqrt{m}]+1}^{\infty} \frac{\varphi_{r-1}(n)}{n^{k+2}}\right\} + 0(m^{k+1/2})$$

$$= \alpha_{r,k} m^{k+1} + m^{(k+1)/2}\left\{m^{(k+1)/2} \sum_{n>\sqrt{m}} 0\left(\frac{1}{n^{k+1}}\right)\right\} + 0(m^{k+1/2})$$

$$= \alpha_{r,k} m^{k+1} + 0(m^{(k+2)/2}) + 0(m^{k+1/2}) = \alpha_{r,k} m^{k+1} + 0(m^{k+1/2}).$$

If we substitute these estimates of S_1, S_2 and S_3 in (3.15) we get part b of Theorem 9. The proof of part a is similar with the following changes. We have to replace $\varphi_{r-1}(n)/n$ by $\varphi^*_{r-1}(n, \beta_{r-1}(n))/\beta_{r-1}(n)$ and use Corollary 2 instead of Corollary 1 to get an estimate similar to (3.14). The proof is complete.

We deduce a few corollaries to our theorem.

COROLLARY 5. *If $\sigma(n)$ denotes the sum of the divisors of n then*

$$\sum_{n=1}^{m} \sigma(n) \sim \frac{\pi^2}{12} m^2.$$

COROLLARY 6. *If $\sigma_{r,k}(n)$ denotes the sum of the kth powers of the divisors of n then*

$$\sum_{n=1}^{m} \sigma_{1,k}(n) = \frac{\zeta(k+1)}{k+1} m^{k+1} + 0(m^{k+1/2})$$

where ζ is Riemann's ζ function.

COROLLARY 7. *If $\sigma_{2,1}(n)$ denotes the sum of the unitary divisors of n then*

$$\sum_{n=1}^{m} \sigma_{2,1}(n) \sim \frac{\pi^2 m^2}{12\zeta(3)} + 0(m^{3/2}).$$

PROOFS. Corollary 5 follows from Theorem 9 if we estimate α_{11}. Clearly

$$\alpha_{11} = \alpha^*_{11} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12}.$$

Corollary 6 follows if we find $\alpha_{1,k}$ which is $\zeta(k+1)/k+1$. Corollary 7 comes out of an estimate of $\sigma_{2,1}$

$$\sigma_{2,1} = \sigma^*_{2,1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\varphi_1(n)}{n^3} = \frac{1}{2} \frac{\zeta(2)}{\zeta(3)} = \frac{\pi^2}{12\zeta(3)}$$

which is the result due to Cohen.

COROLLARY 8. *For any $k \geqq 1$ we have*

$$\alpha_{2,k} \leqq \alpha_{4,k} \leqq \alpha_{6,k} \leqq \cdots \leqq \alpha_{5,k} \leqq \alpha_{3,k} \leqq \alpha_{1,k}.$$

Also

$$c_1 \leqq c_3 \leqq c_5 \leqq \cdots \leqq c_6 \leqq c_4 \leqq c_2.$$

These follow directly from Theorem 5.

4. Error functions

We finally take up a discussion of error functions associated with the Euler functions. (A similar discussion for $r = 1$ is made in Alladi (1974).)

We first calculate the average value of $e_r(n, x)$ and $e_r^*(n, x)$ for fixed n when x is discrete.

THEOREM 10.

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m} e_r(n, i) = \frac{-\varphi_r(n)}{2n}$$

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m} e_r^*(n, i) = \frac{-\varphi_r^*(n, \beta_r(n))}{2\beta_r(n)}.$$

PROOF. From Theorem 2 we deduce

$$e_r(n, i) + e_r(n, n-i) = \begin{cases} 0 & \text{if } (i, n)_r \neq 1 \\ -1 & \text{if } (i, n)_r = 1 \end{cases}$$

so that we get

$$\sum_{i=1}^{n} e_r(n, i) = -\varphi_r(n)/2.$$

Now Theorem 1 says

$$e_r(n, \lambda n + i) = \frac{\lambda n + i}{n}\varphi_r(n, \lambda n + i) = \frac{\lambda n + i}{n}\varphi_r(n) - \lambda\varphi_r(n) - \varphi_r(n, i)$$
$$= e_r(n, i).$$

Let $m = \lambda n + \mu$ for some non-negative λ, where $0 \leqq \mu < n$. Clearly

$$\frac{1}{m}\sum_{i=1}^{m} e(n, i) = \frac{1}{m}\sum_{i=1}^{n} e_r(n, i) + \cdots \frac{1}{m}\sum_{i=(\lambda-1)n+1}^{\lambda n} e_r(n, i)$$
$$+ \frac{1}{m}\sum_{i=\lambda n+1}^{\lambda n+\mu} e_r(n, i)$$
$$= \frac{-\lambda\varphi_r(n)}{2m} + \frac{1}{m}\sum_{i=1}^{\mu} 0(n^{\epsilon}) = \frac{-\varphi_r(n)}{2n} + 0\left(\frac{1}{m}\right).$$

So that proceeding to the limit $m \to \infty$ we get the first part of the theorem The second part follows on similar reasoning.

