Solutions with prescribed mass for L^2-supercritical NLS equations under Neumann boundary conditions

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In this article, we investigate the following non-linear Schrödinger (NLS) equation with Neumann boundary conditions:

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

coupled with a constraint condition:

$$\int_\Omega |u|^2 dx = c,$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ denotes a smooth bounded domain, ν represents the unit outer normal vector to $\partial\Omega$, c is a positive constant, and λ acts as a Lagrange multiplier. When the non-linearity f exhibits a general mass supercritical growth at

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infinity, we establish the existence of normalized solutions, which are not necessarily positive solutions and can be characterized as mountain pass type critical points of the associated constraint functional. Our approach provides a uniform treatment of various non-linearities, including cases such as $f(u) = |u|^{p-2}u$, $|u|^{q-2}u + |u|^{p-2}u$, and $-|u|^{q-2}u + |u|^{p-2}u$, where $2 < q < 2 + \frac{4}{N} < p < 2^*$. The result is obtained through a combination of a minimax principle with Morse index information for constrained functionals and a novel blow-up analysis for the NLS equation under Neumann boundary conditions.

1. Introduction and main results

This article is concerned with the following Neumann problem for non-linear Schrödinger (NLS) equation:

$$\begin{cases} -\Delta u + \lambda u = f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with a mass constraint

$$\int_\Omega |u|^2 dx = c, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $c > 0$ is a given constant, ν stands for the unit outer normal to $\partial\Omega$, and λ serves as a Lagrange multiplier.

The features of problem (1.1)–(1.2) are the following:

   (i) The presence of the Neumann boundary condition is quite rare in the analysis of solutions with prescribed mass.
   (ii) The reaction exhibits a general mass supercritical growth at infinity.
   (iii) The proof relies on powerful techniques, such as Morse theory and a new blow-up analysis for the NLS equation.
   (iv) The analysis presented in this article can be extended to other classes of stationary problems, including biharmonic elliptic equations and Schrödinger–Poisson equations.

The analysis of solutions with prescribed mass is particularly significant from a physical point of view, in relationship with phenomena arising in non-linear optics, the theory of water waves, etc. Indeed, solutions with prescribed L^2-norm are especially relevant since this quantity is preserved along the time evolution. Moreover, the variational characterization of such solutions is often a strong help to analyse their orbital stability and instability properties, see [15, 33, 34, 54, 55].

The investigation of non-linear Neumann problem (1.1) finds applications in various fields. One of the main motivations stems from the analysis of standing

waves in the form $\psi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{R}$, for the time-dependent NLS equation:

$$i\partial_t \psi + \Delta\psi + g(|\psi|)\psi = 0, \quad (t,x) \in \mathbb{R} \times \Omega \qquad (1.3)$$

subject to Neumann boundary conditions. The model (1.3) plays an important role in non-linear optics and Bose–Einstein condensates (see [3, 5, 23, 24, 40]). It is evident that $\psi(t,x)$ is a solution to (1.3) if and only (u,λ) constitutes a coupled solution to (1.1) with $f(u) = g(|u|)u$. We study Eq. (1.1) by searching for solutions satisfying the mass constraint $\int_\Omega |u|^2 dx = c$. In this context, λ remains unknown and emerges as a Lagrange multiplier, a feature deemed meaningful from a physical standpoint due to mass conservation. The solutions under a L^2 constraint are commonly referred to as normalized solutions. These normalized solutions to (1.1) can be obtained as critical points of the energy functional $J : H^1(\Omega) \to \mathbb{R}$ defined by:

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(u)dx$$

on the L^2-sphere constraint:

$$\mathcal{S}_c := \left\{ u \in H^1(\Omega) : \int_\Omega |u|^2 dx = c \right\},$$

where $F(u) = \int_0^u f(t)dt$.

When $\Omega = \mathbb{R}^N$, problem (1.1)–(1.2) is transformed into

$$\begin{cases} -\Delta u + \lambda u = f(u) \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c. \end{cases} \qquad (1.4)$$

Extensive studies have been conducted in recent years for this problem, particularly when considering non-linearities satisfying $f(u) \sim |u|^{p-2}u$ as $|u| \to +\infty$, where $p \in (2_*, 2^*)$ with $2_* := 2 + \frac{4}{N}$ and $2^* := \frac{2N}{N-2}$. Within this L^2-supercritical range, the corresponding energy functional is unbounded from below on the constraint set $\mathcal{S}_c$. The pioneering work in this direction was carried out by Jeanjean [31], where a scaled functional and mountain pass arguments were introduced to address the L^2-supercritical problems. Bartsch and Soave [6] developed a natural constraint approach to investigate the L^2-supercritical NLS equations and systems on $\mathbb{R}^N$. For more related results on $\mathbb{R}^N$, we refer to [4, 7, 30, 33, 34, 56] and the associated references.

The exploration of normalized solutions for the NLS on bounded domains was initialled in [46]. When $f(u) = |u|^{p-2}u$ with p being L^2-supercritical but Sobolev subcritical, Noris et al. [46] demonstrated the existence of a positive normalized solution on a unit ball with Dirichlet boundary conditions. The case for general bounded domains was addressed in [49]. For NLS systems, one can refer to [47].

In [48], Pellacci et al. investigated normalized solutions of the NLS under both Dirichlet and Neumann boundary conditions, focusing on the concentration of solutions at specific points of Ω as the prescribed mass c varies. Specifically, for the

Neumann problem (1.1)–(1.2) with $f(u) = u^{p-1}$ for $p \in (2_*, 2^*)$, they employed the Lyapunov–Schmidt reduction method to prove the existence of positive normalized solutions when $c \in (0, c_0)$ for some $c_0 > 0$. The solutions concentrate at a point $\xi_0 \in \overline{\Omega}$ as $c \to 0$, where ξ_0 is either a non-degenerate critical point of the mean curvature H of the boundary $\partial\Omega$ or the maximum point of the distance function from $\partial\Omega$. However, the nature of these solutions—whether they correspond to local minimizers or mountain pass-type critical points of the associated functional—remains unclear.

Additionally, it is noteworthy that earlier works on L^2-subcritical Schrödinger–Poisson type systems under Neumann boundary conditions can be found in [2, 50, 51], where the authors established the existence of infinitely many normalized solutions using the Ljusternik–Schnirelmann theory.

The methodologies used for bounded domains differ significantly from those utilized in the entire space. In fact, the approaches for $\mathbb{R}^N$ heavily rely on the scaling transform $(t \star u)(x) = t^{\frac{N}{2}} u(tx)$ and the associated Pohozaev identity. However, due to the lack of invariance under translations and dilation, as well as the emergence of uncontrollable boundary terms in the Pohozaev identity (particularly for non-convex domains), neither of these techniques is applicable to general bounded domains.

The normalized solutions are also explored within the framework of ergodic mean field game (MFG) systems, offering another key motivation for studying Eq. (1.1). MFGs were introduced in influential works by Huang, Caines, and Malham'e [29] and Lasry and Lions [37], independently. The primary objective of MFG is to establish a framework for characterizing Nash equilibria in differential games involving an infinite number of agents that are indistinguishable from one another. For more details, we refer interested readers to [18, 19, 53] and the references therein. From a mathematical perspective, such equilibria can be characterized by an elliptic system that combines a Kolmogorov equation and a Hamilton–Jacobi–Bellman equation. Moreover, this system has to satisfy normalization in $L^1(\Omega)$ as follows:

$$\begin{cases} -\Delta v + H(\nabla v) + \lambda = h(m(x)) & \text{in } \Omega, \\ -\Delta m - div(m\nabla H(\nabla v)) = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0, \quad \dfrac{\partial m}{\partial \nu} + m\nabla H(\nabla v) \cdot \nu = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega m dx = 1, \quad \int_\Omega v dx = 0. \end{cases} \tag{1.5}$$

The Neumann boundary conditions are based on the assumption that agents' trajectories are restricted to Ω by bouncing off the boundary in a normal direction. For the quadratic Hamilton case $H(\nabla v) = |\nabla v|^2$, by using a Hopf–Cole transformation $\phi = e^{-v} / \int e^{-v} = \sqrt{m}$, (1.5) is reduced to

$$\begin{cases} -\Delta\phi = \lambda\phi - h(\phi^2)\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega |\phi|^2 dx = 1, \end{cases} \tag{1.6}$$

which can be viewed as a single NLS equation with prescribed mass. In fact, we can check that (1.6) is equivalent to (1.1) by taking a simple transformation $\phi = \frac{1}{\sqrt{c}}u$ and $f(u) = h(\frac{u^2}{c})u$. In [19], Cirant et al. investigated the existence of the viscous ergodic MFG system with Neumann boundary conditions. They proved the existence of global minimizers in L^2-subcritical and critical cases, as well as local minimizers in L^2-supercritical cases.

When f satisfies the L^2-supercritical growth, the constraint functional $J|_{\mathcal{S}_c}$ exhibits a mountain pass geometry structure. Hence, it is natural to seek mountain pass type normalized solutions to the Neumann problem (1.1)–(1.2). However, in this scenario, the functional J becomes unbounded from below on $\mathcal{S}_c$, rendering the method in [19] ineffective. To the best of our knowledge, there are currently no references addressing this issue. The primary aim of this article is to develop a novel variational technique to explore the existence of mountain pass type normalized solutions for (1.1)–(1.2) under the L^2-supercritical growth conditions.

Furthermore, notice that the current existence results of normalized solutions for (1.4) mainly rely on the following global Ambrosetti–Rabinowitz type condition

$$0 < \alpha F(t) \le f(t)t \text{ for } t \ne 0, \text{ where } \alpha \in (2_*, 2^*), \tag{1.7}$$

(see [6, 30, 31]) or combined non-linearities (see [4, 7, 33, 34, 54–56]). Recently, normalized solutions of (1.4) have also been studied when (1.7) is replaced by certain global monotonicity conditions, as discussed in [10, 17, 35, 42]. However, the exploration of normalized solutions on bounded domains with more diverse non-linearities beyond power non-linearity is currently limited. Our method is designed to be flexible and applies to a broad range of non-linearities, including $f(u) = |u|^{p-2}u$, $|u|^{q-2}u + |u|^{p-2}u$, and $-|u|^{q-2}u + |u|^{p-2}u$, where $2 < q < 2 + \frac{4}{N} < p < 2^*$, in a unified manner. Our results do not require the presence of condition (1.7) or the global monotonicity conditions. Additionally, motivated by the arguments in [49], diverging from the prevalent focus on positive solutions in existing literature, we establish the existence of mountain pass type normalized solutions that are not necessarily positive.

Before stating our results, let us impose the following assumptions on f:

(f_1) $f \in C^1(\mathbb{R})$, $\lim\limits_{|t| \to 0} \frac{f(t)}{t} = 0$;

(f_2) there exist constants $p \in (2_*, 2^*)$ and $a_0 > 0$ such that

$$\lim_{|t| \to \infty} \frac{f(t)}{|t|^{p-2}t} = a_0;$$

(f_3) there exist constants $\mu \ge a_0(p-1)$ and $M > 0$ such that

$$\mu|t|^{p-2} \le f'(t), \ \forall |t| \ge M.$$

Our main result is the following theorem.

THEOREM 1.1. *Suppose (f_1)–(f_3) hold. Then there exists a constant $c^* > 0$ such that for any $0 < c < c^*$, problem (1.1)–(1.2) has a normalized solution pair $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$ of mountain pass type.*

REMARK 1.2. If $f(t)t > 0$ for $t \neq 0$, it allows us to establish the positivity of the Lagrange multiplier λ. In fact, under this condition, we can obtain a positive normalized solution u of (1.1). Subsequently, integrating Eq. (1.1) and using the Neumann boundary value condition, we can deduce that $\lambda > 0$.

A significant challenge in proving theorem 1.1 arises from establishing the boundedness of Palais–Smale sequences. To overcome this obstacle, we combine with a parameterized minimax principle with Morse index information for constrained functionals established recently in [13] (see also [12, 16]) and a new blow-up analysis for the NLS equation under Neumann boundary conditions. In applying this method, the sign condition $\int_\Omega F(u) \geq 0$ for all $u \in H^1(\Omega)$ is crucial to ensure the applicability of the monotonicity trick (see theorem 3.4 in §3). However, in our specific problem, $\int_\Omega F(u)$ is not necessarily non-negative when u is not sufficiently large. Our strategy to tackle this challenge is to utilize a cut-off function and decompose f into two parts, ensuring that one of the parts satisfies this sign condition and allows for the application of the monotonicity trick.

To address the original Neumann problem after deriving solutions for approximation problems, we will propose an innovative blow-up analysis tailored for the NLS subject to Neumann boundary conditions. In contrast to the blow-up arguments applied in the Dirichlet case (see [20, 49]), the blow-up analysis for NLS with Neumann boundary conditions, subject to the L^2-constraint, proves to be considerably more intricate. In fact, in Neumann problems, the solution is not necessarily required to vanish at the boundary $\partial\Omega$, and local extremum points of solutions can exist on the boundary. This is in stark contrast to Dirichlet problems where such extremum points are found in the interior of Ω.

