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GROUPS WITH ALL SUBGROUPS NORMAL-BY-FINITE

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Abstract

A group G has all of its subgroups normal-by-finite if $H/\operatorname{core}_G(H)$ is finite for all subgroups H of G. These groups can be quite complicated in general, as is seen from the so-called Tarski groups. However, the locally finite groups of this type are shown to be abelian-by-finite; and they are then boundedly core-finite, that is to say, there is a bound depending on G only for the indices $|H : \operatorname{core}_G(H)|$.

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1. Introduction

In [5] it was proved that if G is a group such that each of its subgroups has finite index in its normal closure, then G is finite-by-abelian, and so the index of each subgroup in its normal closure is bounded. In this paper we shall be concerned with a dual property. We shall say that a group G is a CF-group (core-finite) if each of its subgroups is normal-by-finite, that is, if $H/\operatorname{core}_G(H)$ is finite for all subgroups H of G. That such groups need not even be abelian-by-finite is indicated by the existence of so-called Tarski groups, for instance the examples due to Rips and Ol'shanskii [7] of infinite groups all of whose proper nontrivial subgroups have prime order. A suitable torsion-free example is provided by the construction due to Adian (unpublished) of a group with infinite cyclic centre and with central factor group a Tarski group. Although such examples may seem at first sight a little extravagant in the context of our discussion, nevertheless we shall see later (in Section 4) that any periodic CF-group which is not

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abelian-by-finite has a finitely generated infinite section in which every subgroup is either finite or of finite index.

Before stating our main result we note that the class of CF-groups is an extension of the class of Dedekind groups, groups in which all subgroups are normal. We recall that a non-abelian Dedekind group is called hamiltonian and that such a group has an abelian (periodic) subgroup of index 2. A complete description of such groups, due to Dedekind and Baer, is given as 5.3.7 of [9]. It is not surprising to find that such groups will turn up during our considerations (see the proof of Lemma 3.1).

For any subgroup H of a group G, denote by $\sigma(H)$ the index of $\mathrm{core}_G H$ in H.

We shall say that a CF-group is BCF(boundedly core-finite) if there is an integer n such that $H/\mathrm{core}_G(H)$ has order at most n for all $H \leq G$. Our main result is as follows.

THEOREM. *Every locally finite* CF-*group is abelian-by-finite and* BCF.

We shall present the proof of this theorem in two main parts. In Section 2 we shall prove that every locally finite, abelian-by-finite CF-group is BCF and in Section 3 that every countable locally finite CF-group is abelian-by-finite. The theorem is then deduced as follows. Suppose that G is a locally finite CF-group and that G is not BCF. Assume to begin with that $\sigma(F) \leq n$, for some fixed integer n and for all finite subgroups F of G. Let H be an arbitrary subgroup of G and let $\{H_\lambda : \lambda \in \Lambda\}$ be the set of all finite subgroups of G. If C_λ denotes the core of H_λ in G, then $|H_\lambda : C_\lambda| \leq n$, for all λ, and it follows easily that every finite subgroup of $H/\mathrm{core}_G H$ has order at most n. Hence $\sigma(H) \leq n$ and G is BCF, a contradiction. Thus there exist finite subgroups $H_1, H_2, \ldots$ of G such that $\sigma(H_1) < \sigma(H_2) < \cdots$ Suppose that $\mathrm{core}_G H_i = \bigcap_{j=1}^{n_i} H_i^{g_{ij}}$, and form the subgroup G^* generated by all H_i and all g_{ij}. Then G^* is countable and $\mathrm{core}_{G^*}(H_i) = \mathrm{core}_G(H_i)$. So G^* is not BCF. But by Section 3, G^* is abelian-by-finite and so by Section 2 is BCF, a contradiction.

We remark here that 'abelian-by-finite' cannot, of course, be replaced by 'finite-by-abelian' in the statement of the theorem – a group A of type C_{p^∞} extended by an inverting two-cycle provides an appropriate example. We note also that CF-groups form a countably recognizable class (see [6]). Thus, if G is a group in which every countable subgroup is CF, then G is CF. We omit the proof (which is quite straightforward).

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2. Abelian-by-finite CF-groups

In this section we shall prove that every locally finite, abelian-by-finite CF-group is BCF. Throughout Section 2, G will be such a group, and we denote by A an abelian normal subgroup of finite index in G. Suppose that, for each prime p, there exists an integer n_p such that $\sigma(H) \leq n_p$ for all p-subgroups H of A. Then, for almost all p, every p-subgroup of A is normal in G; and it follows easily that G is BCF. From now on, therefore, we shall assume that A is a p-group.

DEFINITION. A subgroup H of G is G-hamiltonian if every (cyclic) subgroup of H is normal in G. H is almost G-hamiltonian if there is a G-hamiltonian subgroup of finite index in H.

LEMMA 2.1. *Every residually finite subgroup B of A is almost G-hamiltonian.*

PROOF. Let B be as stated and assume that the result is false for B. Since G is CF, we may assume that B is normal in G. There exists b_1 in B such that $\langle b_1 \rangle$ is not normal in G; since b_1 has only finitely many conjugates, it is contained in a finite, G-invariant subgroup B_1 of B. Since B is residually finite, there exists a subgroup N_1 of finite index in B such that $N_1 \cap B_1 = 1$; by the CF-property, we may suppose that N_1 is G-invariant. Since N_1 is not G-hamiltonian, there exists b_2 in N_1 such that $\langle b_2 \rangle$ is not G-invariant, and once again the normal closure B_2 of $\langle b_2 \rangle$ in G is finite. But $B_2 \leq N$ and there exists a G-invariant subgroup N_2 of B such that $N_2 \cap \langle B_1, B_2 \rangle = 1$. Now choose b_3 in N_2 with properties like those of b_1, b_2; and continue in this way to produce a direct product of finite G-invariant subgroups $B_1, B_2, \ldots$ of A such that each B_i contains a subgroup $\langle b_i \rangle$ which is not normal in G. Writing $H = \langle b_1, b_2, \ldots \rangle$, we obtain the contradiction that $\sigma(H)$ is infinite, thus completing the proof of the lemma.

