Math. Proc. Camb. Phil. Soc. (2023), **174**, 605–630
doi:10.1017/S0305004122000445
First published online 28 December 2022

A zero density estimate and fractional imaginary parts of zeros for GL_2 *L*-functions

BY OLIVIA BECKWITH
Mathematics Department, Tulane University, New Orleans, 6823 *St. Charles Ave., LA* 70118, *U.S.A.*
e-mail: obeckwith@tulane.edu

DI LIU, JESSE THORNER
Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 *West Green Street, Urbana, IL* 61801, *U.S.A.*
e-mails: dil4@illinois.edu, jesse.thorner@gmail.com

AND ALEXANDRU ZAHARESCU
Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 *West Green Street, Urbana, IL* 61801, *U.S.A.*
AND *Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box* 1-764, *RO*-014700 *Bucharest, Romania.*
e-mail: zaharesc@illinois.edu

(*Received* 14 *September* 2021*; revised* 10 *October* 2022*; accepted* 27 *July* 2022)

Abstract

We prove an analogue of Selberg's zero density estimate for $\zeta(s)$ that holds for any GL_2 *L*-function. We use this estimate to study the distribution of the vector of fractional parts of $\gamma\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$ is fixed and γ varies over the imaginary parts of the nontrivial zeros of a GL_2 *L*-function.

2020 Mathematics Subject Classification: 11M41 (Primary)

1. *Introduction and statement of results*

1·1. *Main results*

Let $\zeta(s)$ be the Riemann zeta function, and let $N(\sigma, T)$ denote the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma \geq 0$ and $|\gamma| \leq T$. The asymptotic

$$N(T) := N(0, T) \sim \frac{1}{\pi} T \log T \tag{1·1}$$

follows from the argument principle. The Riemann hypothesis (RH) asserts that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1/2$, so $N(\sigma, T) = 0$ for $\sigma > 1/2$. Selberg [**Tit86**, theorem 9·19C] proved a delicate zero density estimate that recovers the upper bound in (1·1) at $\sigma = 1/2$, namely

$$N(\sigma, T) \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})} \log T, \qquad \sigma \geq \frac{1}{2}. \tag{1·2}$$

(See Baluyot [Bal17, theorem 1·2·1] for an improvement.) Selberg's estimate implies that

$$\frac{1}{N(T)}\#\Big\{\rho=\beta+i\gamma:\ \zeta(\rho)=0,\ |\gamma|\le T,\ \beta\in\Big[\frac{1}{2}-\frac{4\log\log T}{\log T},\frac{1}{2}+\frac{4\log\log T}{\log T}\Big]\Big\}\ll\frac{1}{\log T}.$$

As an application of (1·2), Selberg proved a central limit theorem for $\log|\zeta(1/2+it)|$:

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{T\le t\le 2T:\ \log|\zeta(\tfrac{1}{2}+it)|\ge V\sqrt{\tfrac{1}{2}\log\log T}\Big\}\sim\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}du. \quad (1·3)$$

Let $\mathscr{A}$ be the set of cuspidal automorphic representations of GL_2 over $\mathbb{Q}$ with unitary central character. For $\pi\in\mathscr{A}$, let $L(s,\pi)$ be its standard L-function. Define

$$N_\pi(\sigma,T):=\#\{\rho=\beta+i\gamma:\ \beta\ge\sigma,\ |\gamma|\le T,\ L(\rho,\pi)=0\}.$$

As with $\zeta(s)$, the argument principle can be used to prove that

$$N_\pi(T):=N_\pi(0,T)\sim(2/\pi)T\log T. \quad (1·4)$$

The generalised Riemann hypothesis (GRH) asserts that $L(s,\pi)\neq 0$ for $\mathrm{Re}(s)>1/2$. Selberg [Sel92] observed that analogues of (1·2) and (1·3) should also hold for Hecke–Maaß newforms. When $\pi\in\mathscr{A}$ corresponds with a holomorphic cuspidal newform of even weight $k\ge 2$, Luo [Luo95, theorem 1·1] and Li [FZ15, section 7] proved that $N_\pi(\sigma,T)\ll T^{1-\frac{1}{72}(\sigma-\frac{1}{2})}\log T$. We prove:

THEOREM 1·1. *Let $\theta\in[0,7/64]$ be an admissible exponent toward the generalised Ramanujan conjecture for Hecke–Maaß newforms (see (2·1)), and fix $0<c<1/4-\theta/2$. If $\pi\in\mathscr{A}$, $\sigma\ge 1/2$, and $T\ge 2$, then $N_\pi(\sigma,T)\ll T^{1-c(\sigma-\frac{1}{2})}\log T$. The implied constant depends at most on π.*

Remark 1. Under the generalised Ramanujan conjecture for all Hecke–Maaß newforms, we may take $\theta=0$. In this case, our result is as strong as Selberg's zero density estimate (1·2) for $\zeta(s)$. Currently, the best unconditional bound is $\theta\le 7/64$, so we may choose any $c<25/128$. This noticeably improves the work of Luo and Li, and it holds for any $\pi\in\mathscr{A}$. The constant $1/4-\theta/2$ is, as of now, the supremum over all ϖ for which we can unconditionally prove an asymptotic for the second mollified moment of $L(s,\pi)$ on $\mathrm{Re}(s)=1/2+1/\log T$ with a mollifier of length T^ϖ.

COROLLARY 1·2. *Let $\pi\in\mathscr{A}$. If $V\in\mathbb{R}$, then as $T\to\infty$, we have:*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{T\le t\le 2T:\ \log|L(\tfrac{1}{2}+it,\pi)|\ge V\sqrt{\tfrac{1}{2}\log\log T}\Big\}=\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}du;$$

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\Big\{T\le t\le 2T:\ \arg L(\tfrac{1}{2}+it,\pi)\ge V\sqrt{\tfrac{1}{2}\log\log T}\Big\}=\frac{1}{\sqrt{2\pi}}\int_V^\infty e^{-u^2/2}du.$$

Proof. Bombieri and Hejhal [BH95, theorem B] proved that this follows from a zero density estimate of the quality given by Theorem 1·1.

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

Remark 2. Radziwiłł and Soundararajan [RS17] recently found a second proof of (1·3) which avoids the use of zero density estimates. Their work was recently extended to holomorphic newforms by Das [Das20]. The proof relies on both the generalised Ramanujan conjecture and the Sato–Tate conjecture, neither of which is known for any Hecke–Maaß newform.

We study the distribution of imaginary parts of the nontrivial zeros of $L(s,\pi)$ using Theorem 1·1. First, we give a partial history of such results for $\zeta(s)$. Hlawka [Hla75] proved that if $\alpha \in \mathbb{R}$ is fixed and $h\colon \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\lim_{T\to\infty} \frac{1}{N(T)} \sum_{|\gamma|\le T} h(\alpha\gamma) = \int_{\mathbb{T}} h(t)dt, \tag{1·5}$$

where γ varies over the imaginary parts of the nontrivial zeros of $\zeta(s)$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Thus the sequence of fractional parts $\{\gamma\alpha\}$ is equidistributed modulo 1. However, given a rate of convergence, there exist continuous functions h such that the limit in (1·5) cannot be attained with said rate (see also [FZ05, theorem 7]). Therefore, (1·5) is the best that one can say for arbitrary h.

Ford and Zaharescu [FZ05, corollary 2] established the existence of a second order term, proving that if $h : \mathbb{T} \to \mathbb{C}$ is twice continuously differentiable,[1] then

$$\sum_{|\gamma|\le T} h(\alpha\gamma) = N(T) \int_{\mathbb{T}} h(t)dt + T \int_{\mathbb{T}} h(t)g_\alpha(t)dt + o(T), \tag{1·6}$$

where

$$g_\alpha(t) = \begin{cases} \dfrac{\log p}{\pi} \mathrm{Re} \displaystyle\sum_{k=1}^{\infty} \frac{e^{-2\pi iqkt}}{p^{ak/2}} & \text{if there exists a prime } p \text{ and } a, q \in \mathbb{Z} \text{ such that} \\ & \gcd(a,q) = 1 \text{ and } \alpha = \dfrac{a}{q}\dfrac{\log p}{2\pi}, \\ 0 & \text{otherwise.} \end{cases} \tag{1·7}$$

Despite the limitations on the analytic properties of h, Ford and Zaharescu still conjectured [FZ05, conjecture A] that for any interval $\mathbb{I} \subseteq \mathbb{T}$ of length $|\mathbb{I}|$, we have

$$\sum_{\substack{|\gamma|\le T \\ \{\alpha\gamma\}\in\mathbb{I}}} 1 = |\mathbb{I}|N(T) + T \int_{\mathbb{I}} g_\alpha(t)dt + o(T), \tag{1·8}$$

which implies that

$$D_\alpha(T) := \sup_{\mathbb{I}\subseteq\mathbb{T}} \left| \frac{1}{N(T)} \sum_{\substack{|\gamma|\le T \\ \{\alpha\gamma\}\in\mathbb{I}}} 1 - |\mathbb{I}| \right| = \frac{T}{N(T)} \sup_{\mathbb{I}\subseteq\mathbb{T}} \left| \int_{\mathbb{I}} g_\alpha(t)dt \right| + o\Big(\frac{1}{\log T}\Big). \tag{1·9}$$

Ford, Soundararajan, and Zaharescu [FSZ09] made some progress toward the conjectured asymptotics (1·8) and (1·9). Unconditionally, they proved that

$$D_\alpha(T) \ge \frac{T}{N(T)} \sup_{\mathbb{I}\subseteq\mathbb{T}} \left| \int_{\mathbb{I}} g_\alpha(t)dt \right| + o\Big(\frac{1}{\log T}\Big). \tag{1·10}$$

[1] On RH, absolute continuity suffices.