We note that several studies have conducted blow-up analyses of the NLS under Neumann boundary conditions. It is well known that combining blow-up arguments with suitable Liouville-type theorems is highly effective for deriving a priori estimates. In [39], Lin et al. delivered a comprehensive blow-up analysis to establish the boundedness for positive solutions to the following Neumann problem:

$$\begin{cases} -d\Delta u + u = g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $d > 0$ and g satisfies Sobolev subcritical growth conditions. The Sobolev critical case was subsequently addressed by Adimurthi et al. [1]. In [28], Hu et al. used a blow-up argument for a non-linear Neumann problem involving the p-Laplacian to obtain a priori estimates. For further details on blow-up analyses of elliptic problems under Neumann boundary conditions, the reader is referred to [11, 36].

Blow-up analysis can also establish a connection between the boundedness of solutions and their Morse indices. Consider the following problem:

$$\begin{cases} -\Delta u = g(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where g satisfies superlinear and subcritical growth conditions. Harrabi et al. [27] proved that the L^∞ bounds for solutions to (1.9) are equivalent to bounds on their Morse indices.

In contrast to the arguments presented in [27], our discussion is conducted under a mass constraint and involves a family of non-linearities arising from varying Lagrange multipliers, along with a dense set used in the monotonicity trick. The most significant distinction, however, is that our blow-up analysis requires an exponential decay estimate for the solution sequence. This difference necessitates substantial modifications to the blow-up arguments in the Neumann case.

As we discussed earlier, we present a compelling argument to establish that the solution sequence not only exhibits the commonly observed exponential decay away from the boundary $\partial\Omega$, as found in the fixed frequency NLS in the Neumann case (see [43, 44]), but more significantly, it demonstrates uniform exponential decay away from the blow-up points.

To conduct the blow-up analysis and subsequently derive a contradiction, a critical step involves demonstrating that the solution sequence exhibits uniform exponential decay, especially when the blow-up points are situated on the boundary $\partial\Omega$. However, in such cases, we can typically only establish that the solution sequence exponentially decays uniformly away from the blow-up points within a smaller domain in Ω, which is distant from the boundary. Specifically, as detailed in §4, through a comparison argument, we can only provide an estimation that the solution sequence exponentially decays uniformly away from the blow-up points in the domain $\tilde{\Omega}_\theta \backslash \cup_{i=1}^{k} (B_{R\lambda_n^{-\frac{1}{2}}}(P_n^i) \cap \tilde{\Omega}_\theta)$ (Ω_θ is defined in §4, $\tilde{\Omega}_\theta := \overline{\Omega} \backslash \Omega_\theta$ and $k \in \{1, 2\}$). To be precise, it is demonstrated that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$u_n(x) \le C_1 e^{C_1 R} \lambda_n^{\frac{1}{p-2}} \sum_{i=1}^{k} e^{-C_2 \lambda_n^{\frac{1}{2}} |x - P_n^i|}, \quad \forall x \in \tilde{\Omega}_\theta \backslash \cup_{i=1}^{k} (B_{R\lambda_n^{-\frac{1}{2}}}(P_n^i) \cap \tilde{\Omega}_\theta),$$

where u_n satisfies (4.1), $R > 0$ is a constant, P_n^i is a local extremum point, $i \in \{1, \ldots, k\}$. However, this conclusion falls short of addressing our problem adequately. The limitation arises from the challenging nature of estimating the exponential decay of the solution in the neighbourhood near the boundary $\partial\Omega$.

To overcome this limitation, we formulate a diffeomorphism, denoted as $\Phi : \Omega^\theta \to \Omega_\theta$, in a manner that expands the equation to encompass a new domain, namely, $\overline{\Omega} \cup \Omega^\theta$, which combines the original domain Ω with an adjoining tubular neighbourhood Ω^θ (see §4). Employing the comparison theorem, we deduce that the solution exhibits exponential decay away from the boundary of the newly defined domain $\overline{\Omega} \cup \Omega^\theta$. Subsequently, through a similar argument as mentioned earlier, we observe that the solution exponentially decays uniformly away from the blow-up points in $\overline{\Omega} \backslash \cup_{i=1}^{k} (B_{R\lambda_n^{-1/2}}(P_n^i) \cap \overline{\Omega})$. This establishes the intended conclusion we strive to attain.

This article is organized as follows. In §2, we provide the mountain pass geometry of the parameterized functionals J_ρ uniformly for $\rho \in [\frac{1}{2}, 1]$. Subsequently, in §3, we show the existence of a mountain pass critical point u_ρ with Morse index information for $J_\rho|_{\mathcal{S}_c}$ for almost every $\rho \in [\frac{1}{2}, 1]$. In §4, we perform a blow-up analysis

of solutions with bounded Morse index for the Neumann problem. Finally, in §5, we prove theorem 1.1. For convenience, we denote by $\|\cdot\|_r$ the norm of the spaces $L^r(\Omega)(1 \le r \le +\infty)$, by $\|\cdot\|$ the norm in $H^1(\Omega)$.

2. Mountain pass geometry

This section is dedicated to establishing the uniform mountain pass geometry for a parameterized functional. To achieve this goal, we first decompose f into two parts. By (f_1)–(f_2), we deduce that there exists $R_0 > 0$ such that $f(t)t > 0$ for $|t| \ge R_0$. Utilizing this observation, we define

$$f_1(t) := \eta(t)f(t) \quad \text{and} \quad f_2(t) := (1 - \eta(t))f(t),$$

where η is a smooth cut-off function such that

$$\eta(t) = \begin{cases} 1, & |t| \ge R_0 + 1, \\ 0, & |t| < R_0 \end{cases}$$

with $|\eta'(t)| \le 2$ for $R_0 < |t| < R_0 + 1$. Clearly, $f_1(t)t \ge 0$ for all $t \ne 0$.

Define $J_\rho : H^1(\Omega) \to \mathbb{R}$ by

$$J_\rho(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F_2(u)dx - \rho \int_\Omega F_1(u)dx, \ u \in H^1(\Omega), \ \rho \in \left[\frac{1}{2}, 1\right], \quad (2.1)$$

where $F_1(u) = \int_0^u f_1(t)dt$ and $F_2(u) = \int_0^u f_2(t)dt$.

We recall the Gagliardo–Nirenberg inequality (see [45]): for every $N \ge 3$ and $p \in (2, 2^*)$, there exists a constant $C_{N,r,\Omega}$ depending on N, r and Ω such that

$$\|u\|_r \le C_{N,r,\Omega} \|u\|_2^{1-\gamma_r} \|u\|_{H^1(\Omega)}^{\gamma_r}, \quad \forall u \in H^1(\Omega),$$

where

$$\gamma_r := \frac{N(r-2)}{2r}.$$

Define $\mathcal{B}_\alpha := \{u \in \mathcal{S}_c : \int_\Omega |\nabla u|^2 \le \alpha\}$, $\forall \alpha > 0$. We have the following result.

LEMMA 2.1. *Assume (f_1)–(f_3). Then there exists $c^* > 0$ such that for any $c \in (0, c^*)$, we can find $\alpha^* > 0$ such that*

$$\sup_{u \in \mathcal{B}_{\alpha^*}} J_{\frac{1}{2}}(u) < \inf_{u \in \partial\mathcal{B}_{2\alpha^*}} J_1(u).$$

Proof. By (f_1)–(f_2), for any $\epsilon, \delta > 0$ and $q \in (2, 2_*)$, there exist constants $C'_\epsilon, C_\delta > 0$ such that

$$F_2(u) \ge -\frac{\epsilon}{2}|u|^2 - \frac{C'_\epsilon}{q}|u|^q, \tag{2.2}$$

$$F(u) \le \frac{\delta}{2}|u|^2 + \frac{C_\delta}{p}|u|^p. \tag{2.3}$$

Let $u \in \mathcal{B}_\alpha$ and $v \in \partial\mathcal{B}_{2\alpha}$, where α is to be determined. By (2.2)–(2.3), along with the Gagliardo–Nirenberg inequality, we obtain

$$
\begin{aligned}
& J_1(v) - J_{\frac{1}{2}}(u) \\
& \geq \frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(v) + \int_\Omega F_2(u) \\
& \geq \frac{1}{2} \int_\Omega \left(|\nabla v|^2 - |\nabla u|^2 \right) dx - \frac{\delta + \epsilon}{2} c - \frac{C_\delta C_{N,p}^p}{p} c^{\frac{p(1-\gamma_p)}{2}} \|v\|^{p\gamma_p} - \frac{C'_\epsilon C_{N,q}^q}{q} \\
& \quad \times c^{\frac{q(1-\gamma_q)}{2}} \|u\|^{q\gamma_q} \\
& \geq \frac{\alpha}{2} - \frac{\delta + \epsilon}{2} c - \frac{C_\delta C_{N,p}^p}{p} c^{\frac{p(1-\gamma_p)}{2}} (2\alpha + c)^{\frac{p\gamma_p}{2}} - \frac{C'_\epsilon C_{N,q}^q}{q} c^{\frac{q(1-\gamma_q)}{2}} (\alpha + c)^{\frac{q\gamma_q}{2}}.
\end{aligned}
$$

Due to the arbitrariness of ϵ and δ, we conclude that for some $c^* > 0$ and any $c \in (0, c^*)$, taking $\alpha^* = 4c$, we have

$$
J_1(v) - J_{\frac{1}{2}}(u) \geq c - 9^{\frac{p\gamma_p}{2}} \frac{C_1 C_{N,p}^p}{p} c^{\frac{p}{2}} - 5^{\frac{q\gamma_q}{2}} \frac{C'_1 C_{N,q}^q}{q} c^{\frac{q}{2}} > 0.
$$

Hence, the conclusion follows. $\qquad\square$

For any $c \in (0, c^*)$, we define

$$
m_\rho := \inf_{u \in \mathcal{B}_{2\alpha^*}} J_\rho(u), \ \forall \rho \in \left[\frac{1}{2}, 1 \right].
$$

Then, following the approach outlined in [47, proposition 3.4] (see also [8, theorem 1.6]), we can derive the following result.

LEMMA 2.2. *Assume (f_1)–(f_3) and $c \in (0, c^*)$. Then m_ρ is achieved by some $u_\rho^* \in \mathcal{B}_{2\alpha^*} \setminus \partial\mathcal{B}_{2\alpha^*}$.*

Proof. Fix $c \in (0, c^*)$. By lemma 2.1, for any $\rho \in \left[\frac{1}{2}, 1 \right]$, we have

$$
\sup_{u \in \mathcal{B}_{\alpha^*}} J_\rho(u) \leq \sup_{u \in \mathcal{B}_{\alpha^*}} J_{\frac{1}{2}}(u) < \inf_{u \in \partial\mathcal{B}_{2\alpha^*}} J_1(u) \leq \inf_{u \in \partial\mathcal{B}_{2\alpha^*}} J_\rho(u).
$$

Let $\{u_{\rho,n}\} \subset \mathcal{B}_{2\alpha^*}$ be a minimizing sequence for J_ρ at the level m_ρ. Clearly, $\{u_{\rho,n}\}$ is bounded in $H^1(\Omega)$. Consequently, there exist a subsequence of $\{u_{\rho,n}\}$, still denoted by $\{u_{\rho,n}\}$, and some $u_\rho^* \in H^1(\Omega)$ such that $u_{\rho,n} \rightharpoonup u_\rho^*$ in $H^1(\Omega)$ and $u_{\rho,n} \to u_\rho^*$ strongly in $L^r(\Omega)$ for $r \in [1, 2^*)$. This implies that $u_\rho^* \in \mathcal{B}_{2\alpha^*}$ and thus $J_\rho(u_\rho^*) \geq m_\rho$.

By (f_1)–(f_2), together with the Hölder inequality and the Lebesgue dominated convergence theorem, we have

$$
\int_\Omega \left(F(u_{\rho,n}) - F(u_\rho^*) \right) dx = \int_\Omega F(u_{\rho,n} - u_\rho^*) dx + o_n(1) \to 0.
$$

Therefore, we obtain

$$
J_\rho(u_\rho^*) \leq \liminf_{n \to +\infty} J_\rho(u_{\rho,n}) = m_\rho.
$$

Combining this with the previous inequality, we deduce that $J_\rho(u_\rho^*) = m_\rho$ and $\|\nabla\left(u_{\rho,n} - u_\rho^*\right)\|_2^2 \to 0$. This implies that $u_{\rho,n} \to u_\rho^*$ in $H^1(\Omega)$ and hence u_ρ^* is a minimizer of J_ρ. Thus, the desired conclusion follows. $\qquad\square$

Now we can show the mountain pass geometry of J_ρ uniformly for $\rho \in [\frac{1}{2}, 1]$.