It is clearly the case that every divisible subgroup of A is G-hamiltonian. Indeed, with a little work, we can show considerably more, namely that the finite residual of any abelian subgroup of a periodic CF-group G is G-hamiltonian. This fact is not required here and we postpone consideration of it until Section 4.

Our next result turns out to play a decisive role (and does not require that G be a CF-group).

LEMMA 2.2. *Suppose that B is a G-hamiltonian subgroup of A and let b, c be elements of B with $|b| = p^m \geq p^n = |c|$, $n \geq 1$. Suppose further that $g \in G$ and $b^g = b^\lambda$, $c^g = c^\mu (\lambda, \mu \in \mathbb{N})$. Then $\lambda \equiv \mu \bmod p^n$.*

PROOF. Since b is an element of maximal order in $\langle b, c \rangle$, we have $\langle b, c \rangle = \langle b \rangle \times \langle a \rangle$, for some a of order p^s, say. If $a^g = a^\epsilon$ for some $\epsilon \in \mathbb{N}$, we have $(ba)^g = b^\lambda a^\epsilon$. On the other hand, $(ba)^g = b^k a^k$ for some k, so that $\epsilon \equiv \lambda (\bmod p^s)$. Therefore $a^g = a^\lambda$. It follows that $c^g = c^\lambda$, so that $\lambda \equiv \mu \bmod p^n$.

LEMMA 2.3. *Suppose that there exists an integer k such that $\sigma(\langle x\rangle) \leq p^k$ for all $x \in A$. Then G is* BCF.

PROOF. Let B be a basic subgroup of A. Thus B is a direct product of cyclic subgroups which is pure in A, and A/B is divisible. Suppose that D is any subgroup of A such that DB/B has finite rank r, say. For each finite subgroup F of DB, there exist $x_1, \ldots, x_r \in A$ such that $F = (F \cap B)\langle x_1\rangle \cdots \langle x_r\rangle$. By Lemma 2.1, B has a G-hamiltonian subgroup of finite index p^t, say. Thus $\sigma(F) \leq p^{t+rk}$. Since F was arbitrary (finite), this shows that $\sigma(H)$ is bounded, for all $H \leq DB$.

Now suppose that F_1 is a finite subgroup of A with $\sigma(F_1) = n_1$, say. Then $F_1 \leq D_1B$, for some D_1 such that D_1B/B is divisible of finite rank r_1, say. Write $A/B = D_1B/B \times E_1B/B$, where E_1B/B is possibly trivial. If $\sigma(K)$ is bounded for all $K \leq E_1B$, then we may argue (almost) as above to obtain a bound for all subgroups H of A. Otherwise there exists a finite subgroup F_2 of E_1B such that $\sigma(F_2) = n_2 > n_1$. Assuming that G is not BCF, it is clear that we may construct finite subgroups $F_1, F_2, \ldots$ of A which generate their direct product modulo B and which are such that $\sigma(F_{i+1}) > \sigma(F_i)$, for all i. Set $H = \langle F_1, F_2, \ldots\rangle$. Since HB/B is a direct product of cyclic groups and B is pure in HB, a result of Kulikov (see Theorem 28.2 of [2]) applies to show that B is a direct factor of HB so that HB, and therefore H, is a direct product of cyclic groups and hence, by Lemma 2.1, almost G-hamiltonian. This contradicts the choice of the F_i and so completes the proof of the lemma.

As an easy consequence of Lemma 2.3 we have

COROLLARY 2.4. *If N is a G-invariant subgroup of A such that N has finite exponent and G/N is* BCF, *then G is* BCF.

PROOF. Suppose that $N^{p^t} = 1$ and that (with the obvious notation) $\sigma(HN/N) \leq p^k$ for all $H \leq A$. Let a be any element of A. Then $\langle a^{p^{k+t}}\rangle^G = \langle a^{p^{k+t}}\rangle$, and Lemma 2.3 applies.

Our next result, whose proof utilises Lemma 2.2, will allow us to focus attention on the case where A is reduced.

LEMMA 2.5. *Suppose that $C = C_{p^\infty}$ is a subgroup of A and that B is any subgroup of A such that $\sigma(H) \leq p^k$ for all $H \leq B$, for some fixed k. Suppose further that the group E generated by B and C is their direct product. Then, for some fixed l, $\sigma(H) \leq p^l$ for all $H \leq E$.*

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PROOF. Write $C = \langle c_1, c_2, \dots : c_1^p = 1, c_{i+1}^p = c_i, i = 1, 2, \dots \rangle$ and let F be a finite subgroup of E. Since C is G-hamiltonian and $F = (F \cap B)\langle x \rangle$, for some $x \in E$, it suffices to prove that, for some t, $\sigma(\langle x \rangle) \leq p^t$ for all $x \in E$. Replacing E by E^{p^k}, we may assume that B is G-hamiltonian.