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

Assuming RH, they proved that

$$\Big|\sum_{\substack{|\gamma|\le T\\ \{\alpha\gamma\}\in\mathbb{I}}} 1-|\mathbb{I}|N(T)-T\int_{\mathbb{I}} g_\alpha(t)dt\Big|\le\Big(\frac{\alpha}{2}+o(1)\Big)T. \tag{1·11}$$

Along with making some appealing connections between the conjectured asymptotics (1·8) and (1·9) and other intriguing open problems like pair correlation of zeros of $\zeta(s)$ and the distribution of primes in short intervals, they proved analogues for other L-functions of (1·10) (assuming a zero density estimate of the form (1·2)) and (1·11) (assuming GRH).

In this paper, we extend the work in [**FZ05, FSZ09, FMZ17, LZ21**] to $L(s,\pi)$ for any $\pi\in\mathscr{A}$ using Theorem 1·1. Let $n\geqslant 1$. Consider the $\boldsymbol{\alpha}\in\mathbb{R}^n$ for which there exists a constant $C_{\boldsymbol{\alpha}}>0$ such that[2]

$$|\boldsymbol{m}\cdot\boldsymbol{\alpha}|\ge C_{\boldsymbol{\alpha}}e^{-\|\boldsymbol{m}\|_2}\qquad\text{for all }\boldsymbol{m}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}, \tag{1·12}$$

where $\|\boldsymbol{m}\|_p$ is the ℓ^p norm on $\mathbb{R}^n$ for $1\le p\le\infty$. This is a technical artifact of our extension to $\mathbb{R}^n$; when $n=1$, the condition reduces to $\alpha\neq 0$. Our density function $g_{\pi,\boldsymbol{\alpha}}(\boldsymbol{t})$, which extends (1·7) for $n\ge 2$, is identically zero unless there exists a matrix $M=(b_{jk})\in\mathcal{M}_{r\times n}(\mathbb{Z})$ with linearly independent row vectors $\boldsymbol{b}_j$ and $\gcd(b_{j1},\dots,b_{jn})=1$ for all $1\le j\le r$; fully reduced rationals $a_1/q_1,\dots,a_r/q_r$; and distinct primes $p_1,\dots,p_r$ such that

$$M\boldsymbol{\alpha}^{\mathsf{T}}=\Big(\frac{a_1}{q_1}\frac{\log p_1}{2\pi},\dots,\frac{a_r}{q_r}\frac{\log p_r}{2\pi}\Big)^{\mathsf{T}}. \tag{1·13}$$

Among such possible matrices M, choose one with maximal r, which uniquely determines the row vectors $\boldsymbol{b}_j=(b_{j1},\dots,b_{jn})$. If such an M exists, then define

$$g_{\pi,\boldsymbol{\alpha}}(\boldsymbol{t}):=-\frac{2}{\pi}\Re\sum_{j=1}^{r}\sum_{l=1}^{\infty}\frac{\Lambda_\pi(p_j{}^{a_jl})}{p_j{}^{a_jl/2}}e^{-2\pi iq_jl(\boldsymbol{b}_j\cdot\boldsymbol{t})}, \tag{1·14}$$

where

$$-\frac{L'(s,\pi)}{L(s,\pi)}=\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s},\qquad \mathrm{Re}(s)>1.$$

Let $\pi\in\mathscr{A}$, $n\ge 1$, and $\mathbb{B}\subseteq\mathbb{T}^n$ be a product of n subintervals of $\mathbb{T}$. The conjecture in (1·8) can be extended to $\pi\in\mathscr{A}$ as follows:

$$\sum_{\substack{|\gamma|\le T\\ \{\gamma\boldsymbol{\alpha}\}\in\mathbb{B}}} 1=\mathrm{vol}(\mathbb{B})N_\pi(T)+T\int_{\mathbb{B}} g_{\pi,\boldsymbol{\alpha}}(\boldsymbol{t})\mathrm{d}\boldsymbol{t}+o(T), \tag{1·15}$$

where γ ranges over the imaginary parts of the nontrivial zeros of $L(s,\pi)$. As progress toward (1·15), we prove an unconditional n-dimensional version of (1·6) for GL_2

[2] The set of vectors $\boldsymbol{\alpha}$ for which there exists $C_{\boldsymbol{\alpha}}>0$ such that (1·12) holds have full Lebesgue measure in $\mathbb{R}^n$ by work of Kemble [**Kem05**] and Khintchine [**Khi24**].

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

L-functions. In what follows, let $C^u(\mathbb{T}^n)$ be the set of u-times continuously differentiable functions $h\colon \mathbb{T}^n \to \mathbb{R}$. Let γ vary over the imaginary parts of the nontrivial zeros of $L(s,\pi)$.

THEOREM 1·3. *Let* $\pi \in \mathscr{A}$. *Let* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *satisfy* (1·12). *If* $h \in C^{n+2}(\mathbb{T}^n)$, *then*

$$\sum_{|\gamma|\le T} h(\gamma\boldsymbol{\alpha}) = N_\pi(T)\int_{\mathbb{T}^n} h(\boldsymbol{t})\mathrm{d}\boldsymbol{t} + T\int_{\mathbb{T}^n} h(\boldsymbol{t})g_{\pi,\boldsymbol{\alpha}}(\boldsymbol{t})\mathrm{d}\boldsymbol{t} + o(T), \tag{1·16}$$

where $\mathrm{d}\boldsymbol{t}$ *is Lebesgue measure on* $\mathbb{T}^n$. *The implied constant depends on* π, *h, and* $\boldsymbol{\alpha}$.

Remark 3. With extra work, we may allow $h \in C^{n+1}(\mathbb{T}^n)$. Also, under some extra conditions on $\boldsymbol{\alpha}$ which exclude a density zero subset of $\mathbb{R}^n$, we expect to further quantify the $o(T)$ error term in (1·16).

Define the discrepancy

$$D_{\pi,\boldsymbol{\alpha}}(T) := \sup_{\mathbb{B}\subseteq\mathbb{T}^n}\left|\frac{1}{N_\pi(T)}\sum_{\substack{|\gamma|\le T\\ \{\gamma\boldsymbol{\alpha}\}\in\mathbb{B}}} 1 - \mathrm{vol}(\mathbb{B})\right|.$$

Our next result follows quickly from Theorem 1·3.

COROLLARY 1·4. *Let* $\pi \in \mathscr{A}$. *If* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *satisfies* (1·12), *then*

$$D_{\pi,\boldsymbol{\alpha}}(T) \ge \frac{T}{N_\pi(T)}\int_{\mathbb{T}^n} g_{\pi,\alpha}(\boldsymbol{t})\mathrm{d}\boldsymbol{t} + o\Big(\frac{1}{\log T}\Big).$$

When $n=1$, we recover an unconditional analogue of (1·10) for all $\pi \in \mathscr{A}$.This special case of Corollary 1·4 was proved in [**FSZ09**] under the hypothesis of a zero density estimate of the form proved in Theorem 1·1.

Remark 4. Let $d \ge 3$ be an integer, and let $L(s,\pi)$ be the standard L-function associated to a cuspidal automorphic representation π of GL_d over $\mathbb{Q}$. Let $N_\pi(\sigma,T)$ be the number of nontrivial zeros $\beta+i\gamma$ of $L(s,\pi)$ with $\beta \ge \sigma$ and $|\gamma| \le T$. If there exist constants $c_\pi > 0$ and $d_\pi > 0$ such that $N_\pi(\sigma,T) \ll T^{1-c_\pi(\sigma-\frac{1}{2})}(\log T)^{d_\pi}$, then one can prove an analogue of Theorem 1·3 and Corollary 1·4 for $L(s,\pi)$. Such an estimate for $N_\pi(\sigma,T)$ is not yet known for any $d \ge 3$, and it appears to be quite difficult to prove.

1·2. *Application to "zero races"*

In a letter to Fuss, Chebyshev observed that the generalised Riemann hypothesis implies that primes $p \equiv 3 \pmod 4$ tend to be more numerous than primes $p \equiv 1 \pmod 4$. Using the generalised Riemann hypothesis and other hypotheses, Rubinstein and Sarnak [**RS94**] began a systematic study of "prime number races" in which they determine how often $\pi(x;4,3) > \pi(x;4,1)$, where $\pi(x;q,a)$ equals $\#\{p \le x\colon p \equiv a \pmod q\}$. They proved that

$$\lim_{X\to\infty}\frac{1}{\log X}\int_{\substack{2\le t\le X\\ \pi(t;4,3)>\pi(t;4,1)}}\frac{dt}{t} = 0.9959\ldots$$

Thus the "bias" toward primes of the form $4n+3$ is quite strong. The literature on prime number races which study such inequities is quite vast. See, for instance, the work of Fiorilli, Ford, Harper, Konyagin, Lamzouri and Martin [**FM13, FK02, FLK13, FHL19**].