LEMMA 2.3. *Assume* (f_1)–(f_3). *Then, for any* $c \in (0, c^*)$, *there exist* $w_1, w_2 \in \mathcal{S}_c$ *such that*

$$c_\rho := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\rho(\gamma(t)) > \max\{J_\rho(w_1), J_\rho(w_2)\}, \quad \forall \rho \in \left[\frac{1}{2}, 1\right],$$

where

$$\Gamma := \{\gamma \in C([0,1], \mathcal{S}_c) : \gamma(0) = w_1, \ \gamma(1) = w_2\}. \tag{2.4}$$

Proof. Let $B_r(x)$ denote a ball in $\mathbb{R}^N$, centred at $x \in \mathbb{R}^N$ with radius $r > 0$. Take $\phi \in C_0^\infty(B_1(0))$ with $\phi > 0$ in $B_1(0)$ such that $\int_{B_1(0)} \phi^2 = 1$. For $n \in \mathbb{N}$, $x_0 \in \Omega$, define

$$\varphi_n(x) := c^{\frac{1}{2}} n^{\frac{N}{2}} \phi(n(x - x_0)), \ x \in \Omega.$$

We can verify that $\varphi_n \in \mathcal{S}_c$ and $supp(\varphi_n) \subset B_{\frac{1}{n}}(x_0) \subset \Omega$ for sufficiently large n.

By (f_1)–(f_2), taking $R_0 > 0$ larger if necessary, there exist constants $C_{R_0}, C'_{R_0}, C_p > 0$ such that

$$F_1(t) \geq \frac{C_p a_0}{2p}|t|^p, \quad \forall |t| \geq 2R_0, \quad F_1(t) \geq -C_{R_0}, \quad \forall |t| \leq 2R_0,$$
$$F_2(t) \geq -C'_{R_0}, \forall t \in \mathbb{R}.$$

Set $\Omega_{n,2R_0} := \{x \in \Omega : |\varphi_n| \leq 2R_0\}$. Then, for sufficiently large n such that $\max_{x\in\overline{\Omega}} |\varphi_n| > 2R_0$, we have for all $\rho \in [\frac{1}{2}, 1]$,

$$\begin{aligned}
J_\rho(\varphi_n) &\leq \frac{1}{2}\int_\Omega |\nabla\varphi_n(x)|^2 + (C_{R_0} + C'_{R_0})|\Omega| - \rho\int_{\Omega\setminus\Omega_{n,2R_0}} \frac{C_p a_0}{2p}|\varphi_n|^p \\
&= \frac{cn^2}{2}\int_{B_1(0)} |\nabla\phi(x)|^2 + (C_{R_0} + C'_{R_0})|\Omega| + \frac{2^p C_P a_0}{4p}R_0^p|\Omega| \\
&\quad - \frac{C_P a_0}{4p}c^{\frac{p}{2}}n^{\frac{pN-2N}{2}}\int_{B_1(0)} |\phi(x)|^p \to -\infty.
\end{aligned}$$

Hence, there exists $n_0 > 0$ sufficiently large such that

$$J_\rho(\varphi_{n_0}) < J_1(u_{\frac{1}{2}}^*) \leq J_\rho(u_{\frac{1}{2}}^*), \ \forall \rho \in \left[\frac{1}{2}, 1\right].$$

Choose $w_1 = u_{\frac{1}{2}}^*$ and $w_2 = \varphi_{n_0}$. Clearly, $u_{\frac{1}{2}}^* \in \mathcal{B}_{2\alpha^*} \setminus \partial\mathcal{B}_{2\alpha^*}, \varphi_{n_0} \notin \mathcal{B}_{2\alpha^*}$. By continuity, for any $\gamma \in \Gamma$, there exists $t_\gamma \in [0, 1]$ such that $\gamma(t_\gamma) \in \partial\mathcal{B}_{2\alpha^*}$. Thus, by lemma 2.1, it follows that

$$\max_{t\in[0,1]} J_\rho(\gamma(t)) \geq J_\rho(\gamma(t_\gamma)) \geq \inf_{u\in\partial\mathcal{B}_{2\alpha^*}} J_\rho(u) > \sup_{u\in\mathcal{B}_{\alpha^*}} J_{\frac{1}{2}}(u)$$

$$\geq J_\rho(u^*_{\frac{1}{2}}) = \max\{J_\rho(w_1), J_\rho(w_2)\}, \ \forall\rho \in \left[\frac{1}{2}, 1\right].$$

The proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3. Existence of MP solutions for a dense set

In this section, we establish the existence of a bounded Palais–Smale sequence at level c_ρ for almost every $\rho \in [\frac{1}{2}, 1]$. Our approach involves applying a recently developed min-max principle on the L^2-sphere, as detailed in [13]. This principle integrates the monotonicity trick presented in [32] with the min-max theorem enriched by second-order insights from Fang and Ghoussoub [21], which is also elaborated upon in [25, Chapter 11].

For a domain $D \subset \mathbb{R}^N$ and $\phi, u \in H^1(D)$, we consider

$$Q_{\lambda,\rho}(\phi; u; D) := \int_D |\nabla\phi|^2 dx + \lambda \int_D |\phi|^2 dx - \int_D f_2'(u)\phi^2 dx - \int_D \rho f_1'(u)\phi^2 dx,$$

where $\lambda \in \mathbb{R}, \rho \in [\frac{1}{2}, 1]$. The Morse index of u, denote by $m(u)$, is the maximum dimension of a subspace $W \subset H^1(D)$ such that $Q_{\lambda,\rho}(\phi; u; D) < 0$ for all $\phi \in W\backslash\{0\}$.

To state the abstract minimax theorem, we recall a general setting introduced in [9]. Let $(E, \langle\cdot,\cdot\rangle)$ and $(H, (\cdot,\cdot))$ be two infinite dimensional Hilbert spaces such that

$$E \hookrightarrow H \hookrightarrow E'$$

with continuous injections. The continuous injection $E \hookrightarrow H$ has a norm at most 1 and E is identified with its image in H. For $u \in E$, we denote $\|u\|^2 = \langle u, u\rangle$ and $|u|^2 = (u, u)$. For $a \in (0, +\infty)$, we define $S(a) := \{u \in E, |u|^2 = a\}$. We denote by $\|\cdot\|_*$ and $\|\cdot\|_{**}$, respectively, the operator norm of $\mathcal{L}(E, \mathbb{R})$ and of $\mathcal{L}(E, \mathcal{L}(E, \mathbb{R}))$.

DEFINITION 3.1. [13]   *Let* $\phi : E \to \mathbb{R}$ *be a* C²*-functional on* E *and* $\alpha \in (0, 1]$. *We say that* ϕ' *and* ϕ'' *are* α*-Hölder continuous on bounded sets if for any* R > 0 *one can find* $M = M(R) > 0$ *such that for any* $u_1, u_2 \in B(0, R)$:

$$\|\phi'(u_1) - \phi'(u_2)\|_* \leq M\|u_1 - u_2\|^\alpha, \quad \|\phi''(u_1) - \phi''(u_2)\|_{**} \leq M\|u_1 - u_2\|^\alpha. \ (3.1)$$

DEFINITION 3.2. [13]   *Let* ϕ *be a* C²*-functional on* E, *for any* $u \in E$ *define the continuous bilinear map:*

$$D^2\phi(u) := \phi''(u) - \frac{\phi'(u)\cdot u}{|u|^2}(\cdot, \cdot).$$

DEFINITION 3.3. [13]   *For any* $u \in S(a)$ *and* $\theta > 0$, *we define an approximate Morse index by*

$$\tilde{m}_\theta(u) := \sup\big\{dim\ L | L \text{ is a subspace of } T_u S(a) \text{ such that } D^2\phi(u)[\phi, \phi] < -\theta\|\phi\|^2,$$
$$\forall\varphi \in L\backslash\{0\}\big\}.$$

If u is a critical point for the constrained functional $\phi|_{S(a)}$ and $\theta = 0$, then $\tilde{m}_\theta(u)$ is the Morse index of u as a constrained critical point.

THEOREM 3.4. ([13], theorem 1.5). *Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form:*

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \rho \in I,$$

where $B(u) \geq 0$ for every $u \in E$, and

$$\text{either} \quad A(u) \to +\infty \quad \text{or} \quad B(u) \to +\infty \quad \text{as} \quad u \in E \quad \text{and} \quad \|u\| \to +\infty.$$

Suppose moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0, 1]$. Finally, suppose that there exist $w_1, w_2 \in S(a)$ (independent of ρ) such that, set

$$\Gamma := \{\gamma \in C([0, 1], S(a)) : \gamma(0) = w_1, \gamma(1) = w_2\},$$

we have

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\rho(\gamma(t)) > \max\{\Phi_\rho(w_1), \Phi_\rho(w_2)\}, \quad \forall \rho \in I. \qquad (3.2)$$

Then, for almost every $\rho \in I$, there exist sequence $\{u_n\} \subset S(a)$ and $\zeta_n \to 0^+$ such that, as $n \to \infty$,

(i) $\Phi_\rho(u_n) \to c_\rho$;
(ii) $\|\Phi'_\rho|_{S(a)}(u_n)\| \to 0$;
(iii) $\{u_n\}$ *is bounded in* E;
(iv) $\tilde{m}_{\zeta_n}(u) \leq 1$.

Define $h_\rho(t) = \rho f_1(t) + f_2(t)$ for $t \in \mathbb{R}$ and $\rho \in [\frac{1}{2}, 1]$. In the following, we obtain the main result of this section.

THEOREM 3.5. *Assume (f_1)–(f_3) and $c \in (0, c^*)$. Then, for almost every $\rho \in [\frac{1}{2}, 1]$, there exists a critical point u_ρ of J_ρ on S_c at level c_ρ, which solves the following problem*

$$\begin{cases} -\Delta u_\rho + \lambda_\rho u_\rho = h_\rho(u_\rho) & \text{in } \Omega, \\ \frac{\partial u_\rho}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \qquad (3.3)$$

for some $\lambda_\rho \in \mathbb{R}$. Moreover, the Morse index of $\{u_\rho\}$ satisfies $m(u_\rho) \leq 2$.

Proof. We will apply theorem 3.4 to the family of functionals J_ρ, where $E = H^1(\Omega)$, $H = L^2(\Omega)$, $S(a) = \mathcal{S}_c$ and Γ is defined by (2.4). Specifically, we set

$$A(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F_2(u)dx \quad \text{and} \quad B(u) = \int_\Omega F_1(u)dx.$$

Thus, we have $J_\rho(u) = A(u) - \rho B(u)$. Given that $u \in \mathcal{S}_c$ and considering the boundedness of $\int_\Omega F_2(u)dx$, we deduce that

$$A(u) \to \infty \quad \text{as} \quad \|u\| \to +\infty.$$

Moreover, by assumptions (f_1)–(f_2), it follows that J_ρ' and J_ρ'' are locally Hölder continuous on $\mathcal{S}_c$. By lemma 2.3, we can apply theorem 3.4 to produce a bounded Palais–Smale sequence $\{u_n\} \subset H^1(\Omega)$ for the constrained functional $J_\rho|_{\mathcal{S}_c}$ at level c_ρ for almost every $\rho \in [\frac{1}{2}, 1]$. Additionally, there exists a sequence $\zeta_n \to 0^+$ such that $\tilde{m}_{\zeta_n}(u_n) \le 1$.

Since $\|J_\rho'|_{\mathcal{S}_c}(u_n)\| \to 0$, and by the boundedness of $\{u_n\}$, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that for any $\varphi \in H^1(\Omega)$, we have

$$\int_\Omega \nabla u_n \nabla \varphi dx + \lambda_n \int_\Omega u_n \varphi dx - \int_\Omega h_\rho(u_n)\varphi dx = o(1). \tag{3.4}$$

This implies that

$$\int_\Omega |\nabla u_n|^2 dx + \lambda_n c - \int_\Omega h_\rho(u_n)u_n dx \to 0.$$

Using (f_1)–(f_2) again, we deduce that $\{\lambda_n\}$ is bounded. Therefore, up to a subsequence, we may assume that $\lambda_n \to \lambda_\rho \in \mathbb{R}$ and $u_n \rightharpoonup u_\rho$ weakly in $H^1(\Omega)$. By (3.4), we obtain

$$\int_\Omega \nabla u_\rho \nabla \varphi dx + \lambda_\rho \int_\Omega u_\rho \varphi dx - \int_\Omega h_\rho(u_\rho)\varphi dx = 0,$$

which implies that u_ρ weakly solves (3.3). By the compact embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for $r \in [1, 2^*)$ and standard arguments, we obtain that $u_n \to u_\rho$ strongly in $H^1(\Omega)$.