Write $B = D \times R$, where D is divisible and R is reduced. If R has finite exponent p^m, then $x^{p^m} \in C \times D$ for all $x \in E$, and hence $\langle x^{p^m} \rangle \lhd G$. Thus we may assume that R has infinite exponent. We claim that E is then G-hamiltonian, and thus the statement of the lemma holds. All we need do in order to establish the claim is to prove the following:

$$(*) \text{If } c \in C, b \in B, |c| = |b| \text{ and } c^g = c^\lambda, \text{ then } b^g = b^\lambda.$$

In order to prove (*) we may replace B by any subgroup of B of infinite exponent. Since any basic subgroup of R has infinite exponent, we may then assume that $B = \langle b_1 \rangle \times \langle b_2 \rangle \times \cdots$, where $|b_i| = p^i$ for each i. We may further assume that $c = c_i, b = b_i$, for some i. Suppose that $c^g = c^\lambda$ for some $g \in G$ and write $H = \langle c_1 b_1, c_2 b_2, \dots \rangle$. Then, using the defining relations for C, we obtain $H \cap B = \langle b_1^{-1} b_2^p, b_2^{-1} b_3^p, \dots \rangle$. Furthermore, it is easy to see that $\langle b_j \rangle \cap H = 1$ for $j = 1, 2, \dots$ Using bars to denote factor groups modulo $H \cap B$, we have $\bar{E} = \bar{C} \times \bar{B}$, where $\bar{C} \cong \bar{B} \cong C_{p^\infty}$. Thus $\bar{E}$ is G-hamiltonian and $|\bar{c}| = |\bar{b}| = p^i$, and so, by Lemma 2.2, $\bar{b}^{\bar{g}} = \bar{b}^\lambda$, that is, $b^g = b^\lambda h$ for some $h \in H \cap B$. But $\langle b \rangle$ is normal in G and $\langle b \rangle \cap H = 1$ and so $h = 1$, as required.

Most of the remainder of the proof is occupied with establishing the next assertion.

LEMMA 2.6. *If A is reduced then G is* BCF.

PROOF. Assume that A is reduced and let B be a basic subgroup of A. If A has finite exponent, then Lemma 2.1 applies. Otherwise B has infinite exponent. Suppose first that there exists an infinite sequence $C_1, C_2, \dots,$ of subgroups of A with the following properties:

(i) each C_i contains B and C_i/B is divisible of finite rank;
(ii) the subgroups C_i generate their direct product modulo B;
(iii) there exist finite subgroups F_i of C_i, $i = 1, 2, \dots,$ such that $\sigma(F_1) < \sigma(F_2) < \cdots$

Then, as in the proof of Lemma 2.3, we may consider the subgroup $\langle B, F_1, F_2, \dots \rangle$ and obtain a contradiction. It follows that there exists an integer k and subgroups C and D of A with the following properties.

(iv) $B \leq C \cap D$ and C/B is divisible of finite rank;
(v) $A/B = C/B \times D/B$, (where, possibly, C or D is equal to B);

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(vi) $\sigma(H) \leq p^k$, for all $H \leq D$.

Let $R = R(C)$ be the finite residual of C and let C_0 be a G-invariant subgroup of finite index in C. Then $R(C_0) = R(C) \leq R(A) = \bigcap_{n=1}^{\infty} A^{p^n}$, and hence $R \cap B = 1$ and R is finite (since C/B has finite rank and A is reduced).

By Corollary 2.4, it is enough to show that G/R is BCF, so we may assume that G is residually finite, and hence almost G-hamiltonian, by Lemma 2.1. Then there exists an integer t such that $\sigma(H) \leq p^t$ for all $H \leq C$. Let $n = \max(t, k)$ and write $X = A^{p^n}$, $Y = C^{p^n}$, $Z = D^{p^n}$. Then $Y \cap Z \geq B^{p^n}$, which has infinite exponent, while Y and Z are G-hamiltonian. Let x be an arbitrary element of X and write $x = yz$ for some $y \in Y$, $z \in Z$. If $|y| \geq |z|$, choose $b \in B^{p^n}$ such that $|b| = |y|$ and suppose that $y^g = y^\lambda$ for some $g \in G$, $\lambda \in \mathbb{N}$. By Lemma 2.2, $b^g = b^\lambda$, and then $z^g = z^\lambda$ and $x^g = x^\lambda$. A similar argument applies if $|y| \leq |z|$. We have thus shown that X is G-hamiltonian and hence that $\langle a^{p^n} \rangle \lhd G$ for all $a \in A$. Lemma 2.3 gives the desired result.

2.7 THE FINAL STEP. With the usual hypotheses, suppose now that $A = D \times R$, where D is divisible and R is reduced. We may assume that R is normal in G. If D has finite rank, then Lemmas 2.6 and 2.5 (and an easy induction) give the result. Suppose then that D has infinite rank and, for a contradiction, that G is not BCF. Given any $n_1 \in \mathbb{N}$ then, as we saw in the introduction, there exists a finite subgroup F_1 of A such that $\sigma(F_1) \geq n_1$. We have $F_1 \leq R \times D_1$, for some finite rank direct factor D_1 of D. Write $D = D_1 \times B_1$. Then there exists a finite subgroup F_2 of $R \times B_1$ such that $\sigma(F_2) \geq n_2 > n_1$. Continuing this process, we obtain a sequence $F_1, F_2, \ldots$ of finite subgroups such that $H = \langle R, F_1, F_2, \ldots \rangle$ is reduced and $\sigma(F_i) < \sigma(F_{i+1})$ for all i. Again applying Lemma 2.6 (to a suitable subgroup of G), we obtain our final contradiction.

3. Proof of the Theorem

Let G be a countable locally finite CF-group. Our aim is to prove that G is abelian-by-finite. We first reduce to the case where G is a p-group.

LEMMA 3.1. *If every p-subgroup of G is abelian-by-finite (for all primes p) then G is abelian-by-finite.*

PROOF. Let G be as stated and let $\bar{G}$ be any infinite image of G. Then $\bar{G}$ contains an infinite abelian subgroup (Hall and Kulatilaka, [3]) and hence an infinite normal abelian subgroup. It follows easily that G is hyperabelian-by-finite and thus we may assume that G is hyperabelian and hence locally soluble. Let π be the set of primes p such that G has an element of order p.