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

We use Theorem 1·3 to study inequities not between primes in different residue classes, but between zeros of different L-functions. Define

$$\mathscr{C}^r(\mathbb{T}^n) := \{h \in C^r(\mathbb{T}^n) \colon h \text{ is nonnegative and not identically zero}\}.$$

Let $h \in \mathscr{C}^{n+2}(\mathbb{T}^n)$, and let $\boldsymbol{\alpha}$ satisfy (1·12). We consider two holomorphic cusp forms f_1 and f_2 with trivial nebentypus, where

$$f_j(z) = \sum_{n=1}^{\infty} \lambda_{f_j}(n) n^{\frac{k_j-1}{2}} e^{2\pi i n z} \in S_{k_j}^{\mathrm{new}}(\Gamma_0(q_j)), \qquad j \in \{1, 2\}$$

have trivial nebentypus, even integral weights $k_j \geq 2$ and squarefree levels $q_j \geq 1$. We assume that f_j is normalised so that $\lambda_{f_j}(1) = 1$ and that f_j is an eigenfunction of all of the Hecke operators. We call such cusp forms *newforms* (see [**Ono04**, section 2·5]). It is classical that there exists $\pi_j \in \mathscr{A}$ such that $L(s, f_j) = L(s, \pi_j)$, so $\lambda_{f_j}(n) = \lambda_{\pi_j}(n)$, $g_{f_j,\boldsymbol{\alpha}}(\boldsymbol{t}) = g_{\pi_j,\boldsymbol{\alpha}}(\boldsymbol{t})$, etc.

We say f_1 wins the $(\boldsymbol{\alpha}, h)$-race (against f_2) if for all large T, we have

$$\sum_{|\gamma_1| \leqslant T} h(\boldsymbol{\alpha}\gamma_1) > \sum_{|\gamma_2| \leqslant T} h(\boldsymbol{\alpha}\gamma_2),$$

where γ_j runs over the imaginary parts of the nontrivial zeros of $L(s, f_j)$. If neither f_1 nor f_2 wins the $(\boldsymbol{\alpha}, h)$-race, we say the race is undecided. The results of [**LZ21**] suggest that the winner of a decided $(\boldsymbol{\alpha}, h)$-race is determined by the levels q_1 and q_2 and the behavior of $g_{f_1,\boldsymbol{\alpha}}$ and $g_{f_2,\boldsymbol{\alpha}}$.

First, we show that if $q_1 = q_2$ and $h \in \mathscr{C}^3(\mathbb{T})$, then the proportion of primes p such that f_1 wins the $((\log p)/(2\pi), h)$-race is $1/2$.

COROLLARY 1·5. *For $j = 1, 2$, let q_j be squarefree and $f_j \in S_{k_j}^{new}(\Gamma_0(q_j))$ be non-CM newforms. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive Dirichlet characters χ. If $h \in \mathscr{C}^3(\mathbb{T})$, then*

$$\frac{\#\{p \leqslant X \colon f_1 \textit{ wins the } (\frac{\log p}{2\pi}, h)\textit{-race}\}}{\#\{p \leq X\}} = \frac{1}{2} + O\Big(\frac{\sqrt{\log\log\log X}}{(\log\log X)^{1/4}}\Big).$$

Next we look at the distribution of

$$H(f_1, f_2, h, \alpha) := \lim_{T \to \infty} \frac{\sum_{|\gamma_1| < T} h(\alpha\gamma_1) - \sum_{|\gamma_2| < T} h(\alpha\gamma_2)}{T} - \frac{1}{\pi} \log \frac{q_1}{q_2} \int_{\mathbb{T}} h(t) dt$$

as α varies over values of $\log p/2\pi$ for prime values of p. Given $I \subseteq [-2, 2]$, let $\mu_{\mathrm{ST}}(I)$ be the Sato–Tate measure $1/(2\pi) \int_I \sqrt{4 - t^2} dt$, and let $\mu_{ST,2}$ be the product measure defined on boxes $I_1 \times I_2 \subseteq [-2, 2] \times [-2, 2]$ by $\mu_{\mathrm{ST},2}(I_1 \times I_2) = \mu_{\mathrm{ST}}(I_1)\mu_{\mathrm{ST}}(I_2)$. For $\mathcal{I} \subseteq [-4, 4]$, we define

$$\nu(\mathcal{I}) := \mu_{ST,2}(\{(x, y) \in [-2, 2]^2 : x - y \in \mathcal{I}\}). \tag{1·17}$$

For $h \in \mathscr{C}^3(\mathbb{T})$, we set

$$k_h := \int_0^1 h(t) \cos(2\pi t) dt.$$

In the following statement and throughout the paper, for any interval I and $\beta \in \mathbb{R}$, we let $\beta I = \{\beta x : x \in I\}$.

COROLLARY 1·6. *For $j = 1, 2$, let q_j be squarefree and $f_j \in S_{k_j}^{new}(\Gamma_0(q_j))$ be normalised holomorphic non-CM newforms with trivial nebentypus. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive Dirichlet characters χ. Let*

$$\varepsilon_X := \frac{(\log\log\log X)^{1/4}}{(\log\log X)^{1/8}}.$$

Assume $h \in \mathscr{C}^3(\mathbb{T})$ is such that $k_h \neq 0$. Then for any interval $\mathcal{I} \subseteq [-4, 4]$, we have

$$\frac{\#\{p \in [(1-\varepsilon_X)X, X] \colon \frac{\log X}{\sqrt{X}} H(f_1, f_2, h, \frac{\log p}{2\pi}) \in \mathcal{I}\}}{\#\{p \in [(1-\varepsilon_X)X, X]\}} = \nu\Big(\frac{\pi}{2k_h}\mathcal{I}\Big) + O(\varepsilon_X)$$

with an implied constant independent of $\mathcal{I}$.

If $q_1 > q_2$, then in contrast to Corollary 1·5, f_1 wins the $((\log p)/(2\pi), h)$-race for all except finitely many primes p.

COROLLARY 1·7. *For $j = 1, 2$, let $f_j \in S_{k_j}^{new}(\Gamma_0(q_j))$ be normalised holomorphic newforms with trivial nebentypus. If $q_1 > q_2$, then for any $h \in \mathscr{C}^3(\mathbb{T})$, there are at most finitely many primes p such that f_2 wins the $((\log p)/(2\pi))$-race.*

Corollary 1·7 shows that it is rare for f_1 to win an (α, h) race against f_2 if $q_2 > q_1$, but we can show this occurs infinitely often. Our result can be stated neatly in terms of *local races* rather than in terms of the (α, h)-races described above. For $t_0 \in \mathbb{T}$, we say that f_1 wins the local (α, t_0)-race against f_2 if there exists a neighbourhood U of $t_0 \in \mathbb{T}$ such that the (α, h)-race is won by f_1 for all $h \in \mathscr{C}^3(\mathbb{T})$ which are supported on U.

COROLLARY 1·8. *For $j = 1, 2$, let q_j be squarefree and $f_j \in S_{k_j}^{new}(\Gamma_0(q_j))$ be normalised holomorphic non-CM newforms with trivial nebentypus. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive Dirichlet characters χ. Fix $t_0 \in [0, 1)$. Let q_1 be sufficiently large, and let $q_2 \in (q_1, q_1 + q_1^{1/2}]$, and let k_1 and k_2 be fixed. There exists $\alpha \in \mathbb{R}$ such that f_1 wins the local (α, t_0)-race against f_2.*

In addition to Theorem 1·3, the proofs of these corollaries rely on a quantifiable understanding of the joint distribution of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ as p varies over the primes. Such an understanding follows from the effective version of the Sato–Tate conjecture which counts the number of primes $p \leq X$ such that $(\lambda_{f_1}(p), \lambda_{f_2}(p)) \in I_1 \times I_2$ proved by the third author in [**Tho21**] (see Theorem 2·1 below). One can prove analogues of Corollaries 1·5, 1·6 and 1·8 for Dirichlet L-functions by replacing the effective Sato–Tate estimates with results on primes in arithmetic progressions.

2. *Preliminaries*

2·1. GL_2 *L-functions over* $\mathbb{Q}$

Let $\pi \in \mathscr{A}$. Here, we state the essential properties of L-functions of $L(s, \pi)$ that we use throughout our proofs. See [**IK04**, chapter 5] for a convenient summary. Given $\pi \in \mathscr{A}$ with

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

level q_π, there exist suitable complex numbers $\alpha_{1,\pi}(p)$ and $\alpha_{2,\pi}(p)$ such that

$$L(s,\pi)=\prod_{p\text{ prime}}\prod_{j=1}^{2}(1-\alpha_{j,\pi}(p)p^{-s})^{-1}=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s}.$$

The sum and product both converge absolutely for $\mathrm{Re}(s)>1$. There also exist spectral parameters $\kappa_\pi(1)$ and $\kappa_\pi(2)$ such that if we define

$$L(s,\pi_\infty)=\pi^{-s}\Gamma\Big(\frac{s+\kappa_\pi(1)}{2}\Big)\Gamma\Big(\frac{s+\kappa_\pi(2)}{2}\Big),$$

then the completed L-function $\Lambda(s,\pi):=q_\pi^{s/2}L(s,\pi)L(s,\pi_\infty)$ is entire of order 1.