It remains to show that $m(u_\rho) \le 2$. Since $T_u\mathcal{S}_c$ has codimension 1, noting that $d^2|_{\mathcal{S}_c}J_\rho$ and $T_{u_n}\mathcal{S}_c$ vary with continuity, by $\tilde{m}_{\zeta_n}(u_n) \le 1$ it follows that $\tilde{m}_0(u_\rho) \le 1$. Then, we can use similar arguments as in [12, proposition 3.5] to show $m(u_\rho) \le 2$. In fact, since the tangent space $T_u\mathcal{S}_c$ has codimension 1, it suffices to show that $u_\rho \in \mathcal{S}_c$ has Morse index at most 1 as a constrained critical point. If this were not the case, by definition 3.3, we may assume by contradiction that there exists a subspace $W_0 \subset T_u\mathcal{S}_c$ with $dim W_0 = 2$ such that

$$D^2 J_\rho(u_\rho)[w, w] < 0 \quad \text{for all } w \in W_0 \backslash \{0\}. \tag{3.5}$$

Since W_0 is finite-dimensional, there exists a constant $\beta > 0$ such that

$$D^2 J_\rho(u_\rho)[w, w] < -\beta \text{ for all } w \in W_0 \backslash \{0\} \text{ with } \|w\| = 1.$$

Using the homogeneity of $D^2 J_\rho(u_\rho)$, we deduce that

$$D^2 J_\rho(u_\rho)[w, w] < -\beta \|w\|^2 \text{ for all } w \in W_0 \backslash \{0\}.$$

Now, since J'_ρ and J''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0, 1]$, it follows that there exists a sufficient small $\delta_1 > 0$ such that, for any $v \in \mathcal{S}_c$ satisfying $\|v - u_\rho\| \leq \delta_1$,

$$D^2 J_\rho(v)[w, w] < -\frac{\beta}{2} \|w\|^2 \text{ for all } w \in W_0 \backslash \{0\}. \tag{3.6}$$

Hence, using the fact that $\|u_n - u_\rho\| \leq \delta_1$ for sufficiently large $n \in \mathbb{N}$, and in view of (3.5), (3.6), and $\zeta_n \to 0^+$, we obtain

$$D^2 J_\rho(u_n)[w, w] < -\frac{\beta}{2} \|w\|^2 < \zeta_n \|w\|^2 \text{ for all } w \in W_0 \backslash \{0\}$$

for any such large n. Since $\dim W_0 > 1$, this provides a contradiction with theorem 3.4 (iv), recalling that $\zeta_n \to 0$. $\qquad \square$

REMARK 3.6. Note that for any $\rho \in [\frac{1}{2}, 1]$, the constant function $u_c := \left(\frac{c}{|\Omega|}\right)^{\frac{1}{2}}$ is always a solution of (3.3) on $\mathcal{S}_c$ for $\lambda = \frac{h_\rho(u_c)}{u_c}$. Under the assumptions (f_1)–(f_2), we can compute the constraint Morse index $\tilde{m}_0(u_c)$ corresponding to J_ρ, as demonstrated in [16, proposition 2.1] (see also [14, proposition 4.1]). Specifically, for any $c \in (0, c^*)$ with some proper $c^* > 0$, we have $\tilde{m}_0(u_c) = 0$ for all $\rho \in [\frac{1}{2}, 1]$. This implies that u_c is a local minimizer of J_ρ for every $\rho \in [\frac{1}{2}, 1]$. Based on this observation, we may select $w_1 = u_c$ in lemma 2.3 to construct the uniform mountain pass geometry for J_ρ. Furthermore, the mountain pass type solution u obtained in theorem 1.1 cannot be a constant function, as this would contradict the fact that u_c is a local minimizer.

4. Blow-up analysis

In this section, we develop a blow-up analysis for the sequence $\{u_{\rho n}\}$. The goal of this analysis is to prove that $\{u_{\rho n}\}$ is bounded in $H^1(\Omega)$. Consequently, we aim to show that $\{u_{\rho n}\}$ converges strongly in $H^1(\Omega)$ to a constrained critical point of J_1 as $\rho_n \to 1^-$.

For simplicity, we denote $u_n := u_{\rho n}$, $\lambda_n := \lambda_{\rho n}$, $c_n := c_{\rho n}$ in the following discussion. Here, u_n weakly solves the following problem

$$\begin{cases} -\Delta u_n + \lambda_n u_n = h_{\rho_n}(u_n) & \text{in } \Omega, \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_\Omega |u_n|^2 dx = c, \end{cases} \qquad (4.1)$$

where $\lambda_n \in \mathbb{R}$ and $\rho_n \to 1^-$. By theorem 3.5, we have $m(u_n) \leq 2$. Using standard regularity arguments, we obtain $u_n \in C^2(\overline{\Omega})$.

LEMMA 4.1. *There exists a constant C such that $\lambda_n \geq C$ for all n.*

Proof. We assume by contradiction that $\lambda_n \to -\infty$. Let V be a subspace of $H^1(\Omega)$ with dimension k, where $k > 2$. Define $\Omega_{M_0} := \{x \in \Omega : |u_n(x)| \leq M_0\}$, where $M_0 = \max\{M, R_0 + 1\}$ and R_0 is given in §2. By assumptions (f_1) and (f_3), there exist constants $C_{M_0}, C'_{M_0} > 0$ such that for any $\phi \in H^1(\Omega)$,

$$\begin{aligned} \int_\Omega h'_{\rho_n}(u_n)\phi^2 dx &= \int_{\Omega_{M_0}} h'_{\rho_n}(u_n)\phi^2 dx + \int_{\Omega\setminus\Omega_{M_0}} h'_{\rho_n}(u_n)\phi^2 dx \\ &\geq \int_{\Omega_{M_0}} -C_{M_0}\phi^2 dx + \int_{\Omega\setminus\Omega_{M_0}} h'_{\rho_n}(u_n)\phi^2 dx \\ &\geq \int_{\Omega_{M_0}} -C_{M_0}\phi^2 dx + \int_{\Omega\setminus\Omega_{M_0}} \left(\rho_n\mu|u_n|^{p-2} - C'_{M_0}\right)\phi^2 dx. \end{aligned}$$

Taking $\varphi \in V \setminus \{0\}$, we obtain

$$\begin{aligned} Q_{\lambda_n,\rho_n}(\varphi; u_n; \Omega) &\leq \int_\Omega |\nabla\varphi|^2 dx + \lambda_n \int_\Omega \phi^2 dx + \int_{\Omega_{M_0}} C_{M_0}\varphi^2 dx \\ &\quad - \int_{\Omega\setminus\Omega_{M_0}} \left(\mu|u_n|^{p-2} - C'_{M_0}\right)\varphi^2 dx \\ &\leq \|\varphi\|^2 + \left(\lambda_n + C_{M_0} + C'_{M_0} - 1\right) \int_\Omega \varphi^2 dx. \end{aligned}$$

This implies that $Q_{\lambda_n,\rho_n}(\varphi; u_n; \Omega)$ is negative definite on V for sufficiently large n, which contradicts the fact that $m(u_n) \leq 2$. $\qquad\square$

LEMMA 4.2. *If $\lambda_n \to +\infty$, then $\|u_n\|_{L^\infty} \to +\infty$.*

Proof. By (4.1) and assumptions (f_1)–(f_2), there exists a constant $C_1 > 0$ such that

$$\lambda_n c \leq \int_\Omega f(u_n)u_n \leq c + \int_\Omega C_1|u_n|^p dx \leq c + C_1|\Omega|\|u_n\|_{L^\infty}^p.$$

This implies that

$$\|u_n\|_{L^\infty} \geq \left(\frac{c}{C_1|\Omega|}\right)^{\frac{1}{p}} (\lambda_n - 1)^{\frac{1}{p}} \to +\infty.$$

$\qquad\square$

In the following, we will analyse the asymptotic behaviour of the solution to (4.1) as $\lambda_n \to +\infty$. For simplicity, we may assume without loss of generality that $\max\limits_{x \in \overline{\Omega}} u_n(x) > 0$.

We begin by providing a local description of the blow-up points.

LEMMA 4.3. *Suppose that $\lambda_n \to +\infty$. Let $P_n \in \overline{\Omega}$ be such that, for some $R_n \to \infty$,*

$$|u_n(P_n)| = \max_{B_{R_n \tilde{\epsilon}_n}(P_n) \cap \overline{\Omega}} |u_n(x)| \quad where \ \ \tilde{\epsilon}_n = a_0^{-\frac{1}{2}} |u_n(P_n)|^{-\frac{p-2}{2}} \to 0.$$

Set $\epsilon_n = \lambda_n^{-\frac{1}{2}}$. Then

$$\left(\frac{\tilde{\epsilon}_n}{\epsilon_n}\right)^2 \to \tilde{\lambda} \in (0, a_0]. \tag{4.2}$$

Suppose moreover that

$$\limsup_{n \to +\infty} \frac{dist(P_n, \partial\Omega)}{\tilde{\epsilon}_n} = +\infty. \tag{4.3}$$

Then, passing to a subsequence if necessary, we have

(i) $P_n \to P \in \Omega$;

(ii) $\frac{dist(P_n, \partial\Omega)}{\epsilon_n} \to +\infty$ *as* $n \to +\infty$, *and the scaled sequence*

$$v_n(x) := a_0^{\frac{1}{p-2}} \epsilon_n^{\frac{2}{p-2}} u_n(\epsilon_n x + P_n) \ \ for \ \ x \in \Omega_n := \frac{\Omega - P_n}{\epsilon_n} \tag{4.4}$$

converges to some $v \in H^1(\mathbb{R}^N)$ in $C^2_{loc}(\mathbb{R}^N)$, where v satisfies

$$\begin{cases} -\Delta v + v = |v|^{p-2} v & in \ \mathbb{R}^N, \\ |v(0)| = \max\limits_{x \in \mathbb{R}^N} v, \\ v(x) \to 0 \ \ as \ \ |x| \to +\infty; \end{cases} \tag{4.5}$$

(iii) *there exists $\phi_n \in C_0^\infty(\Omega)$, with $supp\phi_n \subset B_{R\epsilon_n}(P_n)$ for some R > 0, such that $Q_{\lambda_n, \rho_n}(\phi_n; u_n; \Omega) < 0$;*

(iv) *for all R > 0 and $q \geq 1$,*

$$\lim_{n \to \infty} \lambda_n^{\frac{N-2}{2} - \frac{q}{p-2}} \int_{B_{R\epsilon_n}(P_n)} |u_n|^q dx = \lim_{n \to \infty} \int_{B_R(0)} |v_n|^q dy = \int_{B_R(0)} |v|^q dy.$$

Instead of (4.3), we suppose that

$$\limsup_{n \to +\infty} \frac{dist(P_n, \partial\Omega)}{\tilde{\epsilon}_n} < +\infty. \tag{4.6}$$

Then, passing to a subsequence if necessary, the following results hold:

(i) $P_n \to P \in \partial\Omega$;

(ii) $\frac{dist(P_n, \partial\Omega)}{\epsilon_n} \to d_0 \geq 0$ *as* $n \to +\infty$, *and the scaled sequence* $\{v_n\}$ *defined in* (4.4) *converges to some* $v \in H^1(\mathbb{R}_+^N)$ *in* $C_{loc}^2(\mathbb{R}_+^N)$ *as* $n \to \infty$, *where* v *satisfies*

$$\begin{cases} -\Delta v + v = |v|^{p-2}v & \text{in } \mathbb{R}_+^N, \\ \dfrac{\partial v}{\partial x_N} = 0 & \text{on } \partial\mathbb{R}_+^N, \\ |v(0)| = \max\limits_{x \in \mathbb{R}_+^N} v, \\ v(x) \to 0 & \text{as } |x| \to +\infty; \end{cases} \tag{4.7}$$

(iii) *there exists* $\phi_n \in C_0^\infty(\Omega)$, *with* $supp\phi_n \subset B_{R\epsilon_n}(P_n) \cap \overline{\Omega}$ *for some* R > 0, *such that* $Q_{\lambda_n, \rho_n}(\phi_n; u_n; \Omega) < 0$;

(iv) *for all* R > 0 *and* $q \geq 1$,

$$\lim_{n \to \infty} \lambda_n^{\frac{N-2}{2} - \frac{q}{p-2}} \int_{B_{R\epsilon_n}(P_n) \cap \Omega} |u_n|^q dx = \lim_{n \to \infty} \int_{B_R(0) \cap \Omega_n} |v_n|^q dy$$
$$= \int_{B_R(0) \cap \mathbb{R}_+^N} |v|^q dy.$$

Proof. Since u_n may change sign, P_n can be either a positive local maximum or a negative local minimum point. For simplicity, we focus on the case where P_n is a positive local maximum point; the arguments for the negative local minimum case are analogous.