Suppose first that $\pi = \{p_1, \ldots, p_k\}$, a finite set. For each $i = 1, \ldots, k$, let P_i be a maximal p_i-subgroup of G. By hypothesis, there is a G-invariant abelian subgroup A_i of finite index P_i. Then $A = A_1 \times \cdots \times A_k$ is abelian and G/A is finitely generated and hence finite. Thus we may assume that $\pi = \{p_1, p_2, \ldots\}$ is infinite. For each pair $\{p, q\}$ of primes in π we shall say that $p \sim q$ if and only if all p-elements of G commute with all q-elements of G.

Let Ω be any infinite subset of π and write $\Omega = \Omega_1 \dot{\cup} \Omega_2$, where both Ω_1 and Ω_2 are infinite. Thus $\Omega_2 \subseteq \Omega_1' = \pi - \Omega_1$. Since G is countable and locally (finite soluble) we may apply a result of Schenkman to deduce that there exists an Ω_1-subgroup S and an Ω_1'-subgroup T of G such that $G = ST$. (The above (unpublished) result constitutes a rather straightforward generalisation of the corresponding result of P. Hall for finite soluble groups.) By the CF-property, there exists G-invariant subgroups S^*, T^* of finite index in S, T respectively. Hence there are (distinct) primes p, q in Ω_1, Ω_2 respectively such that $p \sim q$. Then every infinite set Ω contains distinct primes p, q with $p \sim q$. Applying Ramsey's Theorem [11], we deduce that Ω contains an infinite subset Λ such that $p \sim q$ for all distinct $p, q \in \Lambda$.

Now suppose that there is an infinite subset $\Sigma = \{q_1, q_2, \ldots\}$ of π such that a maximal q_i-subgroup Q_i (say) fails to be normal in G, for each i. We may assume $q_i \sim q_j$ for all $i \neq j$. Let $Q = \langle Q_i : i = 1, 2, \ldots \rangle = \mathrm{Dr}\, Q_i$ and let Q^* be a G-invariant subgroup of finite index in Q. Then Q^* contains all but finitely many of the Q_i, which are therefore normal in G, a contradiction. Hence, for almost all $p_i \in \pi$, the maximal p_i-subgroups P_i are normal in G. Indeed, by the CF-property again, it is easy to see that almost all of the P_i are G-hamiltonian and hence almost all are abelian. Let $P = \langle P_i : P_i \lhd G \text{ and } P_i \text{ abelian} \rangle$. As before, there is a subgroup H of G such that $G = PH$, where $\pi(P) \cap \pi(H) = \emptyset$. Since $\pi(H)$ is finite, H is abelian-by-finite. This concludes the proof of the lemma.

From now on we assume that G is a p-group. Our next reduction is to the case where G is soluble.

LEMMA 3.2. *Let A be a normal subgroup of G and let X be the subgroup consisting of all elements of A which have only finitely many conjugates in G. If A is a direct product of cyclic subgroups, then A/X is finite.*

PROOF. Let A, X be as stated and suppose, for a contradiction, that A/X is infinite. We distinguish two cases.

(i) A/X has finite rank. Using the CF-property, we may as well assume that $A/X \cong C_{p^\infty}$. Since every subgroup of A is also a direct product of cycles (Kulikov, see [2]) we may choose elements $a_1, a_2, \ldots$ of A such that $A = \langle X, a_1, a_2, \ldots \rangle$ and such that $\langle a_1, a_2, \ldots \rangle = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots$. Further, we may suppose that $|a_i X| < |a_{i+1} X|$, for all i. Partition the set of all a_i into infinitely many, disjoint infinite subsets S_j and

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let A_j be the subgroup generated by S_j. Then each A_j has a G-invariant subgroup B_j of finite index. If every B_j fails to be G-hamiltonian then, for each j, we may choose $b_j \in B_j$, $g_j \in G$ such that $\langle b_j \rangle \neq \langle b_j \rangle^{g_j}$. Set $B = \langle b_j : j = 1, 2, \dots \rangle$. Since the B_j generate their direct product, we see that B has infinite index over its core, a contradiction. Thus some B_j is G-hamiltonian and hence contained in X. However, B_j has finite index in A_j and $\langle A_j, X \rangle = A$. This gives a contradiction.

(ii) A/X has infinite rank. We may assume that A/X has exponent p, and choose elements $a_1, a_2, \dots$ of A such that $\langle a_1, a_2, \dots \rangle = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots$ and such that A/X is the direct product of the $\langle a_i X \rangle$. Now let A_j, B_j be defined as in (i). An identical argument shows that some B_j is contained in X, resulting in the contradiction that $|\langle A_j, X \rangle : X|$ is finite. The lemma is thus proved.

The above lemma shows that certain abelian subgroups are "almost" contained in the FC-centre of G. If every element of G has finitely many conjugates then of course G is said to be an FC-group. The following result is true for any CF-group G: it is a special case of Theorem 7.20 of Tomkinson [10].

LEMMA 3.3. *If G is an* FC-*group (and* CF*), then G is centre-by-finite.*

We are now able to prove the following.

LEMMA 3.4. *If every soluble section of G is abelian-by-finite then G is abelian-by-finite.*

PROOF. Let N be the subgroup generated by all normal abelian subgroups of G and suppose that N is abelian-by-finite. If G/N is infinite then it contains an infinite normal abelian subgroup B/N, by [3] and the CF-property. But B is soluble and so, by hypothesis, abelian-by-finite. This contradicts the definition of N. We may thus assume that $N = G$. In particular, G is then a Fitting group (that is, the normal closure of every element is nilpotent).