However the mean over the continuous variable vanishes. To be more precise

$$\int_0^n e_r(n, x) = 0; \quad \int_0^{\beta_r(n)} e_r^*(n, x)dx = 0.$$

The above statement is an immediate consequence of the following statement:

If f is Riemann integrable in $[0, m]$ and $f(x)+f(m-x)=0$, for all but a finite number of $x \in [0, m]$ then $\int_0^m f(x)dx = 0$. Clearly

$$\int_0^m f(x)dx = \int_0^m f(m-x)dx = \frac{1}{2}\int_0^1 f(x)+f(m-x)dx = 0.$$

Note that $e_r(n, x)+e_r(n, n-x)=0$ for all x except when $(x, n)_r = 1$ and $e_r^*(n, x)+e_r^*(n, \beta_r(n)-x)=0$ except when $(n, x)_r = 1$.

We now study the properties of the additive error functions associated with φ_r and φ_r^*. Define for $s \geqq 2$

$$e_r(n, \alpha_1, \alpha_2, \cdots, \alpha_s) = \varphi_r\left(n, \sum_{i=1}^{s} \alpha_i\right) - \sum_{i=1}^{s} \varphi_r(n, \alpha_i)$$

and

$$e_r^*(n, \alpha_1, \alpha_2, \cdots, \alpha_s) = \varphi_r^*\left(n, \sum_{i=1}^{s} \alpha_i\right) - \sum_{i=1}^{s} \varphi_r^*(n, \alpha_i).$$

We begin by showing

THEOREM 11.

(a)
$$\lim_{m\to\infty} \sum_{n=1}^{m} e_r(n, \alpha_1, \alpha_2, \cdots, \alpha_s) = \sum_{n=1}^{\alpha_1+\alpha_2+\cdots+\alpha_s} \frac{\varphi_r^*(n, \beta_r(n))}{\beta_r(n)} - \sum_{i=1}^{s}\sum_{n=1}^{\alpha_i} \frac{\varphi_r^*(n, \beta_r(n))}{\beta_r(n)}$$

and

(b)
$$\lim_{m\to\infty} \frac{1}{m}\sum_{n=1}^{m} e_r^*(n, \alpha_1, \alpha_2, \cdots, \alpha_s) = \sum_{n=1}^{\alpha_1+\alpha_2+\cdots+\alpha_s} \frac{\varphi_r(n)}{n} - \sum_{i=1}^{s}\sum_{n=1}^{\alpha_i} \frac{\varphi_r(n)}{n}.$$

PROOF. We only prove the first part. The proof of (b) is similar. We know

$$(4.1)\quad \frac{1}{m}\sum_{n=1}^{m} e_r(n, \alpha_1, \alpha_2, \cdots, \alpha_s) = \frac{1}{m}\sum_{n=1}^{m} \varphi_r\left(n, \sum_{i=1}^{s}\alpha_i\right) - \frac{1}{m}\sum_{n=1}^{m}\sum_{i=1}^{s} \varphi_r(n, \alpha_i).$$

For any integer j we have

$$(4.2)\quad \begin{aligned} \sum_{n=1}^{m} \varphi_r(n, j) &= \sum_{n=1}^{m} \sum_{\substack{i=1 \\ (i,n)_r=1}}^{j} 1 = \sum_{n=1}^{j} \sum_{\substack{i=1 \\ (n,i)_r=1}}^{m} 1 = \sum_{n=1}^{j} \varphi_r^*(n, m) \\ &= m\sum_{n=1}^{j} \frac{\varphi_r^*(n, \beta_r(n))}{\beta_r(n)} + 0(1). \end{aligned}$$

This implies that

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$$(4.3)\qquad \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\varphi_r(n,j)=\sum_{n=1}^{j}\frac{\varphi_r^*(n,\beta_r(n))}{\beta_r(n)}.$$

If in (4.3) we set j as $\Sigma\,\alpha_i$, and as α_i and then use (4.1) and proceed to the limit $m\to\infty$ we get Theorem 11 part (a). Part (b) follows by observing that

$$(4.4)\qquad \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\varphi_r^*(n,j)=\sum_{n=1}^{j}\frac{\varphi_r(n)}{n}.$$

Note that the right hand side of (b) and (a) are of the form

$$g_r\left(\sum_{i=1}^{s}\alpha_i\right)-\sum_{i=1}^{s}g_r(\alpha_i)$$

and

$$g_r^*\left(\sum_{i=1}^{s}\alpha_i\right)-\sum_{i=1}^{s}g_r^*(\alpha_i)$$

which resembles remarkably the forms of $e_r(n,\alpha_1,\alpha_2,\cdots,\alpha_s)$ and $e_r^*(n,\alpha_1,\alpha_2,\cdots,\alpha_s)$.