By (4.1), we get

$$0 \leq \frac{-\Delta u_n(P_n)}{u_n(P_n)} = \frac{f_2(u_n(P_n)) + \rho_n f_1(u_n(P_n))}{u_n(P_n)} - \lambda_n.$$

Using (f_2) and lemma 4.1, we deduce that

$$\frac{\lambda_n}{|u_n(P_n)|^{p-2}} \to \tilde{\lambda} \in [0, a_0] \text{ as } n \to \infty.$$

Next, we show that $\tilde{\lambda} > 0$.
Define the rescaled function

$$\tilde{u}_n(x) := a_0^{\frac{1}{p-2}} \tilde{\epsilon}_n^{\frac{2}{p-2}} u_n(\tilde{\epsilon}_n x + P_n) \text{ for } x \in \tilde{\Omega}_n := \frac{\Omega - P_n}{\tilde{\epsilon}_n}.$$

Clearly, $\tilde{u}_n$ satisfies

$$\begin{cases} -\Delta\tilde{u}_n + \lambda_n\tilde{\epsilon}_n^2\tilde{u}_n = a_0^{\frac{1}{p-2}}\tilde{\epsilon}_n^{\frac{2p-2}{p-2}}h_{\rho_n}(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n) & \text{in } \tilde{\Omega}_n, \\ |\tilde{u}_n(x)| \le |\tilde{u}_n(0)| = 1 & \text{in } \tilde{\Omega}_n, \\ \dfrac{\partial\tilde{u}_n}{\partial\nu} = 0 & \text{on } \partial\tilde{\Omega}_n. \end{cases} \qquad (4.8)$$

Let $d_n = dist(P_n, \partial\Omega)$. Then

$$\frac{d_n}{\tilde{\epsilon}_n} := L \in [0, +\infty] \ \text{ and } \ \tilde{\Omega}_n \to \begin{cases} \mathbb{R}^N & \text{if } L = +\infty; \\ \mathbb{H} & \text{if } L < +\infty, \end{cases}$$

where $\mathbb{H}$ denotes a half-space such that $0 \in \overline{\mathbb{H}}$ and $d(0, \partial\mathbb{H}) = L$. By regularity arguments, up to a subsequence, $\tilde{u}_n \to \tilde{u}$ in $C^2_{loc}(\overline{D})$, where $\tilde{u}$ solves

$$\begin{cases} -\Delta\tilde{u} + \tilde{\lambda}\tilde{u} = |\tilde{u}|^{p-2}\tilde{u} & \text{in } D, \\ |\tilde{u}(x)| \le |\tilde{u}(0)| = 1 & \text{in } D, \\ \dfrac{\partial\tilde{u}}{\partial\nu} = 0 & \text{on } \partial D, \end{cases} \qquad (4.9)$$

where D is either $\mathbb{R}^N$ or $\mathbb{H}$.

We claim that $m(\tilde{u}) \le 2$. To see this, suppose for contradiction that there exists $k > 2$ such that there are k positive functions $\phi_1, \ldots, \phi_k \in H^1(D)$, orthogonal in $L^2(\Omega)$, satisfying

$$\int_D |\nabla\phi_i|^2 dx + \tilde{\lambda}\int_D \phi^2 dx - \int_D (p-1)|\tilde{u}|^{p-2}\phi^2 dx < 0 \qquad (4.10)$$

for every $i \in \{1, \ldots, k\}$.

Define the rescaled functions

$$\phi_{i,n}(x) := \tilde{\epsilon}_n^{-\frac{N-2}{2}}\phi_i\left(\frac{x - P_n}{\tilde{\epsilon}_n}\right).$$

Additionally, let

$$\tilde{\Omega}_{n,M_0} := \{x \in \tilde{\Omega}_n : |a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n| \le M_0\}, \ \ \tilde{\Omega}_{n,M_0}^c := \tilde{\Omega}_n \backslash \tilde{\Omega}_{n,M_0},$$

where M_0 is as given in lemma 4.1.

By direct computations, there exist constants $\tilde{C}_{M_0}, \tilde{C}'_{M_0} > 0$ such that the following estimates hold:

$$\int_\Omega f_1'(u_n)\phi_{i,n}^2 dx = \int_{\tilde{\Omega}_n} \tilde{\epsilon}_n^2 \phi_i^2 f_1'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)dx$$

$$= \int_{\tilde{\Omega}_{n,M_0+1}} [\tilde{\epsilon}_n^2 \phi_i^2 f'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)\eta(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)$$

$$+ \tilde{\epsilon}_n^2 \phi_i^2 f(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)\eta'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)]dx$$

$$+ \int_{\tilde{\Omega}_{n,M_0}^c} \tilde{\epsilon}_n^2 \phi_i^2 f'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)\eta(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)dx$$

$$\geq \int_{\tilde{\Omega}_{n,M_0}\cap supp\{\phi_i\}} -\tilde{C}_{M_0}\tilde{\epsilon}_n^2 \phi_i^2 dx$$

$$+ \int_{\tilde{\Omega}_{n,M_0}^c} \tilde{\epsilon}_n^2 \phi_i^2 \mu|a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n|^{p-2}dx$$

$$= \int_{\tilde{\Omega}_{n,M_0}\cap supp\{\phi_i\}} -\tilde{C}_{M_0}\tilde{\epsilon}_n^2 \phi_i^2 dx$$

$$+ \int_{\tilde{\Omega}_n} \tilde{\epsilon}_n^2 \phi_i^2 \mu|a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n|^{p-2}dx$$

$$- \int_{\tilde{\Omega}_{n,M_0}} \tilde{\epsilon}_n^2 \phi_i^2 \mu|a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n|^{p-2}dx$$

and

$$\int_\Omega f_2'(u_n)\phi_{i,n}^2 dx = \int_{\tilde{\Omega}_n} \tilde{\epsilon}_n^2 \phi_i^2 f_2'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)dx$$

$$= \int_{\tilde{\Omega}_{n,M_0}\cap supp\{\phi_i\}} [\tilde{\epsilon}_n^2 \phi_i^2 f'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)$$

$$\times (1 - \eta(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n))$$

$$- \tilde{\epsilon}_n^2 \phi_i^2 f(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)\eta'(a_0^{-\frac{1}{p-2}}\tilde{\epsilon}_n^{-\frac{2}{p-2}}\tilde{u}_n)]dx$$

$$\geq \int_{\tilde{\Omega}_{n,M_0}\cap supp\{\phi_i\}} -\tilde{C}'_{M_0}\tilde{\epsilon}_n^2 \phi_i^2 dx.$$

Note that $a_0^{-\frac{1}{p-2}}\tilde{u}_n \leq \tilde{\epsilon}_n^{\frac{2}{p-2}}M_0$ for $x \in \tilde{\Omega}_{n,M_0}$, by (4.10), it follows that

$$Q_{\lambda_n,\rho_n}(\phi_{i,n}; u_n; \Omega) = \int_\Omega |\nabla\phi_{i,n}|^2 dx + \lambda_n \int_\Omega \phi_{i,n}^2 dx - \int_\Omega f_2'(u_n)\phi_{i,n}^2 dx - \rho_n$$

$$\times \int_\Omega f_1'(u_n)\phi_{i,n}^2 dx$$

$$\leq \int_{\tilde\Omega_n} |\nabla\phi_i|^2 dx + \lambda_n \tilde\epsilon_n^2 \int_{\tilde\Omega_n} \phi_i^2 dx$$

$$+ \int_{\tilde\Omega_{n,M_0} \cap supp\{\phi_i\}} (\rho_n \tilde{C}_{M_0} + \tilde{C}_{M_0}')\tilde\epsilon_n^2 \phi_i^2 dx$$

$$- \rho_n \int_{\tilde\Omega_n} \tilde\epsilon_n^2 \phi_i^2 \mu |a_0^{-\frac{1}{p-2}} \tilde\epsilon_n^{-\frac{2}{p-2}} \tilde u_n|^{p-2} dx + \rho_n$$

$$\times \int_{\tilde\Omega_{n,M_0}} \phi_i^2 \mu a_0^{-1} |\tilde u_n|^{p-2} dx$$

$$\to \int_D |\nabla\phi_i|^2 dx + \tilde\lambda \int_D \phi^2 dx - \int_D a_0^{-1} \mu |\tilde u|^{p-2} \phi^2 dx$$

$$\leq \int_D |\nabla\phi_i|^2 dx + \tilde\lambda \int_D \phi^2 dx - \int_D (p-1)|\tilde u|^{p-2} \phi^2 dx < 0.$$

This implies that $m(u_n) \geq k > 2$ for sufficiently large n, thereby yielding a contradiction. Thus, the claim is valid.

Having established that $\tilde u$ is a finite Morse non-trivial solution of (4.9), we can apply either [22, theorem 2] or [57, proposition 2.1]. This allows us to conclude that the occurrence of $\tilde\lambda = 0$ is ruled out, regardless of whether D is a half-space or a whole space. Consequently, we can assert that $\tilde\lambda \in (0, a_0]$.

In the sequel, we consider the sequence $\{v_n\}$ defined by (4.4). Clearly, v_n satisfies

$$\begin{cases} -\Delta v_n + v_n = a_0^{\frac{1}{p-2}} \epsilon_n^{\frac{2p-2}{p-2}} h_{\rho_n}(a_0^{-\frac{1}{p-2}} \epsilon_n^{-\frac{2}{p-2}} v_n) & \text{in } \Omega_n, \\ |v_n(x)| \leq |v_n(0)| = (\frac{\epsilon_n}{\tilde\epsilon_n})^{\frac{2}{p-2}} \to \tilde\lambda^{-\frac{1}{p-2}} & \text{in } \Omega_n, \\ \frac{\partial v_n}{\partial \nu} = 0 & \text{on } \partial\Omega_n. \end{cases} \quad (4.11)$$

Up to a subsequence, we have $v_n \to v$ in $C_{loc}^2(\overline{D})$, where D is either $\mathbb{R}^N$ or a half space $\mathbb{H}$, and v solves

$$\begin{cases} -\Delta v + v = |v|^{p-2}v & \text{in } D, \\ |v(x)| \leq |v(0)| = \tilde\lambda^{-\frac{1}{p-2}} & \text{in } D, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \quad (4.12)$$

Arguing as above implies that $m(v) \leq 2$.

More precisely, we distinguish the following three cases:

(1) If $\limsup_{n\to+\infty} \frac{dist(P_n, \partial\Omega)}{\tilde\epsilon_n} = +\infty$, then $D = \mathbb{R}^N$. Using $m(v) \leq 2$, by standard regularity arguments, $v \in C^2(\mathbb{R}^N)$, $|v(x)| \to 0$ as $|x| \to +\infty$. If $v \geq 0$,

then using the strong maximum principle and [20, theorem 1.1] we know that $v > 0, m(v) = 1$, and it coincides with the unique radial ground-state solution U_0 to $-\Delta u + u = u^{p-1}$ in $\mathbb{R}^N$. If v is sign-changing, then $m(v) = 2$ and v admits exactly one local maximum point and one local minimum point in Ω.

(2) If there exists a constant $a > 0$ such that

$$0 < a \leq \limsup_{n \to +\infty} \frac{dist(P_n, \partial\Omega)}{\tilde{\epsilon}_n} < +\infty,$$

we may assume that $\lim_{n \to +\infty} \frac{dist(P_n, \partial\Omega)}{\tilde{\epsilon}_n} = d > 0$. Then, up to subsequences, $v_n \to v$ in $C^2_{loc}(\{x_N > -d\})$, and v weakly solves

$$\begin{cases} -\Delta v + v = |v|^{p-2}v & \text{in } \{x_N > -d\}, \\ |v(x)| \leq |v(0)| = \tilde{\lambda}^{-\frac{1}{p-2}} & \text{in } \{x_N > -d\}, \\ \dfrac{\partial v}{\partial x_N} = 0 & \text{on } \{x_N = -d\}. \end{cases}$$

Let $\tilde{v}(x) = v(x', x_N - d)$, where $x' = (x_1, x_2, \ldots, x_{N-1})$. It is straightforward to verify that $\tilde{v}$ is a finite Morse index solution to the following system:

$$\begin{cases} -\Delta \tilde{v} + \tilde{v} = |\tilde{v}|^{p-2}\tilde{v} & \text{in } \{x_N > 0\}, \\ |\tilde{v}(x)| \leq |\tilde{v}(0, \ldots, 0, d)| = \tilde{\lambda}^{-\frac{1}{p-2}} & \text{in } \{x_N > 0\}, \\ \dfrac{\partial \tilde{v}}{\partial x_N} = 0 & \text{on } \{x_N = 0\}. \end{cases}$$

Next, we extend $\tilde{v}$ by reflection with respect to $\{x_N = 0\}$. Specifically, for $(x', x_N) \in \mathbb{R}^N$, we define

$$\hat{v}(x', x_N) := \begin{cases} \tilde{v}(x', x_N) & \text{if } x_N \geq 0, \\ \tilde{v}(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Consequently, $\hat{v}$ satisfies the equation

$$\begin{cases} -\Delta \hat{v} + \hat{v} = |\hat{v}|^{p-2}\hat{v} & \text{in } \mathbb{R}^N, \\ |\hat{v}(x)| \leq |\hat{v}(0, \ldots, 0, -d)| = |\hat{v}(0, \ldots, 0, d)| = \tilde{\lambda}^{-\frac{1}{p-2}} & \text{in } \mathbb{R}^N, \\ \dfrac{\partial \tilde{v}}{\partial x_N} = 0 & \text{on } \{x_N = 0\}. \end{cases}$$