Let A be any normal abelian subgroup of G and let B be a G-invariant subgroup of finite index in a basic subgroup of A. Let F be the FC-centre of G. By Lemma 3.2, BF/F is finite and so AF/F is divisible-by-finite. By the CF-property, every C_{p^∞}-type subgroup of G/F is normal and hence, because G is a Fitting group, central in G/F. Let Z/F be the centre of G/F. Since G is generated by abelian normal subgroups, we see that G/Z is a product of finite normal subgroups and hence, by Lemma 3.3, G/Z is soluble. But F is also soluble, giving G soluble and hence abelian-by-finite, as claimed.

Thus we need consider only the case where G is soluble. An easy induction allows us to assume that G is metabelian. We shall therefore suppose for the remainder of this section that G is a metabelian p-group. We proceed to dispose of some special cases.

LEMMA 3.5. *Suppose that G is a Fitting group and that G' has infinite exponent. Then G is abelian-by-finite.*

PROOF. Since G' has infinite exponent, it has a homomorphic image isomorphic to C_{p^∞}. By the CF-property, there exists a normal subgroup M of G contained in G' such that $G'/M = C/M \times F/M$, where $C/M \cong C_{p^\infty}$ and F/M is finite. Replacing M by F^G, if necessary, we may assume that $F/M = 1$.

Write $H = G/M$ and let N be a normal subgroup of H maximal subject to intersecting H' trivially. If H/N has infinite rank then, by a result of Kargapolov (see [8]), it has an abelian subgroup $L = K/N$ of infinite rank. So L is a countable abelian p-group of infinite rank. We show that L can be written as a direct product of infinitely many non-trivial groups and therefore of infinitely many infinite groups. If the divisible part of L has infinite rank, this is easy. So we may assume that L is reduced of infinite rank. If L has finite exponent, L is a direct sum of cycles, which is enough for us. Hence we may assume that L has infinite exponent.

By Theorem 5.2 of [1], L has a direct factor which is a direct sum of cycles and is of infinite exponent. Thus, in either case, we have $L = K/N = K_1/N \times K_2/N \times \cdots$, where each K_i/N is infinite. By the CF-property we may assume that each K_i is normal in H. But $K_i \cap H'$ is non-trivial for all i, and we have a contradiction. Thus H/N has finite rank and is therefore (divisible abelian)-by-finite and hence centre-by-finite, since H is a Fitting group. This gives H centre-by-finite and H' finite, a contradiction.

LEMMA 3.6. *If G is nilpotent then G is abelian-by-finite.*

PROOF. By the previous lemma we may assume that G' has finite exponent. An easy induction allows us to assume that this exponent is p. We may also suppose that G has nilpotency class 2, so that G^p is central in G. Suppose for a contradiction that G is not abelian-by-finite. Let Z, F denote the centre and FC-centre of G, respectively. Then, by Lemma 3.3, $|G : F|$ is infinite.

We claim that, given any integer $k \geq 2$, there exist elements $x_1, \dots, x_k$ of G such that, for each $l = 1, \dots, k-1$, $x_{l+1} \notin F\langle x_1, \dots, x_l\rangle$ and such that the elements $[x_i, x_j]$, $1 \leq i < j \leq k$, are linearly independent (over $\mathbb{Z}_p$).

Consider first the case $k = 2$. Choose $x_1 \in G\backslash F$ and note that $F\langle x_1\rangle$ and $C_G(x_1)$ each has infinite index in G. Since no group is the union of two proper subgroups, $G \neq F\langle x_1\rangle \cup C_G(x_1)$ and so we may choose $x_2 \notin F\langle x_1\rangle \cup C_G(x_1)$.

Assume that, for some $k \geq 2$, elements $x_1, \dots, x_k$ have been chosen so as to satisfy the conditions stated in the claim. Let $X = \langle x_1, \dots, x_k\rangle$. Suppose that $g \in F \cap X$

and write $g = x_1^{n_1} \cdots x_r^{n_r} z$ for some non-negative integers n_i, some $r \leq k$ and $z \in Z$. Then $x_r^{n_r} \in F\langle x_1, \dots, x_{r-1}\rangle$ and so p divides n_r. It follows easily that $F \cap X \leq Z$.

Suppose next that, for all $x \in G\backslash FX$, $[x, X]$ has rank less than k, modulo X'. For each such x, the map $\theta_x : X/X \cap Z \to Z/X'$, defined by $\theta_x(y(X \cap Z)) = [x, y]X'$, for all $y \in X$, is a homomorphism with non-trivial kernel. Thus $G = FX \cup \bigcup_{y \in X\backslash Z} C^*(y)$, where $C^*(y) = \{x \in G : [x, y] \in X'\}$. Applying [4], we deduce that $C^*(y)$ has finite index in G for some $y \in X\backslash Z$. If follows that $y \in F \cap X$ and thus that $y \in Z$, a contradiction. Thus, for some element x_{k+1} of $G\backslash FX$, $[x_{k+1}, X]$ has rank at least k modulo X', which means that the set $\{[x_i, x_j] : 1 \leq i < j \leq k+1\}$ is independent. By induction, therefore, the claim is established. We may thus construct an infinite subset $S = \{x_1, x_2, \dots\}$ of G such that $\{[x_i, x_j] : i < j\}$ is independent. Let us call any such (infinite) subset S of G an L-set. Clearly any L-set of G generates a subgroup which is not abelian-by-finite. For each element g and each subgroup H of G, let $\bar{g}$, $\bar{H}$ denote gG' and HG'/G' respectively.