Let $\tilde{\pi}\in\mathscr{A}$ be the contragredient representation. We have $\alpha_{j,\tilde{\pi}}(p)=\overline{\alpha_{j,\pi}(p)}$ and $\kappa_{\tilde{\pi}}(j)=\overline{\kappa_\pi(j)}$ for $j=1,2$. Moreover, there exists a complex number $W(\pi)$ of modulus 1 such that for all $s\in\mathbb{C}$, we have

$$\Lambda(s,\pi)=W(\pi)\Lambda(1-s,\tilde{\pi}).$$

Building on work of Kim and Sarnak [**Kim03**, appendix], Blomer and Brumley [**BB11**] proved that there exists $\theta\in[0,7/64]$ such that we have the uniform bounds

$$\log_p|\alpha_{j,\pi}(p)|,\ -\mathrm{Re}(\kappa_\pi(j))\le\theta. \tag{2·1}$$

The generalised Ramanujan conjecture and the Selberg eigenvalue conjecture assert that (2·1) holds with $\theta=0$.

The Rankin–Selberg L-function[3]

$$L(s,\pi\otimes\tilde{\pi})=\sum_{n=1}^{\infty}\frac{\lambda_{\pi\times\tilde{\pi}}(n)}{n^s}\doteq\prod_{p\nmid q_\pi}\prod_{j=1}^{2}\prod_{j'=1}^{2}(1-\alpha_{j,\pi}(p)\overline{\alpha_{j',\pi}(p)}p^{-s})^{-1},\qquad \mathrm{Re}(s)>1$$

factors as $\zeta(s)L(s,\mathrm{Ad}^2\pi)$, where the adjoint square lift $\mathrm{Ad}^2\pi$ is an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$. Thus, $L(s,\mathrm{Ad}^2\pi)$ is an entire automorphic L-function. This fact and the bound $|\lambda_\pi(n)|^2\le\lambda_{\pi\times\tilde{\pi}}(n)$ [**JLW21**, lemma 3·1] enable us to prove via contour integration that

$$\sum_{n\le X}|\lambda_\pi(n)|^2\le\sum_{n\le X}\lambda_{\pi\times\tilde{\pi}}(n)\ll_\pi X. \tag{2·2}$$

It follows from [**IK04**, theorem 5·42] applied to $\zeta(s)$ and $L(s,\mathrm{Ad}^2\pi)$ that there exists an effectively computable constant $c_\pi>0$ such that $L(s,\pi\times\tilde{\pi})\ne0$ in the region

$$\mathrm{Re}(s)\ge1-\frac{c_\pi}{\log(|\mathrm{Im}(s)|+3)}. \tag{2·3}$$

[3] The $\doteq$ suppresses the more complicated Euler factors at primes $p|q_\pi$. Their explicit description does not arise in our proofs.

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

2·2. *Holomorphic newforms*

Many of our corollaries pertain specifically to $\pi \in \mathscr{A}$ corresponding to holomorphic newforms. As above, let

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \in S_k^{\mathrm{new}}(\Gamma_0(q))$$

be a holomorphic cuspidal newform (normalised so that $\lambda_f(1) = 1$) of even integral weight $k \geq 2$, level $q \geq 1$, and trivial nebentypus. If $\pi_f \in \mathscr{A}$ corresponds with f, then $L(s, f) = L(s, \pi_f)$, $\lambda_f(n) = \lambda_{\pi_f}(n)$, etc.

For these newforms, it follows from Deligne's proof of the Weil conjectures that the generalised Ramanujan conjecture holds. Thus, for f a holomorphic cuspidal newform as above, we may take $\theta = 0$ in (2·1). Since we assume that f has trivial central character, Deligne's bound implies that there exists $\theta_p \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos\theta_p.$$

The Sato–Tate conjecture, now a theorem due to Barnet-Lamb, Geraghty, Harris, and Taylor [**BLGHT11**], states that the sequence (θ_p) is equidistributed in the interval $[-2, 2]$ with respect to the measure $2/\pi(\sin t)^2 dt$. In other words, if $I \subseteq [0, \pi]$ is a subinterval, then

$$\lim_{X\to\infty} \frac{\#\{p \leq X \colon \theta_p \in I\}}{\#\{p \leq X\}} = \frac{2}{\pi}\int_I (\sin t)^2 dt, \qquad \pi(X) := \#\{p \text{ prime} :\ p \leq X\}.$$

After a change of variables, this implies that for any interval $I \subseteq [-2, 2]$, we have

$$\lim_{X\to\infty} \frac{\#\{\lambda_f(p) \in I : p \leqslant X\}}{\#\{p \leq X\}} = \frac{1}{2\pi}\int_I \sqrt{4 - t^2}\,dt =: \mu_{\mathrm{ST}}(I). \tag{2·4}$$

A recent paper by Thorner [**Tho21**] provides both an unconditional and a GRH-conditional rate of convergence in (2·4).

Our corollaries of Theorem 1·3 require a natural refinement of the Sato–Tate conjecture. For $j = 1, 2$, let $f_j \in S_{k_j}^{\mathrm{new}}(\Gamma_0(q_j))$ be a holomorphic cuspidal newform as above. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive nontrivial Dirichlet characters χ. Building on work of Harris [**Har09**], Wong [**Won19**] proved that the sequences $(\lambda_{f_1}(p))$ and $(\lambda_{f_2}(p))$ exhibit a joint distribution: If $I_1, I_2 \subseteq [-2, 2]$, then

$$\lim_{X\to\infty} \frac{\#\{p \leq X \colon \lambda_{f_1}(p) \in I_1,\ \lambda_{f_2}(p) \in I_2\}}{\#\{p \leq X\}} = \mu_{\mathrm{ST}}(I_1)\mu_{\mathrm{ST}}(I_2). \tag{2·5}$$

Our corollaries of Theorem 1·3 require a nontrivial unconditional bound on the rate of convergence in (2·5). Such a bound was recently proved in [**Tho21**, theorem 1·2].

THEOREM 2·1. *For $j = 1, 2$, let q_j be squarefree and $f_j \in S_{k_j}^{new}(\Gamma_0(q_j))$ be a normalised holomorphic cuspidal newform with trivial nebentypus. Suppose that $f_1 \neq f_2 \otimes \chi$ for all primitive nontrivial Dirichlet character χ. Let $I_1, I_2 \subseteq [-2, 2]$ be subintervals. There exists an absolute and effectively computable constant $c > 0$ such that*

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

$$\left|\frac{\#\{p \leqslant X \colon \lambda_{f_1}(p) \in I_1, \lambda_{f_2}(p) \in I_2\}}{\#\{p \leq X\}} - \mu_{ST}(I_1)\mu_{ST}(I_2)\right| \leq c\frac{\log\log(k_1k_2q_1q_2\log X)}{\sqrt{\log\log X}}.$$

We use the following result to prove our corollaries of Theorem 1·3.

Corollary 2·2. *Let f_1 and f_2 be as in Theorem 2·1. Recall the definition of ν in (1·17). If $\mathcal{I} \subseteq [-4, 4]$, then*

$$\frac{\#\{p \leqslant X : \lambda_{f_1}(p) - \lambda_{f_2}(p) \in \mathcal{I}\}}{\#\{p \leq X\}} = \nu(\mathcal{I}) + O\Big(\frac{\sqrt{\log\log\log X}}{(\log\log X)^{1/4}}\Big),$$

with an implied constant independent of $\mathcal{I}$.

Proof. Let

$$g(X) := \frac{(\log\log X)^{1/4}}{\sqrt{\log\log\log X}}, \qquad S := \{(x, y) \in [-2, 2]^2 : x - y \in \mathcal{I}\}.$$

We will define rectangles whose unions approximate S. Let $x_j = -2 + j\lfloor 4/g(X)\rfloor$. If $\mathcal{I} = (c, d)$, set $R_j = [x_j, x_{j+1}] \times [x_j - d, x_{j+1} - c]$, and similarly $T_j := [x_j, x_{j+1}] \times [x_{j+1} - d, x_j - c]$. By construction, we have $\cup_j T_j \subseteq S \subseteq \cup_j R_j$, which implies that

$$\begin{aligned}\#\{p \leqslant X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \cup T_j\} &\leqslant \#\{p \leqslant X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in S\}\\ &\leqslant \#\{p \leqslant X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \cup_j R_j\}.\end{aligned} \tag{2·6}$$

We apply Theorem 2·1 to count the primes p with $(\lambda_{f_1}(p), \lambda_{f_2}(p))$ in $\cup_j R_j$.

$$\begin{aligned}\frac{\#\{p \leqslant X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \cup_j R_j\}}{\#\{p \leq X\}} &= \sum_{j=1}^{g(X)} \Big(\mu_{ST,2}(R_j) + O\Big(\frac{\log\log\log X}{\sqrt{\log\log X}}\Big)\Big)\\ &= \mu_{ST,2}(\cup_j R_j) + O\Big(g(X)\frac{\log\log\log X}{\sqrt{\log\log X}}\Big).\end{aligned}$$

Since the area of $\cup_j R_j \setminus S$ is at most $g(X)\lfloor 4/g(X)\rfloor^2$, we have

$$\mu_{ST,2}(\cup_j R_j) - \mu_{ST,2}(S) = \mu_{ST,2}((\cup_j R_j) \setminus S) = O(g(X)^{-1}),$$

and we conclude that

$$\begin{aligned}&\frac{\#\{p \leqslant X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in \cup_j R_j\}}{\#\{p \leq X\}}\\ &= \nu(\mathcal{I}) + O\Big(g(X)^{-1} + g(X)\frac{\log\log\log X}{\sqrt{\log\log X}}\Big) = \nu(\mathcal{I}) + O\Big(\frac{\sqrt{\log\log\log X}}{(\log\log X)^{1/4}}\Big).\end{aligned} \tag{2·7}$$

The same argument shows that (2·7) holds with R_j replaced with T_j on the left-hand side. The lemma now follows from (2·6).