Thus, $\hat{v}$ is a bounded function that solves the equation $-\Delta\hat{v} + \hat{v} = |\hat{v}|^{p-2}\hat{v}$ in $\mathbb{R}^N$ in the weak sense. By applying Schauder interior and boundary estimates (see [26]), we conclude that $\hat{v} \in C^{2,\alpha}_{loc}(\mathbb{R}^N)$ for $\alpha \in (0, 1)$, and the regularity extends up to the hyperplane $\{x_N = 0\}$. Since $\hat{v}$ is symmetric across $\{x_N = 0\}$, it follows that the second derivatives of $\hat{v}$ are continuous everywhere,

including at the boundary $\{x_N = 0\}$. Consequently, by applying elliptic regularity theory and considering that $\hat{v}$ is a solution with finite Morse index, we can show that $\hat{v} \in C^2(\mathbb{R}^N)$ and $|\hat{v}(x)| \to 0$ as $|x| \to +\infty$. Therefore, $\tilde{v}$ must be a sign-changing solution, a situation that can occur according to [41, theorem 1]. In fact, if $\hat{v}$ is a positive solution, then by applying [20, theorem 1.1] once again, we deduce that $\hat{v}$ coincides with the unique radial ground-state solution U_0 to $-\Delta u + u = u^{p-1}$ in $\mathbb{R}^N$. It is well known that U_0 has a unique global maximum and is a radially strictly decreasing function. This would contradict the fact that $\max_{x \in \mathbb{R}^N} \hat{v} = |\hat{v}(0, \ldots, 0, -d)| = |\hat{v}(0, \ldots, 0, d)|$.

(3) If $\limsup\limits_{n \to +\infty} \frac{dist(P_n, \partial\Omega)}{\bar{\epsilon}_n} = 0$, then, up to a subsequence, $\Omega_n \to \mathbb{R}^N_+(\{x_N > 0\})$. By similar arguments as above, $v_n \to v$ in $C^2_{loc}(\mathbb{R}^N_+)$, where v is a finite Morse index of the following equation

$$\begin{cases} -\Delta v + v = |v|^{p-2}v & \text{in } \{x_N > 0\}, \\ \dfrac{\partial v}{\partial x_N} = 0 & \text{on } \{x_N = 0\}, \\ |v(y)| \le |v(0)| = \tilde{\lambda}^{-\frac{1}{p-2}}. \end{cases}$$

Extending v by reflection with respect to $\{x_N = 0\}$ and defining

$$\hat{v}(x', x_N) := \begin{cases} v(x', x_N) & \text{if } x_N \ge 0, \\ v(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then $\hat{v}$ satisfies

$$\begin{cases} -\Delta\hat{v} + \hat{v} = |\hat{v}|^{p-2}\hat{v} & \text{in } \mathbb{R}^N, \\ |\hat{v}(x)| \le |\hat{v}(0)| & \text{in } \mathbb{R}^N, \\ \dfrac{\partial v}{\partial x_N} = 0 & \text{on } \{x_N = 0\}. \end{cases}$$

Using arguments analogous to those in case (1), we deduce that if $\hat{v}$ is positive, then $m(\hat{v}) = 1$, and thus $\hat{v}$ coincides with U_0. Using [41, theorem 1], it is established that $\hat{v}$ may exhibit sign-changing behaviour, and in such instance, $m(v) = 2$.

All in all, we get that (ii) holds. Employing similar reasoning to [20, theorem 3.1], we conclude that (iii) and (iv) are also satisfied. $\qquad\square$

We now proceed to provide a comprehensive global blow-up analysis. The following result offers a detailed global description of the asymptotic behaviour of $\{u_n\}$ as $\lambda_n \to +\infty$.

LEMMA 4.4. *There exists $k \in \{1, 2\}$ and sequences of points $\{P_n^1\}, \ldots, \{P_n^k\}$, such that*

$$|u_n(P_n^i)| = \max_{B_{R_n\lambda_n^{-1/2}}(P_n^i)\cap\overline{\Omega}} |u_n| \text{ for some } R_n \to \infty, \text{ for every } i, \qquad (4.13)$$

$$\lambda_n |P_n^i - P_n^j|^2 \to +\infty, \quad \forall i \neq j, \quad as \ \ n \to \infty, \tag{4.14}$$

and moreover,

$$\lim_{R \to \infty} \left(\limsup_{n \to +\infty} \lambda_n^{-\frac{1}{p-2}} \max_{d_n(x) \geq R\lambda_n^{-1/2}} |u_n(x)| \right) = 0, \tag{4.15}$$

where $d_n(x) = \min\{|x - P_n^i| : i = 1, \ldots, k\}$ is the distance function from $\{P_n^1, \ldots, P_n^k\}$ for $x \in \overline{\Omega}$.

Proof. Take $P_n^1 \in \overline{\Omega}$ such that $u_n(P_n^1) = \max_{\overline{\Omega}} |u_n(x)|$. If (4.15) is satisfied for P_n^1, then we get $k = 1$. It is evident that P_n^1 satisfies (4.13).

Otherwise, if P_n^1 does not satisfy (4.15), we suppose that there exists $\delta > 0$ such that

$$\lim_{R \to \infty} \left(\limsup_{n \to +\infty} \lambda_n^{-\frac{1}{p-2}} \max_{|x - P_n^1| \geq R\lambda_n^{-1/2}} |u_n(x)| \right) \geq 4\delta. \tag{4.16}$$

For sufficiently large R, up to a subsequence, it holds

$$\lambda_n^{-\frac{1}{p-2}} \max_{|x - P_n^1| \geq R\lambda_n^{-1/2}} |u_n(x)| \geq 2\delta. \tag{4.17}$$

Let $P_n^2 \in \overline{\Omega} \backslash B_{R\lambda_n^{-1/2}}(P_n^1)$ such that

$$|u_n(P_n^2)| = \max_{\overline{\Omega} \backslash B_{R\lambda_n^{-1/2}}(P_n^1)} |u_n|.$$

Then, (4.17) yields that $|u_n(P_n^2)| \to +\infty$ as $n \to +\infty$.

We claim that

$$\lambda_n |P_n^1 - P_n^2|^2 \to +\infty. \tag{4.18}$$

If (4.18) is not true, then up to subsequence $\lambda_n^{\frac{1}{2}} |P_n^1 - P_n^2| \to R' \geq R$. Define

$$v_{n,1}(x) := a_0^{\frac{1}{p-2}} \lambda_n^{-\frac{1}{p-2}} u_n(\lambda_n^{-1/2} x + P_n^1).$$

As in lemma 4.3, we can deduce that $v_{n,1} \to v$ in $C_{loc}^2(\overline{D})$, where $D = \mathbb{R}^N$ or D is a half-space. Then, up to subsequences,

$$\lambda_n^{-\frac{1}{p-2}} u_n(P_n^2) = v_{n,1}(\lambda^{1/2}(P_n^2 - P_n^1)) \to v(x'), \ |x'| \geq R' > R.$$

Since $v(x) \to 0$ as $|x| \to \infty$, taking R larger if necessary, it follows that

$$|v(x)| \leq a_0^{\frac{1}{p-2}} \delta, \ \forall |x| \geq R.$$

This contradicts to (4.17), which proves the claim (4.18).

In the following, we shall show that

$$|u_n(P_n^2)| = \max_{\overline{\Omega} \cap B_{R_{n,2}\lambda_n^{-1/2}}(P_n^2)} |u_n(x)|, \text{ for some } R_{n,2} \to +\infty. \tag{4.19}$$

Let $\tilde{\epsilon}_{n,2} = a_0^{-\frac{1}{2}} |u_n(P_n^2)|^{-\frac{p-2}{2}}$. Clearly, $\tilde{\epsilon}_{n,2} \to 0$. By (4.17), we get $\tilde{\epsilon}_{n,2} \leq (2\delta)^{-\frac{p-2}{2}} \lambda_n^{-\frac{1}{2}}$. From (4.18), we can assert that

$$\tilde{R}_{n,2} := \frac{|P_n^1 - P_n^2|}{2\tilde{\epsilon}_{n,2}} \geq \frac{(2\delta)^{\frac{p-2}{2}}}{2} \lambda_n^{\frac{1}{2}} |P_n^1 - P_n^2| \to +\infty \text{ as } n \to \infty.$$

On the other hand, for any $x \in B_{\tilde{R}_{n,2}\tilde{\epsilon}_{n,2}}(P_n^2)$ and $R > 0$, we have

$$|x - P_n^1| \geq |P_n^2 - P_n^1| - |x - P_n^2| \geq \frac{1}{2}|P_n^2 - P_n^1| \geq R\lambda_n^{-\frac{1}{2}}$$

for arbitrarily large n. Consequently,

$$\overline{\Omega} \cap B_{\tilde{R}_{n,2}\tilde{\epsilon}_{n,2}}(P_n^2) \subset \overline{\Omega} \backslash B_{R\lambda_n^{-\frac{1}{2}}}(P_n^1),$$

which implies that

$$|u_n(P_n^2)| = \max_{\overline{\Omega} \cap B_{\tilde{R}_{n,2}\tilde{\epsilon}_{n,2}}(P_n^2)} |u_n|. \tag{4.20}$$

We define

$$\tilde{u}_{n,2} := a_0^{\frac{1}{p-2}} \tilde{\epsilon}_{n,2}^{\frac{2}{p-2}} u_n(\tilde{\epsilon}_{n,2}x + P_n^2), \quad x \in \tilde{\Omega}_{n,2} := \frac{\Omega - P_n^2}{\tilde{\epsilon}_{n,2}}.$$

Set $d_{n,2} = \text{dist}(P_n^2, \partial\Omega)$. Then, passing to subsequences if necessary,

$$\frac{\tilde{\epsilon}_{n,2}}{d_{n,2}} \to L_2 \in [0, +\infty] \text{ and } \tilde{\Omega}_{n,2} \to \begin{cases} \mathbb{R}^N & \text{if } L_2 = +\infty, \\ \mathbb{H} & \text{if } L_2 < +\infty. \end{cases}$$

Then $\tilde{u}_{n,2}$ satisfies the following equation

$$\begin{cases} -\Delta\tilde{u}_{n,2} + \lambda_n\tilde{\epsilon}_{n,2}^2\tilde{u}_{n,2} = a_0^{\frac{1}{p-2}} \tilde{\epsilon}_{n,2}^{\frac{2p-2}{p-2}} h_{\rho_n}(a_0^{-\frac{1}{p-2}} \tilde{\epsilon}_{n,2}^{-\frac{2}{p-2}} \tilde{u}_{n,2}) & \text{in } \tilde{\Omega}_{n,2}, \\ |\tilde{u}_{n,2}(x)| \leq |\tilde{u}_{n,2}(0)| = 1 & \text{in } \tilde{\Omega}_{n,2}, \\ \frac{\partial\tilde{u}_{n,2}}{\partial\nu} = 0 & \text{on } \partial\tilde{\Omega}_{n,2}. \end{cases}$$

Since P_n^2 is a local maximum or a local minimum point, by lemma 4.1, we get

$$\frac{\lambda_n}{|u_n(P_n^2)|^{p-2}} \to \tilde{\lambda}^{(2)} \in [0, a_0], \text{ as } n \to \infty.$$

Using the similar argument as in lemma 4.3, we deduce $\tilde{\lambda}^{(2)} > 0$, namely

$$\lim_{n \to +\infty} \lambda_n^{\frac{1}{2}} \tilde{\epsilon}_{n,2} > 0.$$

Set

$$v_{n,2} := a_0^{\frac{1}{p-2}} \epsilon_n^{\frac{2}{p-2}} u_n(\epsilon_n x + P_n^2) \text{ for } x \in \Omega_{n,2} := \frac{\Omega - P_n^2}{\epsilon_n},$$

where $\epsilon_n = \lambda_n^{-\frac{1}{2}}$ is defined in (4.4). Up to a subsequence, there exists a function $v^{(2)} \in H^1(D)$ such that $v_{n,2} \to v^{(2)}$ in $C_{loc}^2(\overline{D})$, where D is either $\mathbb{R}^N$ or a half space. Moreover, $v^{(2)}$ solves the following problem

$$\begin{cases} -\Delta v^{(2)} + v^{(2)} = |v^{(2)}|^{p-2} v^{(2)} & \text{in } D, \\ |v^{(2)}(x)| \le |v^{(2)}(0)| = (\tilde{\lambda}^{(2)})^{-\frac{1}{p-2}} & \text{in } D, \\ \dfrac{\partial v^{(2)}}{\partial \nu} = 0 & \text{on } \partial D. \end{cases} \qquad (4.21)$$

Then, by a similar discussion as above, we conclude that either $\frac{d_{n,2}}{\epsilon_n} \to +\infty$ or $\frac{d_{n,2}}{\epsilon_n}$ remains bounded. Define $R_{n,2} = \tilde{R}_{n,2} \lambda_n^{\frac{1}{2}} \tilde{\epsilon}_{n,2}$. Clearly, $R_{n,2} \to +\infty$ as $n \to +\infty$. Hence, (4.19) holds.