Our aim now is to construct an L-set T such that $\overline{\langle T\rangle}$ is the direct product of all the $\overline{\langle t\rangle}$ such that $t \in T$. Let S be defined and suppose that $U_k = \{x_1, \dots, x_k\}$ is such that $\overline{\langle U_k\rangle}$ is the direct product of the $\overline{\langle x_i\rangle}$, $(i = 1, \dots, k)$. If there is an L-set T which contains U_k and is such that, for some $t \in T\backslash U_k$, $\overline{\langle t\rangle} \cap \overline{\langle U_k\rangle} = 1$, then we may write $x_{k+1} = t$, $U_{k+1} = U_k \cup \{t\}$, thus extending our direct product by an appropriate cyclic factor. But, assuming that there is no such L-set, we may assume (relabelling if necessary) that, for all $i, j > k$, $\overline{\langle X_i\rangle} \cap \overline{\langle U_k\rangle} = \overline{\langle X_j\rangle} \cap \overline{\langle U_k\rangle}$. Further, if $q_i = p^{r_i}$ is the order of x_i mod $\overline{\langle U_k\rangle}$, we may assume that $\bar{x}_i^{q_i} = \bar{x}_j^{q_j}$ for all $i, j > k$. Among all L-sets S satisfying these further properties, choose one containing an element $x \notin U_k$ such that $\bar{x}$ has minimal order p^r, say. Now choose $y \in S\backslash(U_k \cup \{x\})$. If $|\bar{y}| = p^s$, then $s \geq r$. Also, $\bar{x}^{p^{r-1}} = \bar{y}^{p^{s-1}}$ (by the manner in which S was constructed and since each of these elements has order p). But now we have $(\bar{x}(\bar{y}^{-1})^{p^{s-r}})^{p^{r-1}} = 1$ and, since replacing the pair $\{x, y\}$ by $x(y^{-1})^{p^{s-r}}$ still leaves us with an L-set, we have a contradiction to the choice of S. Thus we may certainly extend our set U_k to an appropriate set U_{k+1}, and by induction we may construct an L-set S such that $\overline{\langle S\rangle} = \operatorname{Dr}_{x \in S} \overline{\langle x\rangle}$. Again write $S = \{x_1, x_2, \dots\}$ and, for each i, let $n_i = p^{s_i}$ be the order of $\bar{x}_i$. We may as well assume that $n_i \leq n_{i+1}$ for all i. We shall say that an element σ of $\langle S\rangle'$ *involves* $[x_k, x_l] (k < l)$ if $[x_k, x_l]$ appears (nontrivially) in the expression for σ as a reduced word in the basis elements $[x_i, x_j] (i < j)$. Set $\Omega = \{[x_i, x_j] : i < j\}$, and write $G' = \langle\Omega\rangle \times N$ (for some N).

Now write $x_1^{n_1} = \sigma_1 z_1$, say, where $\sigma_1 \in \langle\Omega\rangle$, $z_1 \in N$. Let X_1 be a (finite) subset of S satisfying the following:

(i) $X_1 = \{x_1, \dots, x_{k_1}\}$ for some k_1.
(ii) If σ_1 involves $[x_i, x_j]$, then $i < j \leq k_1$.

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(iii) There exists $w_1 \in X_1 \backslash \{x_1\}$ such that no (nontrivial) $[w_1, x_j]$ or $[x_j, w_1]$ is involved in σ_1.

Write $y_1 = x_1$, $m_1 = n_1$, $\Omega_1 = \{[x_i, x_j] : i < j \leq k_1\}$, $A_1 = \langle \Omega_1 \rangle$ and $B_1 = \langle \Omega_1' \rangle$, so $G' = A_1 \times B_1 \times N$. Since infinitely many of the $x_i^{n_i}$, $i > k_1$, are congruent mod $B_1 N$, we may assume that they all are.

We need to simplify our notation a little for the next step. Write $y_2 = x_{k_1+1}$, $m_2 = n_{k_1+1}$, $v = x_{k_1+2}$, $l = n_{k_1+2}$. So we have $y_2^{m_2} \equiv v^l \bmod B_1 N$, which gives $(y_2 v^{-l/m_2})^{m_2} \in B_1 N$. (This is immediate if $m_2 > 2$, for then $(gh)^{m_2} = g^{m_2} h^{m_2}$ for all $g, h \in G$. For $m_2 = 2$ we obtain $(gh)^2 = g^2 h^2 [g, h]$, but $[y_2, v] \in B_1$, by the definition of B_1.) Replacing the pair $\{v, y_2\}$ by $y_2 v^{-l/m_2}$ if necessary, we may assume that $y_2^{m_2} \in B_1 N$. Now write $y_2^{m_2} = \sigma_2 z_2$, where $\sigma_2 \in B_1$, $z_2 \in N$, and let X_2 be a (finite) subset of S satisfying the following.

(i) $X_1 \subseteq X_2 = \{x_1, \ldots, x_{k_2}\}$ for some k_2.
(ii) If σ_2 involves $[x_i, x_j]$, then $i < j \leq k_2$.
(iii) There exists $w_2 \in X_2 \backslash (X_1 \cup \{y_2\})$ such that no (nontrivial) $[w_2, x_j]$ or $[x_j, w_2]$ is involved in σ_2.

Write $\Omega_2 = \{[x_i, x_j] : i < j \leq k_2\}$, $A_2 = \langle \Omega_2 \rangle$ and $B_2 = \langle \Omega_2' \rangle$; so $G' = A_2 \times B_2 \times N$.