Finally, we require a refinement of (1·4), namely

$$N_f(T) = T\log\Big(q_f\Big(\frac{T}{2\pi e}\Big)^2\Big) + O(\log(k_f q_f T)). \tag{2·8}$$

This is [**IK04**, theorem 5·8] applied to $L(s,f)$.

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

3. *Proof of Theorem* 1·1

Let $\pi \in \mathscr{A}$. We detect the zeros of $L(s,\pi)$ by estimating a mollified second moment of $L(s,\pi)$ near the line $\mathrm{Re}(s) = 1/2$. To describe our mollifier, we define $\mu_\pi(n)$ by the convolution identity

$$\sum_{d|n} \mu_\pi(d)\lambda_\pi(n/d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases} \tag{3·1}$$

so that

$$\frac{1}{L(s,\pi)} = \sum_{n=1}^{\infty} \frac{\mu_\pi(n)}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

Let $T > 0$ be a large parameter, and let $0 < \varpi < 1/4$. Define $P(t)$ by

$$P(t) = \begin{cases} 1 & \text{if } 0 \le t \le T^{\varpi/2}, \\ 2(1 - \frac{\log t}{\log T^{\varpi}}) & \text{if } T^{\varpi/2} < t \le T^{\varpi}, \\ 0 & \text{if } t > T^{\varpi}. \end{cases} \tag{3·2}$$

Our mollifier is

$$M_\pi(s, T^{\varpi}) = \sum_{n \le T^{\varpi}} \frac{\mu_\pi(n)}{n^s} P(n). \tag{3·3}$$

As a proxy for detecting zeros of $L(s,\pi)$ near $\mathrm{Re}(s) = 1/2$, where $L(s,\pi)$ oscillates wildly, we detect zeros of the mollified L-function $L(s,\pi)M_\pi(s,T^{\varpi})$ near $\mathrm{Re}(s) = 1/2$. To this end, we let $w : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function whose support is a compact subset of $[T/4, 2T]$ and whose j-th derivative satisfies $|w^{(j)}(t)| \ll_{w,j} ((\log T)/T)^j$ for all $j \ge 0$. Also, let $w(t) = 1$ for $t \in [T/2, T]$. We will estimate

$$I_f(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) L(\tfrac{1}{2} + \alpha + it, \pi) L(\tfrac{1}{2} + \beta - it, \tilde{\pi}) |M_\pi(\tfrac{1}{2} + \tfrac{1}{\log T} + it, T^{\varpi})|^2 dt,$$

eventually choosing α and β to equal $1/\log T$. Define

$$\begin{aligned}
G(s) &= e^{s^2} \frac{(\alpha+\beta)^2 - (2s)^2}{(\alpha+\beta)^2}, \\
g_{\alpha,\beta}(s,t) &= \frac{L(\frac{1}{2} + \alpha + s + it, \pi_\infty) L(\frac{1}{2} + \beta + s - it, \tilde{\pi}_\infty)}{L(\frac{1}{2} + \alpha + it, \pi_\infty) L(\frac{1}{2} + \beta - it, \tilde{\pi}_\infty)}, \\
V_{\alpha,\beta}(s,t) &= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=1} \frac{G(s)}{s} g_{\alpha,\beta}(s,t) x^{-s} ds, \\
X_{\alpha,\beta}(t) &= \frac{L(\frac{1}{2} - \alpha - it, \pi_\infty) L(\frac{1}{2} - \beta + s + it, \tilde{\pi}_\infty)}{L(\frac{1}{2} + \alpha + it, \pi_\infty) L(\frac{1}{2} + \beta - it, \tilde{\pi}_\infty)}.
\end{aligned}$$

As in [**Ber15, AT21**], it follows from the approximate functional equation (see [**IK04**, section 5.2] also) that

$$I_f(\alpha,\beta) = \sum_{a,b \le T^{\varpi}} \frac{\mu_\pi(a)\overline{\mu_\pi(b)}}{\sqrt{ab}} \frac{P(a)P(b)}{(ab)^{\frac{1}{\log T}}} (D^+_{a,b}(\alpha,\beta) + D^-_{a,b}(\alpha,\beta) + N^+_{a,b}(\alpha,\beta) + N^-_{a,b}(\alpha,\beta)),$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

where we have split the sum into diagonal terms

$$D^{+}_{a,b}(\alpha,\beta)=\sum_{am=bn}\frac{\lambda_\pi(m)\overline{\lambda_\pi(n)}}{m^{\frac12+\alpha}n^{\frac12+\beta}}\int_{-\infty}^{\infty}w(t)V_{\alpha,\beta}(mn,t)\,dt,$$
$$D^{-}_{a,b}(\alpha,\beta)=\sum_{am=bn}\frac{\lambda_\pi(m)\overline{\lambda_\pi(n)}}{m^{\frac12-\beta}n^{\frac12-\alpha}}\int_{-\infty}^{\infty}w(t)X_{\alpha,\beta}(t)V_{-\beta,-\alpha}(mn,t)\,dt$$

and off-diagonal terms

$$N^{+}_{a,b}(\alpha,\beta)=\sum_{am\neq bn}\frac{\lambda_\pi(m)\overline{\lambda_\pi(n)}}{m^{\frac12+\alpha}n^{\frac12+\beta}}\int_{-\infty}^{\infty}w(t)\Big(\frac{bn}{am}\Big)^{it}V_{\alpha,\beta}(mn,t)\,dt,$$
$$N^{-}_{a,b}(\alpha,\beta)=\sum_{am\neq bn}\frac{\lambda_\pi(m)\overline{\lambda_\pi(n)}}{m^{\frac12-\beta}n^{\frac12-\alpha}}\int_{-\infty}^{\infty}w(t)\Big(\frac{bn}{am}\Big)^{it}X_{\alpha,\beta}(t)V_{-\beta,-\alpha}(mn,t)\,dt.$$

LEMMA 3·1. *Let $\varepsilon>0$. If $\alpha,\beta\in\mathbb{C}$ satisfy $|\alpha|,|\beta|\ll 1/\log T$ and $|\alpha+\beta|\gg 1/\log T$, then for any integers $a,b\geq 1$, we have that $N^{\pm}_{a,b}(\alpha,\beta)\ll_\varepsilon (ab)^{\frac12}T^{\frac12+\theta}(abT)^{\varepsilon}$.*

Proof. This is [**AT21**, proposition 3·4].

COROLLARY 3·2. *Fix $0<\varpi<1/4-\theta/2$. There exists a constant $\delta>0$ such that*

$$\Big|\sum_{a,b\leq T^{\varpi}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}}{\sqrt{ab}}\frac{P(a)P(b)}{(ab)^{\frac{1}{\log T}}}(N^{+}_{a,b}(\alpha,\beta)+N^{-}_{a,b}(\alpha,\beta))\Big|\ll T^{1-\delta}.$$

Proof. Let $\varepsilon>0$. First, observe that $|\mu_\pi(a)|\ll 1+|\lambda_\pi(a)|\ll 1+|\lambda_\pi(a)|^2$. We then apply Lemma 3·1 and bound everything else trivially to obtain

$$\sum_{a,b\leq T^{\varpi}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}}{\sqrt{ab}}\frac{P(a)P(b)}{(ab)^{\frac{1}{\log T}}}\sum_{\pm}N^{\pm}_{a,b}(\alpha,\beta)\ll T^{\frac12+\theta+\varepsilon}\Big(\sum_{a\leq T^{\varpi}}1+\sum_{a\leq T^{\varpi}}|\lambda_\pi(a)|^2\Big)^2.$$

By (2·2), this is $\ll_{f,\varepsilon} T^{\frac12+\theta+2\varpi+\varepsilon}$. If $\varpi\leq 1/4-\theta/2-\varepsilon$, then the above display is $\ll T^{1-\varepsilon}$.

PROPOSITION 3·3. *If T is sufficiently large and $\varpi\in(0,1/4-\theta/2)$ is fixed, then*

$$\int_{T/2}^{T}|L(\tfrac12+\tfrac{1}{\log T}+it,\pi)M_\pi(\tfrac12+\tfrac{1}{\log T}+it,T^{\varpi})|^2dt\ll T.$$

Proof. In light of Corollary 3·2, it remains to bound the diagonal contribution. We first note by a calculation identitcal to [**Ber15**, lemma 11] that

$$\sum_{a,b\leq T^{\varpi}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}}{\sqrt{ab}}\frac{P(a)P(b)}{(ab)^{\frac{1}{\log T}}}(D^{+}_{a,b}(\alpha,\beta)+D^{-}_{a,b}(\alpha,\beta))$$
$$=\sum_{a,b\leq T^{\varpi}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}}{\sqrt{ab}}\frac{P(a)P(b)}{(ab)^{\frac{1}{\log T}}}(D^{+}_{a,b}(\alpha,\beta)+T^{-2(\alpha+\beta)}D^{+}_{a,b}(-\alpha,-\beta))+O\Big(\frac{T}{\log T}\Big).$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

So it suffices for us to estimate

$$I^D(\alpha,\beta)=\sum_{a,b\le T^{\varpi}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}}{\sqrt{ab}}\frac{P(a)P(b)}{(ab)^{\frac{1}{\log T}}}D^+_{a,b}(\alpha,\beta)$$

with $\alpha=\beta=1/\log T$.