In fact as (4.3) and (4.4) are true the following can be shown without too much trouble.

$$(4.5)\qquad \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}e_r(n,i)=2c_ri-\sum_{n=1}^{i}\frac{\varphi_r^*(n,\beta_r(n))}{\beta_r(n)}=o(i)$$

and

$$(4.6)\qquad \lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}e_r^*(n,i)=2c_ri-\sum_{n=1}^{i}\frac{\varphi_r(n)}{n}=o(i)$$

by virtue of Theorem 7. Compare (4.5) and (4.6) with Theorem 10.

We conclude by proving a necessary and sufficient condition for a number n to be a power of a prime using $e_r(n,\alpha_1,\alpha_2)$.

THEOREM 12. *A necessary and sufficient condition for n to be a power of a prime is that*

$$(4.7)\qquad e_r(n,\alpha_1,\alpha_2)\leqq 0\,\forall\alpha_1,\alpha_2\in Z^+=\{1,2,3,\cdots\}.$$

PROOF. The necessity part is easy to establish. We know that

$$\varphi_r(n,\alpha_1+\alpha_2)=\alpha_1+\alpha_2-\left[\frac{\alpha_1+\alpha_2}{p^{f_r(m)}}\right]$$

$$\varphi_r(n,\alpha_1)=\alpha_1-\left[\frac{\alpha_1}{p^{f_r(m)}}\right];\quad \varphi_r(n,\alpha_2)=\alpha_2-\left[\frac{\alpha_2}{p^{f_r(m)}}\right]$$

where $n=p^m$, p prime. Now as $[x+y]\geqq[x]+[y]$ the necessity part follows directly.

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To prove sufficiency let (4.7) hold and let $n = \prod_{i=1}^{s} p_i^{\beta_i}$, $s > 1$. We shall get a contradiction. Consider the two numbers $p_i^{f_r(\beta_i)}$ and $p_j^{f_r(\beta_j)}$ for distinct i, j with $1 \leqq i < j \leqq s$. As these numbers are relatively prime, there exist integers x and y positive so that

$$|xp_i^{f_r(\beta_i)} - yp_j^{f_r(\beta_j)}| = 1. \tag{4.8}$$

Without loss of generality let $yp_j^{f_r(\beta_j)} > xp_i^{f_r(\beta_i)}$. Consider now an integer m satisfying

$$m \equiv 0 (\bmod\, p_1^{f_r(\beta_1)} \cdots p_s^{f_r(\beta_s)}) \tag{4.9}$$

and let

$$m' = \prod_{i=1}^{s} p_i^{f_r(\beta_i)}. \tag{4.10}$$

One can show that $(a, n)_r = 1$ if and only if

$$(\lambda m' + a, n)_r = (\lambda m' - a, n)_r = 1. \tag{4.11}$$

Now consider the intervals $(0, yp_j^{f_r(\beta_j)}]$ and $(m - 2, m + yp_j^{f_r(\beta_j)} - 2]$. It is evident from (4.11) that for every a with $0 < a \leqq yp_j^{f_r(\beta_j)} - 2$ and $(a, n)_r = 1$ we have equivalently an $m + a$ satisfying $m < m + a \leqq m + yp_j^{f_r(\beta_j)} - 2$ and $(m + a, n)_r = 1$. But by Theorem 1 neither $xp_i^{f_r(\beta_i)}$ or $yp_j^{f_r(\beta_j)}$ are prime to n order r. Yet as $(1, n)_r = 1$ we have $(m - 1, n)_r = 1$. Thus

$$\varphi_r(n, m - 2, m + yp_j^{f_r(\beta_j)} - 2) = \varphi_r(n, yp_j^{f_r(\beta_j)}) + 1 \tag{4.12}$$

which is the same as saying

$$e_r(n, \alpha_1, \alpha_2) = 1 > 0$$

if we set $\alpha_1 = m - 2$ and $\alpha_2 = p_j^{f_r(\beta_j)}$ in (4.12), a contradiction to our assumption (4.7). Thus $s = 1$ which establishes sufficiency. The proof is complete.

Acknowledgement

I am thankful to the referee for his comments and suggestions.

I also thank Professors E. G. Straus, H. G. Niederreiter and Ian Richards for their criticisms and for stimulating discussions.

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