Continuing in the obvious manner (and with the obvious notation) we obtain infinite subsets $\{w_1, w_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$. Let $H = \langle y_1, y_2, \ldots \rangle$, $C = \operatorname{core}_G H$. Certainly HZ/Z is infinite, and so $C \leq Z$. Choose $c \in C \backslash Z$ and write $c = y_{i_1}^{\alpha_1} \ldots y_{i_t}^{\alpha_t} z$, where $i_1 < \cdots < i_t$, $0 < \alpha_j < p$ for all j, and $z \in Z$. Let $r = i_1$, $\alpha = \alpha_1$. Then $[w_r, c] = [w_r, y_r]^\alpha [w_r, y]$, for some $y \in S \backslash X_r$; indeed we have $[w_r, y] \in \langle \Omega_r' \rangle = B_r$, while $[w_r, y_r] \in A_r$. This means that $[y_r, w_r]$ is involved in $[w_r, c]$. Now $[w_r, c] \in H \cap G'$, which by construction is just KH', where $K = \langle y_i^{m_i} : i \geq 1 \rangle$. It follows that $[y_r, w_r]$ is involved either in some $[y_i, y_j]$ $(i < j)$ or in some $y_i^{m_i}$. The first possibility is ruled out by linear independence and the fact that $w_r \in S \backslash H$. For $i > r$ we have $y_i^{m_i} \in B_r$, while $[y_r, w_r] \in A_r$. For $i < r$ we have $y_i^{m_i} \in A_i$, while $[y_r, w_r] \in B_i$ (since $y_r, w_r \notin X_{r-1}$). This leaves only the possibility that $[y_r, w_r]$ is involved in $y_r^{m_r}$. Since w_r was chosen so as to avoid this possibility, we obtain a contradiction that completes the proof of the lemma.

LEMMA 3.7. *If G' has finite exponent, then G is abelian-by-finite.*

PROOF. As before we may assume that $(G')^p = 1$. For each $i = 1, 2, \ldots$, set $Z_i = Z_i(G)$ and $X_i = Z_i \cap G'$. Then $X_2 = X_1 \times Y_1$ for some Y_1, while Y_1 contains a G-invariant subgroup W_1 of finite index (in Y_1). Now $[W_1, G] \leq W_1 \cap Z_1 = 1$ and so $W_1 \leq X_1$. Hence $W_1 = 1$ and Y_1 is finite. Similarly, $X_3 = X_1 \times Y_1 \times Y_2$ for some Y_2 which must be finite. Indeed we see that X_i/X_1 is finite, for all i. Let $A/X_1 = Z(G/X_1)$ and $B = A \cap G'$. Thus $B \leq X_2$ and B/X_1 is finite. If G/X_1 is abelian-by-finite then the result follows from Lemma 3.6. By Lemma 3.3, therefore,

G'/X_1 may be assumed infinite. Hence, by the previous remarks, we may factor by X_1, if necessary, and thus assume that each X_i is finite. Using the CF-property, we easily construct a subgroup V of finite index in G' such that $V = V_1 \times V_2 \times \cdots$, where each V_i is infinite and normal in G (using, of course, the fact that every subgroup of G' is a direct factor). By the residual finiteness of V and the CF-property, we see that each V_i has a G-invariant series of subgroups of finite index with trivial intersection. Choose k so that $V_j \frown Z(G) = 1$, for all $j > k$. For each j, there exists N_j of finite index in V_j and normal in G such that $[V_j, G] \not\leq N_j$. Write $N = V_1 \times \cdots \times V_k \times M$, where $M = \langle N_j : j > K \rangle$ and let $\bar{G} = G/N$. Certainly $\bar{G}'$ lies in the hypercentre of $\bar{G}$. But the argument at the beginning of the proof remains valid for the group $\bar{G}$ and so $Z_1(\bar{G}) \frown \bar{G}'$ has finite index in $Z_2(\bar{G}) \frown \bar{G}'$. From the structure of $\bar{V}$ and the fact that it has finite index in $\bar{G}'$ we deduce that almost all of the $\bar{V}_j$ are central in $\bar{G}$, contradicting the choice of the subgroups N_j. This completes the proof of the lemma.

Our final requirement is as follows.

LEMMA 3.8. *Suppose that A is an abelian subgroup of G which is G-hamiltonian and has infinite exponent. The the centralizer of A has index at most* 2 *in G.*

PROOF. Assume first of all that p is odd and let a be an element of A of order $p^n, n \geq 1$. The Sylow p-subgroup of $\mathrm{Aut}\langle a \rangle$ is generated by the map $\theta : a \to a^{1+p}$, which has order p^{n+1}. For $n \geq 2$ define $x = \theta^{p^{n-2}}$. Then $a^x = a^\mu$, where $\mu \equiv 1 \bmod p^{n-1}$. Now suppose that g has order at most p modulo $C_G(A)$ and choose b of order p^{n+1} in A. Write $b^g = b^\lambda, \lambda \in \mathbb{N}$. Then, as above, $\lambda \equiv 1 \bmod p^n$. By Lemma 2.2, $a^g = a^\lambda$ and thus $A \leq Z(G)$.

The argument for $p = 2$ is similar, except that we need to note that, for a of order $2^n (n \geq 2)$, the Sylow 2-subgroup of $\mathrm{Aut}\langle a \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, the generator of $\mathbb{Z}_2$, of course, inverting the elements of $\langle a \rangle$. The details are omitted.

3.9 CONCLUSION OF THE PROOF. Let R be the finite residual of G'. Then R is G-hamiltonian (see Lemma 4.1). If R has finite exponent p^k, then $R \leq Z_k(G)$. Otherwise $|G : C_G(R)| \leq 2$, by Lemma 3.8. Using 3.6, we may thus assume that G' is residually finite. Now let H be the Fitting subgroup of G.

By Lemmas 3.5 and 3.7 we may assume that $|G : H|$ is infinite and G' has infinite exponent. Let g, h be arbitrary elements of $G \backslash H$. We shall show that $gh^{-1} \in H$ and thus obtain the contradiction that $|G : H| \leq 2$. Write $J = \langle g, h \rangle$. By Lemma 2.1 there is a subgroup of A of finite index in G' such that A is J-hamiltonian. Then $C = C_J(A)$ has index at most 2 in J, by Lemma 3.8. Since $[G', C]$ is finite, we see that CG' is nilpotent and thus contained in H. But $|JG' : CG'| \leq 2$ and so $gh^{-1} \in H$. This contradiction completes the proof of the theorem.