Let $\sigma_0=1/2+1/\log T$. We observe via the Mellin inversion that $I^D(\alpha,\beta)$ equals

$$\frac{4}{(\log(T^{\varpi}))^2}\int_{-\infty}^{\infty}\frac{1}{(2\pi i)^3}\int_{(1)}\int_{(1)}\int_{(1)}T^{\frac{\varpi(u+v)}{2}}(T^{\frac{u\varpi}{2}}-1)(T^{\frac{v\varpi}{2}}-1)\frac{G(s)}{s}g_{\alpha,\beta}(s,t)$$
$$\times\sum_{\substack{a,b,m,n\ge1\\ am=bn}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}\lambda_\pi(m)\overline{\lambda_\pi(n)}}{a^{\sigma_0+v}b^{\sigma_0+u}m^{\frac12+\alpha+s}n^{\frac12+\beta+s}}ds\frac{du}{u^2}\frac{dv}{v^2}dt.$$

By a computation identical to [**Ber15**, lemma 6], there exists a product of half-planes containing an open neighborhood of the point $u=v=s=0$ and an Euler product $A_{\alpha,\beta}(u,v,s)$, absolutely convergent for (u,v,s) in said product of half-planes, such that

$$\sum_{\substack{a,b,m,n\ge1\\ am=bn}}\frac{\mu_\pi(a)\overline{\mu_\pi(b)}\lambda_\pi(m)\overline{\lambda_\pi(n)}}{a^{\sigma_0+v}b^{\sigma_0+u}m^{\frac12+\alpha+s}n^{\frac12+\beta+s}}=A_{\alpha,\beta}(u+\tfrac{1}{\log T},v+\tfrac{1}{\log T},s)$$
$$\times\frac{L(1+\alpha+\beta+2s,\pi\otimes\tilde\pi)L(1+\frac{2}{\log T}+u+v,\pi\otimes\tilde\pi)}{L(1+\frac{1}{\log T}+\alpha+u+s,\pi\otimes\tilde\pi)L(1+\frac{1}{\log T}+\beta+v+s,\pi\otimes\tilde\pi)}.$$

By Möbius inversion and a continuity argument, one can prove that $A_{0,0}(0,0,0)=1$ [**Ber15**, lemma 7].

Upon choosing δ sufficiently small and shifting the contours to $\mathrm{Re}(u)=\delta$, $\mathrm{Re}(v)=\delta$, and $\mathrm{Re}(s)=-\delta/2$, we find that

$$I^D(\alpha,\beta)=\frac{4L(1+\alpha+\beta,\pi\otimes\tilde\pi)}{(2\pi i)^2(\log(T^{\varpi}))^2}\int_{\mathbb{R}}w(t)dt\int_{(\delta)}\int_{(\delta)}T^{\frac{\varpi(u+v)}{2}}(T^{\frac{u\varpi}{2}}-1)(T^{\frac{v\varpi}{2}}-1)$$
$$\times\frac{L(1+\frac{2}{\log T}+u+v,\pi\otimes\tilde\pi)A_{\alpha,\beta}(u+\frac{1}{\log T},v+\frac{1}{\log T},0)}{L(1+\frac{1}{\log T}+\alpha+u,\pi\otimes\tilde\pi)L(1+\frac{1}{\log T}+\beta+v,\pi\otimes\tilde\pi)}\frac{du}{u^2}\frac{dv}{v^2}+O\Big(\frac{T}{\log T}\Big).$$

The contribution from the pole at $u=v=0$ determines the magnitude of the double integral over u and v, and this magnitude is $\asymp_\pi(\log T)^2$ for α and β in our prescribed range. (This follows once we push the u- and v-contours to the left using (2·3).) Since $\int_{\mathbb{R}}w(t)dt\asymp T$ by hypothesis, we combine all preceding contributions and choose $\alpha=\beta=1/\log T$ to conclude the desired bound.

COROLLARY 3·4. *If T is sufficiently large and $\varpi\in(0,1/4-\theta/2)$ is fixed, then*

$$\int_{T/2}^{T}|L(\tfrac12+\tfrac{1}{\log T}+it,\pi)M_\pi(\tfrac12+\tfrac{1}{\log T}+it,T^{\varpi})-1|^2dt\ll T.$$

Proof. This follows from Proposition 3·3 and the Cauchy–Schwarz inequality.

We prove a corresponding estimate on a vertical line to the right of $\mathrm{Re}(s)=1$.

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

LEMMA 3·5. *If $\varpi \in (0, 1/4 - \theta/2)$ and $A > \theta$ are fixed and T is sufficiently large, then*

$$\int_{T/2}^{T} |L(1+A+it,\pi)M_{\pi}(1+A+it,T^{\varpi})-1|^2 dt \ll_A T^{1+(\theta-A-\frac{1}{2})\varpi}(\log T)^{15}.$$

Proof. Let $A > \theta$. It follows immediately from [MV74, corollary 3] that if $T > 1$ and (b_n) is any sequence of complex numbers satisfying $\sum_n n|b_n|^2 < \infty$, then

$$\int_{T/2}^{T} \Big|\sum_{n=1}^{\infty} b_n n^{-it}\Big|^2 dt = \sum_{n=1}^{\infty} |b_n|^2 (T/2 + O(n)). \tag{3·4}$$

We define a_n by the identity

$$L(1+A+it,\pi)M_{\pi}(1+A+it,T^{\varpi}) - 1 = \sum_{n=1}^{\infty} a_n n^{-1-A-it}.$$

By (3·1), (3·2), and (3·3), we have that $a_n = 0$ for all $n \le T^{\varpi/2}$ and $|a_n|^2 \ll (n^{\theta} d_4(n))^2 \ll n^{2\theta} d_{16}(n)$, where $d_k(n)$ is the n-th Dirichlet coefficient of $\zeta(s)^k$. The desired result follows once we apply (3·4) with $b_n = a_n n^{-1-A}$.

Proof of Theorem **1·1**. We use Gabriel's convexity principle [Tit86, section 7·8] to interpolate the bounds in Corollary 3·4 and Lemma 3·5. In particular, if $0 < \varpi < 1/4 - \theta/2$ and $A > \theta$ are fixed and $c = \varpi(1+2A-2\theta)/(1+2A)$, then in the range $1/2 + 1/\log T \le \sigma \le 1 + A$, we have

$$\begin{aligned}
&\int_{T/2}^{T} |L(\sigma+it,\pi)M_{\pi}(\sigma+it,T^{\varpi})-1|^2 dt \\
&\ll T^{\frac{1+A-\sigma}{1+A-(\frac{1}{2}+\frac{1}{\log T})}} \left(T^{1+(\theta-A-\frac{1}{2})\varpi}(\log T)^{15}\right)^{1-\frac{1+A-\sigma}{1+A-(\frac{1}{2}+\frac{1}{\log T})}} \ll T^{1-c(\sigma-\frac{1}{2})}.
\end{aligned} \tag{3·5}$$

Define $\Phi(s) := 1 - (1 - L(s,\pi)M_{\pi}(s,T^{\varpi}))^2$. By construction, if $\alpha \in \mathbb{C}$, then

$$\operatorname*{ord}_{s=\alpha} \Phi(s) \ge \operatorname*{ord}_{s=\alpha} L(s,\pi)M_{\pi}(s,T^{\varpi}).$$

For any $M \geqslant 1$, let C_M be the rectangular contour with corners $\sigma + iT/2$, $\sigma + iT$, $M + iT/2$, and $M + iT$. Applying Littlewood's lemma ([Tit58, pp. 132–133]) and letting $M \to \infty$, we have

$$\begin{aligned}
&\int_{\sigma}^{1} (N_{\pi}(\sigma',T) - N_{\pi}(\sigma',\tfrac{T}{2}))d\sigma' \leqslant \lim_{M\to\infty} \frac{1}{2\pi}\int_{C_M} \log \Phi(s) ds \\
&= \frac{1}{2\pi}\int_{T/2}^{T} \log|\Phi(\sigma+it)|dt + \frac{1}{2\pi}\int_{\sigma}^{\infty} \arg(\Phi(x+iT))dx - \frac{1}{2\pi}\int_{\sigma}^{\infty} \arg(\Phi(x+\tfrac{iT}{2}))dx.
\end{aligned}$$

In view of the bound $\log|1+z| \leqslant |z|$, it follows from (3·5) that

$$\frac{1}{2\pi}\int_{T/2}^{T} \log|\Phi(\sigma+it)|dt \le \int_{T/2}^{T} |L(\sigma+it,\pi)M_{\pi}(\sigma+it,T^{\varpi})-1|^2 dt \ll T^{1-c(\sigma-\frac{1}{2})}.$$