If (4.15) does not hold, we can apply similar arguments as before to show that there exists P_n^3 such that (4.13)–(4.14) are satisfied. For $P_n^i, i = 1, 2, 3$, applying lemma 4.3 again, we can find $\phi_n^i \in C_0^\infty(\Omega)$ with $\mathrm{supp}\phi_n^i \subset B_{R\varepsilon_n}(P_n^i) \cap \overline{\Omega}$ for some $R > 0$, such that

$$\int_\Omega |\nabla \phi_n^i|^2 dx + \int_\Omega [(\lambda - (p-1)\rho_n u_n^{p-2})(\phi_n^i)^2] dx < 0.$$

In light of (4.14), we observe that $\phi_n^1, \phi_n^2, \phi_n^3$ are mutually orthogonal for sufficiently large n, which implies $\lim_{n \to +\infty} m(u_n) \ge 3$. This leads to a contradiction with the fact that $m(u_n) \le 2$. $\qquad \square$

In the subsequent analysis, we show that u_n exhibits exponential decay away from the blow-up points.

LEMMA 4.5. *Let $\{P_n^1\}, \ldots, \{P_n^k\}$ be given in lemma 4.4. Then there exist constants $C_1 > 0$, $C_2 > 0$ such that, for some $R > 0$,*

$$u_n(x) \le C_1 e^{C_1 R} \lambda_n^{\frac{1}{p-2}} \sum_{i=1}^k e^{-C_2 \lambda_n^{\frac{1}{2}} |x - P_n^i|}, \ \forall x \in \overline{\Omega} \backslash \cup_{i=1}^k (B_{R\lambda_n^{-\frac{1}{2}}}(P_n^i) \cap \overline{\Omega}). \quad (4.22)$$

Proof. The proof is inspired by [38, lemma 2.1] and [20, theorem 3.2]. For any $\theta > 0$, we set

$$\Omega_\theta := \{x \in \overline{\Omega} : dist(x, \partial\Omega) < \theta\},$$

and define the inner normal bundle

$$(\partial\Omega)_\theta := \{(x,y) : x \in \partial\Omega, y \in (-\theta, 0]\nu_x\},$$

where ν_x is the unit outer normal of $\partial\Omega$ at x.

Since $\partial\Omega$ is a smooth compact submanifold of $\mathbb{R}^N$, by the tubular neighbourhood theorem, it follows that there exists a diffeomorphism Φ_{Ib} from Ω_θ onto $(\partial\Omega)_\theta$. More precisely, for any $x \in \Omega_\theta$, it is easily seen that there exists a unique $\bar{x} \in \partial\Omega$ such that $\text{dist}(x, \bar{x}) = \text{dist}(x, \partial\Omega)$. Hence, we can define $\Phi_{Ib}(x) := (\bar{x}, -\text{dist}(x, \bar{x})\nu_{\bar{x}})$ for any $x \in \Omega_\theta$. Clearly, $\Phi_{Ib}(x) = x$ for $x \in \partial\Omega$.

Similarly, let

$$\Omega^\theta := \{x \in \mathbb{R}^N \backslash \Omega, dist(x, \partial\Omega) < \theta\},$$

and define the outer normal bundle

$$(\partial\Omega)^\theta := \{(x,y) : x \in \partial\Omega, y \in [0, \theta)\nu_x\}.$$

Then there exists a diffeomorphism $\Phi_{Ob} : \Omega^\theta \to (\partial\Omega)^\theta$ defined by $\Phi_{Ob}(x) := (\hat{x}, dist(x, \hat{x})\nu_{\hat{x}}), \forall x \in \Omega^\theta$. Here $\hat{x} \in \partial\Omega$ is the unique element in $\partial\Omega$ such that $\text{dist}(x, \hat{x}) = \text{dist}(x, \partial\Omega)$. Moreover, $\Phi_{Ob}|_{\partial\Omega} = \text{Identity}$.

Consider the reflection $\Phi_C : (\partial\Omega)_\theta \to (\partial\Omega)^\theta$ defined by $\Phi_C((x,y)) := (x, -y)$. Then, $\Phi := \Phi_{Ib}^{-1} \circ \Phi_C^{-1} \circ \Phi_{Ob}$ is a diffeomorphism from Ω^θ onto Ω_θ and $\Phi|_{\partial\Omega} = \text{Identity}$. Furthermore, we take $x = \Phi(z) = (\Phi_1(z), \ldots, \Phi_N(z))$ for $z \in \Omega^\theta$, $z = \Psi(x) = \Phi^{-1}(x) = (\Psi_1(x), \ldots, \Psi_N(x))$ for $x \in \Omega_\theta$, and

$$g_{ij} = \sum_{l=1}^{N} \frac{\partial\Phi_l}{\partial z_i}\frac{\partial\Phi_l}{\partial z_j}, \quad g^{ij} = \sum_{l=1}^{n} \frac{\partial\Psi_i}{\partial x_l}\frac{\partial\Psi_j}{\partial x_l}(\Phi(z)).$$

Then $g_{ij}|_{\partial\Omega} = g^{ij}|_{\partial\Omega} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. For simplicity, denote $G = (g^{ij})$, $g(x) = det(g_{ij})$ and $\hat{u}_n(x) = u_n(\Phi(x))$ for $x \in \Omega^\theta$. Then $\hat{u}_n$ satisfies

$$\begin{cases} -L\hat{u}_n + \sqrt{g}\lambda_n\hat{u}_n = \sqrt{g}h_{\rho_n}(\hat{u}_n) & \text{in } \Omega^\theta, \\ \dfrac{\partial\hat{u}_n}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$L\hat{u}_n = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sqrt{g}\sum_{j=1}^{N} g^{ij}\frac{\partial\hat{u}_n}{\partial x_j}\right).$$

Next we define

$$\bar{u}_n := \begin{cases} u_n(x), x \in \Omega, \\ \hat{u}_n(x), x \in \Omega^\theta, \end{cases} \quad \bar{g}_{ij} := \begin{cases} \delta_{ij}, x \in \bar{\Omega}, \\ g_{ij}, x \in \Omega^\theta, \end{cases} \quad \bar{g}^{ij} := \begin{cases} \delta_{ij}, x \in \bar{\Omega}, \\ g^{ij}, x \in \Omega^\theta, \end{cases}$$

and $\bar{g} := det(\bar{g}_{ij})$. Let $A(x,\xi) = (A_1(x,\xi), \ldots, A_N(x,\xi))$ for $\xi = (\xi_1, \ldots, \xi_N)$ with

$$A_i(x,\xi) = \sqrt{\bar{g}} \sum_{j=1}^{N} \bar{g}^{ij} \xi_j.$$

Then $\bar{u}_n$ weakly solves

$$-div\left(A(x,\nabla \bar{u}_n)\right) + \lambda_n \sqrt{\bar{g}} \bar{u}_n = \sqrt{\bar{g}} g_{\rho n}(\bar{u}_n) \text{ in } \overline{\Omega} \cup \Omega^{\theta}. \qquad (4.23)$$

Given that Φ is a diffeomorphism, the functions $g_{ij}(x)$, $\frac{\partial g_{ii}}{\partial x_i}$ and $\frac{\partial g_{ij}}{\partial x_i}$ are all smooth on the domain Ω^{θ}. Consequently, there exists a constant $C_0 > 0$ such that $|\frac{\partial g_{ii}}{\partial x_i}|, |\frac{\partial g_{ij}}{\partial x_i}| \le C_0$ are both bounded by C_0 for $x \in \Omega^{\theta}$. Moreover, according to the Taylor expansion, as θ approaches 0, the functions g_{ij} and g^{ij} tend to 0, while g_{ii} and g^{ii} tend to 1, and the determinant $g(x)$ converges to 1.

For any $\tilde{x} \in \overline{\Omega}$, there exists $\beta > 0$ such that $B_\beta(\tilde{x}) \subset \overline{\Omega} \cup \Omega^{\theta}$. For $x \in B_\beta(\tilde{x}) \backslash \tilde{x}$, denote $\sigma = |x - \tilde{x}|$. Then, for any smooth increasing function $\phi(\sigma)$, we have

$$\left| \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\sqrt{g} g^{ij}) \frac{\partial \phi}{\partial x_j} \right| = \left| \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\sqrt{g} g^{ij}) \frac{x_j - \tilde{x}_j}{\sigma} \phi' \right|$$

$$\le \sum_{i=1}^{N} \left| \max_{x \in \overline{\Omega} \cup \Omega^{\theta}} \frac{\partial}{\partial x_i}(\sqrt{g} g^{ij}) \right| \phi' \frac{diam(\overline{\Omega} \cup \Omega^{\theta})}{\sigma}$$

$$\le \frac{C_{\Phi,\Omega} \phi'}{\sigma}$$

for some $C_{\Phi,\Omega} > 0$, where $\phi' = \frac{d\phi(\sigma)}{d\sigma}$. Moreover, for $\theta > 0$ small enough, there exist small constants δ_0, δ_1 such that

$$\left| \sqrt{g} g^{ii} \frac{\partial^2 \phi}{\partial x_i^2} \right| = \left| \sqrt{g} g^{ii} \left(\frac{\phi'}{\sigma} + \phi'' \frac{(x_i - \tilde{x}_i)^2}{\sigma^2} - \phi' \frac{(x_i - \tilde{x}_i)^2}{\sigma^3} \right) \right|$$

$$\le (1 + \delta_0) \left(\frac{\phi'}{\sigma} + |\phi''| \frac{(x_i - \tilde{x}_i)^2}{\sigma^2} + \phi' \frac{(x_i - \tilde{x}_i)^2}{\sigma^3} \right),$$

and

$$\left| \sqrt{g} g^{ij} \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right| = \left| \sqrt{g} g^{ij} \left(\phi'' \frac{(x_i - \tilde{x}_i)(x_j - \tilde{x}_j)}{\sigma^2} - \phi' \frac{(x_i - \tilde{x}_i)(x_j - \tilde{x}_j)}{\sigma^3} \right) \right|$$

$$\le |\sqrt{g} g^{ij}| \left(\frac{1}{2} |\phi''| \frac{(x_i - \tilde{x}_i)^2}{\sigma^2} + \frac{1}{2} \phi' \frac{(x_i - \tilde{x}_i)^2}{\sigma^3} \right)$$

$$+ |\sqrt{g} g^{ij}| \left(\frac{1}{2} |\phi''| \frac{(x_j - \tilde{x}_j)^2}{\sigma^2} + \frac{1}{2} \phi' \frac{(x_j - \tilde{x}_j)^2}{\sigma^3} \right)$$

$$\le \delta_1 \left(\frac{1}{2} |\phi''| \frac{(x_i - \tilde{x}_i)^2}{\sigma^2} + \frac{1}{2} \phi' \frac{(x_i - \tilde{x}_i)^2}{\sigma^3} \right)$$

$$+ \delta_1 \left(\frac{1}{2} |\phi''| \frac{(x_j - \tilde{x}_j)^2}{\sigma^2} + \frac{1}{2} \phi' \frac{(x_j - \tilde{x}_j)^2}{\sigma^3} \right), \text{ for } i \ne j.$$

Then

$$\left| \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sqrt{g} \sum_{j=1}^{N} g^{ij} \frac{\partial \phi}{\partial x_j} \right) \right| \leq 2|\phi''| + \frac{2(N-1)}{\sigma} \phi' + \frac{C_{\Phi,\Omega} \phi'}{\sigma}. \qquad (4.24)$$

From now, let us fix $\theta_* \in (0,\theta)$ such that $\frac{1}{2} \leq \sqrt{g} \leq \frac{3}{2}$ and (4.24) holds. Define

$$A_n^* := \{x \in \overline{\Omega} \cup \Omega^{\theta *} : d_n(x) \geq R\lambda_n^{-1/2}\}.$$

In view of (4.5), for every $\epsilon \in (0,1)$ small, to be chosen later, there exist $R^* > 0$ and $n_R \in \mathbb{N}$ large such that

$$\max_{\{x \in \overline{\Omega}: d_n(x) \geq R\lambda_n^{-1/2}\}} |u_n| \leq \lambda_n^{\frac{1}{p-2}} \epsilon, \ \forall R > R^*, \ \forall n \geq n_R.$$

Let $A_n := \{x \in \overline{\Omega} : d_n(x) \geq R\lambda_n^{-1/2}\}$. Clearly,

$$|u_n(x)|^{p-2} \leq \lambda_n \epsilon^{p-2}, \ x \in A_n, \ \forall n \geq n_R.$$

Note that there exists a constant $n_\theta > 0$ large enough such that for $n > n_\theta$, for any $i \in \{1,\ldots,k\}$, we have $B_{\lambda_n^{-1/2}R}(P_n^i) \subset \Omega^{\theta *} \cup \Omega_{\theta_*}$. Then we can deduce that $A_n \subset A_n^*$, for $n > \max\{n_\theta, n_R\} := n_{\theta R}$. From the definition of $\bar{u}_n$, we get

$$|\bar{u}_n(x)|^{p-2} \leq \lambda_n \epsilon^{p-2}, \ x \in A_n^*, \ \forall n > n_{\theta R}.$$

Using (f_1)–(f_2), we conclude that

$$-div\,(A(x, \nabla \bar{u}_n)) + \frac{\lambda_n}{2} \bar{u}_n \leq 0, \ \forall x \in A_n^* \qquad (4.25)$$

holds for sufficiently small $\epsilon > 0$.