4. Concluding remarks

There are some further observations that may be made concerning G-hamiltonian subgroups. The first thing that we need to do is to establish the following result, a special case of which was used in 3.9.

LEMMA 4.1. *Suppose that G is a periodic* CF*-group and that A is an abelian subgroup of G. Then the finite residual B of A is G-hamiltonian.*

PROOF. We may clearly assume that A is a p-group and, by the CF-property, that A is normal in G. For each ordinal $\alpha \geq 0$ we define a characteristic subgroup A_α of A as follows. Set $A_0 = A$, $A_1 = B$ and, in general, let $A_{\alpha+1}$ denote the finite residual of A_α. If α is a limit ordinal, define $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$. Now let b be an arbitrary non-trivial element of B. We wish to show that $\langle b \rangle \lhd G$. Since the series $\{A_\alpha\}$ terminates in a divisible subgroup (possibly trivial) we may assume that $b \in A_\alpha \backslash A_{\alpha+1}$, for some ordinal α. Without loss of generality we suppose $\alpha = 1$, that is $b \notin R(B)$ (the finite residual of B). Since $R(B) = \bigcap_{k=1}^{\infty} B^{p^k}$, we may further assume that $b \notin B^p$.

Now, given any $k_0 \geq 1$, there exists $a_1 \in A \setminus A^p$ such that $a_1^{p^{k_1}} = b$ and $k_1 > k_0$. Then there exists B_1 of finite index in A such that $B \leq B_1$ and $B_1/B \frown \langle a_1 \rangle B/B = 1$. Now choose $a_2 \in B_1 \backslash B^p$ such that $A_2^{p^{k_2}} = b$, where $k_2 > k_1$, and continue in the obvious manner to obtain an infinite subgroup $C = \langle a_1, a_2, \dots \rangle$ such that each a_i has order exactly p^{k_i} modulo$\langle b \rangle$ (where $k_1 < k_2 < \cdots$) and the $\langle a \rangle$ generate their direct product modulo$\langle b \rangle$. By the CF-property, there exists a G-invariant subgroup D of finite index in C and so $R(C) = \bigcap_{j=1}^{\infty} C^{p^j}$ is normal in G. But $R(C) = \langle b \rangle$ and the lemma is proved.

In view of Lemma 2.5 it is reasonable to ask whether the direct product of two G-hamiltonian subgroups A and C is (almost) G-hamiltonian, at least in the case where G is a CF-group. That this is not so may be seen by considering the group $G = \langle A, C, x \rangle$, where A is an infinite elementary abelian p-group, C is of type C_{p^∞} and x is of order 2 acting on $C \times A$ via $c \to c^{-1}$, $a \to a$, for all $c \in C$, $a \in A$.

Routine calculations show that G is BCFbut does not contain a G-hamiltonian subgroup of finite index. (Thus, for instance, the claim introduced during the proof of Lemma 2.5 requires the hypothesis of infinite exponent.)

An easy example of a CF-group which is not BCFis the group $G = \langle C, g \rangle$, where $C \cong C_{p^\infty}$, $c^g = c^{-1}$ for all $c \in C$ and g has infinite order. Note that G is of rank 2, metabelian and abelian-by-finite.

Before continuing we present the following elementary results.

LEMMA 4.2. *In any group G the subgroup generated by all infinite cyclic normal subgroups is abelian.*

PROOF. Suppose that x and y have infinite order and generate normal subgroups of G. If $[x, y] \neq 1$, then $[x, y] = x^{-2}$ and $[y, x] = y^{-2}$, giving $[x, y]$ central and $x^2 = y^{-2}$. Thus $1 = [x, y^2] = [x, y]^2$, giving $x^4 = 1$, a contradiction.

LEMMA 4.3. *Suppose that G is a* CF-*group and that A is the subgroup of G generated by all normal infinite cyclic subgroups. Then, for each $g \in G$, either $a^g = a$ for all $a \in A$ or $a^g = a^{-1}$ for all $a \in A$. Therefore $|G : C_G(A)| \leq 2$.*

PROOF. It suffices to consider elements of A having infinite order. Suppose, for a contradiction, that $a^g = a$ and $b^g = b^{-1}$, where a, b have infinite order. If $\langle a\rangle \cap \langle b\rangle = 1$ then $\langle ab\rangle$ is infinite and $(ab)^g = (ab)^{\pm 1}$, a contradiction. Suppose then that $a^\lambda = b^\mu$, where λ, μ are non-zero. Then $a^\lambda = (a^\lambda)^g = b^{-\mu}$ and so $a^{2\lambda} = 1$, a contradiction.

COROLLARY 4.4. *Every finitely generated soluble* CF-*group G is abelian-by-finite and* BCF.

PROOF. By Lemma 4.2 such a group G is certainly abelian-by-finite and, in particular, has finite rank. The result follows from Lemma 4.3, since every cyclic subgroup of G has bounded index over its core.

Now suppose that G is an arbitrary CF-group and let A be defined as in Lemma 4.3. Let B/A denote the locally finite radical of G/A. Thus B/A is abelian-by-finite. If $B \neq G$ then G/B contains a finitely generated infinite periodic subgroup G_0, and G_0 in turn has an infinite homomorphic image G_1 such that every subgroup of G_1 is either finite or has finite index in G_1. Thus any CF-group which is not metabelian-by-finite involves a rather complicated finitely generated group. Indeed, one sees that in the absence of such a section, a CF-group is almost nilpotent of class (at most) two. There thus appears some motivation for further investigation of CF-groups.

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