For the second and third integrals, we consider the integrals over $x \leqslant 1$ and over $x > 1$ separately. For $x \in (1, \infty)$ we can trivially bound the Dirichlet series as

$$|1 - L(x+iT, \pi)M_\pi(x+iT, T^{\varpi})| \leqslant \sum_{n=2}^{\infty} |a_n| n^{-x} \ll 2^{-x},$$

where (a_n) is a certain sequence of complex numbers such that $|a_n| \ll_\varepsilon n^{\theta+\varepsilon}$ for any fixed $\varepsilon > 0$ and all $n \geq 2$. Thus, we have

$$|\arg(1 - (1 - L(x+iT, \pi)M_\pi(x+iT, T^{\varpi}))^2)| \ll 2^{-x},$$

so

$$\left| \int_1^\infty \arg(\Phi(x+iT))dx \right|, \left| \int_1^\infty \arg(\Phi(x+2iT))dx \right| \ll 1.$$

To handle the integrals for $1/2 + 1/\log T \leq x \leq 1$, we use the trivial bound

$$\left| \int_\sigma^1 \arg(\Phi(x+2iT))dx \right| \leq (1-\sigma) \max_{\sigma \leq x \leq 1} |\arg(\Phi(x+2iT))| \ll \log T.$$

A proof of the final bound is contained within the proof of [IK04, theorem 5·8]. The corresponding integral for $\Phi(x+iT)$ has the same bound.

By the preceding work, it follows by dyadic decomposition that

$$\int_\sigma^1 N_\pi(\sigma', T)d\sigma' \ll T^{1-c(\sigma-\frac{1}{2})}, \qquad \frac{1}{2} + \frac{1}{\log T} \leq \sigma \leq 1.$$

This estimate, the mean value theorem for integrals, and the fact that $N_\pi(\sigma, T)$ is monotonically decreasing as σ increases together imply that

$$\begin{aligned} N_\pi(\sigma, T) &\leq \frac{1}{\sigma - (\sigma - \frac{1}{\log T})} \int_{\sigma - \frac{1}{\log T}}^{\sigma} N_\pi(\sigma', T)d\sigma' \\ &\ll \frac{1}{\sigma - (\sigma - \frac{1}{\log T})} \int_{\sigma - \frac{1}{\log T}}^{1} N_\pi(\sigma', T)d\sigma' \ll T^{1-c(\sigma-\frac{1}{2})} \log T. \end{aligned}$$

If $1/2 \leq \sigma \leq 1/2 + 1/\log T$, then (1·4) implies that $N_\pi(\sigma, T) \ll T \log T \asymp T^{1-c(\sigma-1/2)} \log T$.

To finish the proof, note that if $\theta = 0$, then for all $A > 0$, we have $c = \varpi$. If $\theta > 0$, then fix $0 < \varepsilon < \theta\varpi/(\theta + 1/2)$ and choose $A = \theta\varpi/\varepsilon - 1/2$. With these choices, we find that $c > \varpi - \varepsilon$. Theorem 1·1 now follows.

4. *Proof of Theorem* 1·3

We begin with a few preliminary lemmas. Throughout the section, θ is an admissible exponent toward the generalised Ramanujan conjecture as in Theorem 1·1. Our first result is an n-dimensional version of the Riemann–Lebesgue lemma.

LEMMA 4·1. *Let $J \geq 1$. Suppose that $h \in C^{n+2}(\mathbb{T}^n)$ has the Fourier expansion*

$$h(\boldsymbol{t}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^n} c_{\boldsymbol{m}} e^{2\pi i(\boldsymbol{m}\cdot\boldsymbol{t})}.$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

We have $|c_{\boldsymbol{m}}| \ll_h \|\boldsymbol{m}\|_2^{-n-2}$*, and consequently, we have*

$$h(\boldsymbol{t}) = \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^n \\ \|\boldsymbol{m}\|_2\le J}} c_{\boldsymbol{m}} e^{2\pi i(\boldsymbol{m}\cdot\boldsymbol{t})} + O_h(J^{-2}).$$

Proof. We have

$$c_{\boldsymbol{m}} = \int_{\mathbb{T}^n} h(\boldsymbol{t}) e^{-2\pi i(\boldsymbol{m}\cdot\boldsymbol{t})} \mathrm{d}\boldsymbol{t}. \tag{4·1}$$

Let $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$. Choose $j \in \{1, \ldots, n\}$ such that $|m_j| = \|\boldsymbol{m}\|_\infty$. Integrate (4·1) by parts for $n+2$ times with respect to the coordinate t_j of $\boldsymbol{t}$ so that

$$c_{\boldsymbol{m}} = \frac{(-1)^{n+2}}{(2\pi m_j)^{n+2}} \int_{\mathbb{T}^n} \Big(\frac{\partial^{n+2}}{\partial t_j^{n+2}} h(\boldsymbol{t})\Big) e^{-2\pi i(\boldsymbol{m}\cdot\boldsymbol{t})} \mathrm{d}\boldsymbol{t}. \tag{4·2}$$

Since $\sqrt{n}\|\boldsymbol{m}\|_\infty \ge \|\boldsymbol{m}\|_2$, the desired result follows from the triangle inequality.

LEMMA 4·2. *Let* $x > 1$ *and* $T \geqslant 2$*, and let* $\langle x\rangle$ *being the closest integer to* x*. We have*

$$\sum_{|\gamma|\le T} x^\rho = -\frac{\Lambda_\pi(\langle x\rangle)}{\pi} \cdot \frac{e^{iT\log(x/\langle x\rangle)} - 1}{i\log(x/\langle x\rangle)} + O\Big(x^{1+\theta}(\log(2x) + \log T) + \frac{\log T}{\log x}\Big).$$

Proof. This is [**FSZ09**, lemma 2] with $\varepsilon = \theta$.

Using Theorem 1·1 and Lemma 4·2, we prove an analogue of [**FZ05**, (3·8)].

LEMMA 4·3. *Let* c *be as in Theorem* 1·1*. If* $1 < x < \exp((c/3)(\log T/\log\log T))$*, then*

$$\sum_{|\gamma|\le T} x^{i\gamma} = \sum_{|\gamma|\le T} x^{\rho-\frac{1}{2}} + O\Big(\frac{T(\log x)^2}{\log T} + \frac{T}{(\log T)^2}\Big).$$

Proof. Let $\delta = ((3/c)(\log\log T/\log T))$, so $0 < \delta\log x < 1$. By Theorem 1·1, we have that

$$\Big|\sum_{\substack{|\gamma|\le T \\ |\beta-\frac{1}{2}|\geqslant\delta}} (x^{i\gamma} - x^{\rho-\frac{1}{2}})\Big| \ll \sum_{\substack{|\gamma|\le T \\ \beta\geqslant\frac{1}{2}+\delta}} x^{\beta-\frac{1}{2}} \ll x^\delta N_\pi(\tfrac{1}{2}+\delta, T)$$
$$+ \log x \int_{\frac{1}{2}+\delta}^{1} x^{\sigma-\frac{1}{2}} N_\pi(\sigma, T) d\sigma \ll \frac{T}{(\log T)^2}.$$

By the functional equation for $L(s,\pi)$, $\beta + i\gamma$ is a nontrivial zero if and only if $1 - \beta + i\gamma$ is a nontrivial zero. Therefore, we have

$$\Big|\sum_{\substack{|\gamma|\le T \\ |\beta-\frac{1}{2}|<\delta}} (x^{i\gamma} - x^{\rho-\frac{1}{2}})\Big| = \Big|\sum_{\substack{|\gamma|\le T \\ \frac{1}{2}<\beta<\frac{1}{2}+\delta}} (x^{i\gamma}(1 - x^{\beta-\frac{1}{2}}) + x^{i\gamma}(1 - x^{\frac{1}{2}-\beta})\Big|$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

$$\leq \sum_{\substack{|\gamma|\leq T \\ 0<\beta-\frac{1}{2}<\delta}} |x^{i\gamma}(1-x^{\beta-\frac{1}{2}})+x^{i\gamma}(1-x^{-(\beta-\frac{1}{2})})|$$
$$= \sum_{\substack{|\gamma|\leq T \\ 0<\beta-\frac{1}{2}<\delta}} (x^{\beta-\frac{1}{2}}+x^{-(\beta-\frac{1}{2})}-2). \tag{4·3}$$

Note that $x^{\beta-1/2}+x^{-(\beta-1/2)}-2=(2\sinh(1/2(\beta-1)\log x))^2$. Note that if $0<\beta-1/2<\delta$, then $0<(1/2)(\beta-1/2)\log x<1/2$. Since $y<\sinh(y)<2y$ for $0<y<1/2$, (4·3) is

$$\ll (\log x)^2 \sum_{\substack{|\gamma|\leq T \\ 0<\beta-\frac{1}{2}<\delta}} \Big(\beta-\frac{1}{2}\Big)^2 \ll (\log x)^2 \int_0^\delta \sigma N_\pi(\tfrac{1}{2}+\sigma,T)d\sigma \ll \frac{T(\log x)^2}{\log T}.$$

The desired result follows.