For any $x_0 \in A_n$ such that $B_r(x_0) \subset A_n^*$, consider the function

$$\phi_n(\sigma) = \phi_n(|x - x_0|) = \lambda_n^{\frac{1}{p-2}} \frac{\cosh\left(\gamma \lambda_n^{\frac{1}{2}} \sigma\right)}{\cosh\left(\gamma \lambda_n^{\frac{1}{2}} r\right)}.$$

Clearly, $\phi'(\rho) > 0$ and $\phi''(\rho) > 0$. By direct computations, we obtain

$$2|\phi_n''| + \frac{2(N-1)}{\sigma} \phi_n' + \frac{C_{\Phi,\Omega}}{\sigma} \phi_n' - \frac{\lambda_n}{2} \phi_n$$

$$= \lambda_n \phi_n \left(2\gamma^2 + (2(N-1) + C_{\Phi,\Omega})\gamma^2 \frac{\tanh\left(\gamma \lambda_n^{\frac{1}{2}} \sigma\right)}{\gamma \lambda_n^{\frac{1}{2}} \sigma} - \frac{1}{2} \right) \qquad (4.26)$$

$$\leq \lambda_n \phi_n \left(2\gamma^2 + (2(N-1) + C_{\Phi,\Omega})\gamma^2 - \frac{1}{2} \right) \leq 0,$$

for any $\gamma \in (0, \gamma_*]$, where $\gamma_* = (4N + 2C_{\Phi, \Omega})^{-\frac{1}{2}}$. Then, fixing $\gamma \in (0, \gamma_*]$, by (4.24) and (4.26) it follows that

$$div(A(x, \nabla\phi_n)) - \frac{\lambda_n}{2}\phi_n = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\sqrt{g}\sum_{j=1}^{N}\bar{g}^{ij}\frac{\partial\phi}{\partial x_j}\right) - \frac{\lambda_n}{2}\phi_n \leq 0, \ \forall x \in B_r(x_0).$$
(4.27)

In addition, for $x \in \partial B_r(x_0)$, we have $(\phi_n - \bar{u}_n) \geq \lambda_n^{\frac{1}{p-2}}(1-\epsilon) > 0$. Then, together with (4.25) and (4.27), by the comparison principle ([52, theorem 10.1]) it follows that $\bar{u}_n \leq \phi_n$ in $B_r(x_0)$, which implies that $u_n(x_0) \leq \lambda_n^{\frac{1}{p-2}}e^{-\gamma\lambda_n^{\frac{1}{2}}r}$.

Take $r = dist(x_0, \partial A_n^*)$. We distinguish the following two cases:

(i) $dist(x_0, \partial A_n^*) = dist(x_0, B_{R\lambda_n^{-1/2}}(P_n^i))$ for some $i \in \{1, \ldots, k\}$;
(ii) $dist(x_0, \partial A_n^*) = dist(x_0, \partial(\overline{\Omega} \cup \Omega^{\theta*}))$.

For case (i), we have $|x_0 - P_n^i| = r + R\lambda_n^{-\frac{1}{2}}$. Then

$$u_n(x_0) \leq e^{\gamma R}\lambda_n^{\frac{1}{p-2}}e^{-\gamma\lambda_n^{\frac{1}{2}}r-\gamma R} = e^{\gamma R}\lambda_n^{\frac{1}{p-2}}e^{-\gamma\lambda_n^{\frac{1}{2}}|x_0-P_n^i|}.$$

For case (ii), we deduce that

$$u_n(x_0) \leq \lambda_n^{\frac{1}{p-2}}e^{-\gamma\lambda_n^{\frac{1}{2}}\theta_*} \leq \lambda_n^{\frac{1}{p-2}}e^{-\gamma\lambda_n^{\frac{1}{2}}\frac{\theta_*}{diam(\Omega)}|x_0-P_n^i|}.$$

Hence, since x_0 is arbitrary, we conclude that there exist $C_1, C_2 > 0$ such that

$$u_n(x) \leq C_1 e^{C_1 R}\lambda_n^{\frac{1}{p-2}}\sum_{i=1}^{k}e^{-C_2\lambda_n^{\frac{1}{2}}|x-P_n^i|}, \ \forall x \in A_n.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Proof of theorem 1.1

In this section, we complete the proof of theorem 1.1. We have previously established a sequence of mountain pass type critical points $\{u_{\rho_n}\}$ of J_{ρ_n} on $\mathcal{S}_c$, which have uniformly bounded Morse indices. These critical points are constructed for a sequence $\rho_n \to 1^-$. Building on the blow-up analysis conducted in §4, we now present the following proposition.

PROPOSITION 5.1. *Let* $\{u_n\} \subset H^1(\Omega)$ *be a sequence of solutions to* (4.1), *corresponding to some* $\{\lambda_n\} \subset \mathbb{R}$ *and* $\rho_n \to 1^-$. *Suppose that*

$$\int_{\Omega}|u_n|^2 dx = c, \ m(u_n) \leq 2, \ \forall n,$$

for some $c > 0$, *and the energy levels* $\{c_n := J_{\rho_n}(u_n)\}$ *are bounded. Then, the sequence of solution pairs* $\{(u_n, \lambda_n)\}$ *is bounded in* $H^1(\Omega) \times \mathbb{R}$. *Moreover,* $\{u_n\}$ *is a bounded Palais–Smale sequence for the functional* J *constrained on* $\mathcal{S}_c$ *at level* c_1.

Proof. First, we show that $\{\lambda_n\}$ is bounded. Suppose, by contradiction, that $\lambda_n \to +\infty$. By lemma 4.4, there exist at most k blow-up limits $\{P_n^1\}, \ldots, \{P_n^k\}$ with $k \leq 2$ in $\overline{\Omega}$. In the following, we denote by $\{v_n^i\}$ the scaled sequence around $\{P_n^i\}$, and by v^i the limits of $\{v_n^i\}$. Note that for these blow-up points $\{P_n^1\}, \ldots, \{P_n^k\}$, it may hold

$$\lambda_n^{\frac{1}{2}} dist(P_n^i, \partial\Omega) \to +\infty \ \ or \ \ \lambda_n^{\frac{1}{2}} dist(P_n^i, \partial\Omega) \to d \geq 0.$$

Without loss of generality, assume there exists an integer $k_1 \in \{0, 1, \ldots, k\}$ such that

$$\lambda_n^{\frac{1}{2}} dist(P_n^i, \partial\Omega) \to +\infty \ \ for \ every \ \ i \in \{1, \ldots, k_1\},$$

and

$$\lambda_n^{\frac{1}{2}} dist(P_n^j, \partial\Omega) \to d \geq 0 \ \ for \ every \ \ j \in \{k_1 + 1, \ldots, k\}.$$

On the one hand, we can deduce that for any $R > 0$,

$$\left| \lambda_n^{\frac{N}{2} - \frac{2}{p-2}} \int_\Omega u_n^2 dx - \sum_{i=1}^{k_1} \int_{B_R(0)} |v_n^i|^2 dx - \sum_{i=k_1+1}^{k} \int_{B_R(0) \cap \Omega_n} |v_n^i|^2 dx \right| \to +\infty. \quad (5.1)$$

In fact, since $p \in (2_*, 2^*)$, the first term satisfies

$$\left| \lambda_n^{\frac{N}{2} - \frac{2}{p-2}} \int_\Omega u_n^2 dx \right| = \lambda_n^{\frac{N}{2} - \frac{2}{p-2}} c \to +\infty.$$

By lemma 4.4, we have

$$\sum_{i=1}^{k_1} \int_{B_R(0)} |v_n^i|^2 dx \to \sum_{i=1}^{k} \int_{B_R(0)} |v^i|^2 dx,$$

and

$$\sum_{i=k_1+1}^{k} \int_{B_R(0) \cap \overline{\Omega}_n} |v_n^i|^2 dx \to \sum_{i=1}^{k} \int_{B_R(0) \cap \mathbb{R}_+^N} |v^i|^2 dx,$$

which imply that (5.1) holds.

On the other hand, by lemma 4.5, there exist constants $C, C' > 0$ such that

$$\left| \lambda_n^{\frac{N}{2} - \frac{2}{p-2}} \int_\Omega u_n^2 dx - \sum_{i=1}^{k_1} \int_{B_R(0)} |v_n^i|^2 dx - \sum_{i=k_1+1}^{k} \int_{B_R(0) \cap \Omega_n} |v_n^i|^2 dx \right|$$

$$= \lambda_n^{\frac{N}{2} - \frac{2}{p-2}} \int_{\Omega \setminus \cup_{i=1}^{k} (B_{R\lambda_n^{-1/2}}(P_n^i) \cap \Omega)} u_n^2 dx$$

$$\leq C_1 e^{2C_1 R} \lambda_n^{\frac{N}{2}} \sum_{i=1}^{k} \int_{\mathbb{R}^N \setminus \cup_{i=1}^{k} B_{R\lambda_n^{-1/2}}(P_n^i)} e^{-2C_2 \lambda_n^{\frac{1}{2}} |x - P_n^i|} dx$$

$$\leq C e^{CR} \sum_{i=1}^{k} \int_{\mathbb{R}^N \setminus B_R(0)} e^{-2C_2 |y|} dy \leq C' = C'(R).$$

Taking $n \to +\infty$, we obtain a contradiction to (5.1). Hence, $\{\lambda_n\}$ is bounded.

Now, we show that $\{u_n\}$ is bounded in $H^1(\Omega)$. By contradiction, we suppose that $\|u_n\| \to \infty$. Using standard arguments, we have $\|u_n\|_{L^\infty} \to \infty$. Take $P_n \in \overline{\Omega}$ such that $|u_n(P_n)| = \|u_n\|_{L^\infty}$. Let $\tilde{\epsilon}_n = a_0^{-\frac{1}{2}} |u_n(P_n)|^{-\frac{p-2}{2}}$ and define

$$\tilde{u}_n(x) := a_0^{\frac{1}{p-2}} \tilde{\epsilon}_n^{\frac{2}{p-2}} u_n(\tilde{\epsilon}_n x + P_n) \ \text{ for } \ x \in \tilde{\Omega}_n := \frac{\Omega - P_n}{\tilde{\epsilon}_n}.$$

Clearly, $\tilde{u}_n$ satisfies (4.8). Using the boundedness of $\{\lambda_n\}$, we get $\lambda_n \tilde{\epsilon}_n^2 \to 0$ as $n \to \infty$. Then, up to a subsequence, $\tilde{u}_n \to \tilde{u}$ in $C^2_{loc}(\overline{D})$, where $\tilde{u}$ is a finite Morse index solution of

$$\begin{cases} -\Delta \tilde{u} = |\tilde{u}|^{p-2} \tilde{u} & \text{in } D, \\ |\tilde{u}(x)| \leq |\tilde{u}(0)| = 1 & \text{in } D, \\ \frac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial D, \end{cases} \tag{5.3}$$

where D is either $\mathbb{R}^N$ or a half space. By invoking [22, theorem 2] and [57, proposition 2.1], respectively, we conclude that $\tilde{u} \equiv 0$. This contradicts to the fact that $|\tilde{u}(0)| = 1$. Therefore, we deduce that $\{u_n\}$ is a bounded sequence in $H^1(\Omega)$. Consequently, employing standard arguments, we establish that $\{u_n\}$ is a bounded Palais–Smale sequence for $J|_{\mathcal{S}_c}$ at level c_1. □

Completion of proof of theorem 1.1. Let $\{u_n\}$ be the sequence given in proposition 5.1 for some $c \in (0, c^*)$. By the compact embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for $r \in [1, 2^*)$, and using similar arguments as in §3, we can deduce that $u_n \to u$ strongly in $H^1(\Omega)$. This, in turn, implies that u is a mountain pass type normalized solution of (1.1). □

Acknowledgements

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