Proof of Theorem **1·3.** Let $h\in C^{n+2}(\mathbb{T}^n)$, and let $\boldsymbol{\alpha}$ satisfy (1·12). Let $J\in[1,100\log(eT)]$. We begin with the expansion

$$\sum_{|\gamma|\leqslant T} h(\gamma\boldsymbol{\alpha}) = \sum_{|\gamma|\leqslant T}\sum_{\boldsymbol{m}\in\mathbb{Z}^n} c_{\boldsymbol{m}} e^{2\pi i\gamma(\boldsymbol{m}\cdot\boldsymbol{\alpha})}$$
$$= N_\pi(T)\int_{\mathbb{T}^n} h(\boldsymbol{t})\mathrm{d}\boldsymbol{t} + \sum_{|\gamma|\leqslant T}\Big(\sum_{1\leqslant\|\boldsymbol{m}\|_2\leqslant J} + \sum_{\|\boldsymbol{m}\|_2>J}\Big)c_{\boldsymbol{m}}e^{2\pi i\gamma(\boldsymbol{m}\cdot\boldsymbol{\alpha})}.$$

By Lemma 4·1, we have

$$\sum_{|\gamma|\leqslant T} h(\gamma\boldsymbol{\alpha}) - N_\pi(T)\int_{\mathbb{T}^n} h(\boldsymbol{t})\mathrm{d}\boldsymbol{t} = \sum_{|\gamma|\leqslant T}\sum_{1\leqslant\|\boldsymbol{m}\|_2\leqslant J} c_{\boldsymbol{m}}e^{2\pi i\gamma(\boldsymbol{m}\cdot\boldsymbol{\alpha})} + O\Big(\frac{N_\pi(T)}{J^2}\Big). \tag{4·4}$$

Write $x_{\boldsymbol{m}}=e^{2\pi(\boldsymbol{m}\cdot\boldsymbol{\alpha})}$. Since $x_{-\boldsymbol{m}}^{i\gamma}=x_{\boldsymbol{m}}^{-i\gamma}$ and $c_{-\boldsymbol{m}}=-c_{\boldsymbol{m}}$, we find that (4·4) equals

$$2\Re \sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J \\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}} c_{\boldsymbol{m}} \sum_{0<\gamma\leqslant T} x_{\boldsymbol{m}}^{i\gamma} + O\Big(\frac{N_\pi(T)}{J^2}\Big).$$

Choose J so that $\|\boldsymbol{m}\|_2\leq J$ implies $\log x_{\boldsymbol{m}}<((c/3)(\log T/\log\log T))$ (with c as in Theorem 1·1). By Lemma 4·3 and the above display, (4·4) equals

$$2\Re \sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J \\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}} c_{\boldsymbol{m}} \sum_{|\gamma|\leq T} x_{\boldsymbol{m}}^{\rho-\frac{1}{2}} + O\Big(\frac{N_\pi(T)}{J^2} + \frac{T}{\log T}\sum_{1\leqslant\|\boldsymbol{m}\|_2\leqslant J}\frac{1}{\|\boldsymbol{m}\|_2^{n+2}}\Big((\log x_{\boldsymbol{m}})^2+\frac{1}{\log T}\Big)\Big).$$

Since $\log x_{\boldsymbol{m}}\leq 2\pi\|\boldsymbol{m}\|_2\|\boldsymbol{\alpha}\|_2$, it follows from our preliminary bound for J that (4·4) equals

$$2\Re \sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J \\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}} c_{\boldsymbol{m}} \sum_{|\gamma|\leq T} x_{\boldsymbol{m}}^{\rho-\frac{1}{2}} + O\Big(\frac{N_\pi(T)}{J^2} + \frac{T\log\log T}{\log T}\Big). \tag{4·5}$$

We apply Lemma 4·2 to conclude that (4·5) equals

$$-\frac{2T}{\pi}\Re\sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}}\frac{c_{\boldsymbol{m}}\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)}{\sqrt{x_{\boldsymbol{m}}}}\cdot\frac{e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1}{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}+\mathcal{E}, \tag{4·6}$$

where $\mathcal{E}$ satisfies (note that $\log x_{\boldsymbol{m}} = 2\pi(\boldsymbol{m}\cdot\boldsymbol{\alpha})$)

$$|\mathcal{E}| \ll \sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}} x_{\boldsymbol{m}}^{1+\theta}(\log(2x_{\boldsymbol{m}})+\log T)+\sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}}\frac{\log T}{\boldsymbol{m}\cdot\boldsymbol{\alpha}}+\Big(\frac{N_\pi(T)}{J^2}+\frac{T\log\log T}{\log T}\Big).$$

Our choice of J ensures that $\log(2x_{\boldsymbol{m}}) \ll \log T$, so it follows from (1·12) that

$$|\mathcal{E}| \ll \sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}}(x_{\boldsymbol{m}}^{1+\theta}+e^{\|\boldsymbol{m}\|_2})\log T+\frac{N_\pi(T)}{J^2}+\frac{T\log\log T}{\log T}.$$

Since $x_{\boldsymbol{m}}^{1+\theta} = e^{2\pi(1+\theta)(\boldsymbol{m}\cdot\boldsymbol{\alpha})} \le e^{4\pi\|\boldsymbol{m}\|_2\|\boldsymbol{\alpha}\|_2}$, it follows that

$$|\mathcal{E}| \ll e^{(n+4\pi\|\boldsymbol{\alpha}\|_2)J}\log T+\frac{N_\pi(T)}{J^2}+\frac{T\log\log T}{\log T}. \tag{4·7}$$

We choose $J = (\log T)^{2/3}$. Since $N_\pi(T) \ll T\log T$, (4·6) equals

$$-\frac{2T}{\pi}\Re\sum_{\substack{1\leqslant\|\boldsymbol{m}\|_2\leqslant J\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0}}\frac{c_{\boldsymbol{m}}\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)}{\sqrt{x_{\boldsymbol{m}}}}\cdot\frac{e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1}{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}+O\Big(\frac{T}{(\log T)^{1/3}}\Big). \tag{4·8}$$

Observe that if $x_{\boldsymbol{m}} \ne \langle x_{\boldsymbol{m}}\rangle$, then $|e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1| \le |iT\log(x_{\boldsymbol{m}}/\langle x_{\boldsymbol{m}}\rangle)|$. Also, since $0 \le \theta < 1/2$, it follows that $|\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)| \ll \sqrt{x_{\boldsymbol{m}}}$. The proof of Lemma 4·1 ensures that $|c_{\boldsymbol{m}}| \ll \|\boldsymbol{m}\|_2^{-n-2}$, so the sum over $\boldsymbol{m}$ converges absolutely. By the decay of $|c_{\boldsymbol{m}}|$, our choice of J, and (4·7), (4·8) equals

$$-\frac{2T}{\pi}\Re\Big(\sum_{\substack{\boldsymbol{m}\cdot\boldsymbol{\alpha}>0\\ x_{\boldsymbol{m}}=\langle x_{\boldsymbol{m}}\rangle}}\frac{c_{\boldsymbol{m}}\Lambda_\pi(x_{\boldsymbol{m}})}{\sqrt{x_{\boldsymbol{m}}}}+\sum_{\substack{\boldsymbol{m}\cdot\boldsymbol{\alpha}>0\\ x_{\boldsymbol{m}}\ne\langle x_{\boldsymbol{m}}\rangle}}\frac{c_{\boldsymbol{m}}\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)}{\sqrt{x_{\boldsymbol{m}}}}\cdot\frac{e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1}{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}\Big)+O\Big(\frac{T}{(\log T)^{1/3}}\Big). \tag{4·9}$$

In particular, each sum over $\boldsymbol{m}$ converges absolutely.

To handle the sum over $\boldsymbol{m}$ such that $x_{\boldsymbol{m}} \ne \langle x_{\boldsymbol{m}}\rangle$, we note (by absolute convergence) that for all $\varepsilon > 0$, there exists $M_\varepsilon = M_\varepsilon(\boldsymbol{\alpha}, h) > 0$ such that

$$\Big|\sum_{\substack{\|\boldsymbol{m}\|_2>M_\varepsilon\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0\\ x_{\boldsymbol{m}}\ne\langle x_{\boldsymbol{m}}\rangle}}\frac{c_{\boldsymbol{m}}\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)}{\sqrt{x_{\boldsymbol{m}}}}\cdot\frac{e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1}{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}\Big| < \varepsilon.$$

Consequently, we have

$$\Big| -\frac{2T}{\pi}\Re \sum_{\substack{\|\boldsymbol{m}\|_2\ge 1\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0\\ x_{\boldsymbol{m}}\ne\langle x_{\boldsymbol{m}}\rangle}} \frac{c_{\boldsymbol{m}}\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)}{\sqrt{x_{\boldsymbol{m}}}}\cdot\frac{e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1}{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}\Big|$$
$$\le \Big|\Re \sum_{\substack{\|\boldsymbol{m}\|_2\le M_\varepsilon\\ \boldsymbol{m}\cdot\boldsymbol{\alpha}>0\\ x_{\boldsymbol{m}}\ne\langle x_{\boldsymbol{m}}\rangle}} \frac{c_{\boldsymbol{m}}\Lambda_\pi(\langle x_{\boldsymbol{m}}\rangle)}{\sqrt{x_{\boldsymbol{m}}}}\cdot\frac{e^{iT\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}-1}{\log\frac{x_{\boldsymbol{m}}}{\langle x_{\boldsymbol{m}}\rangle}}\Big| + \varepsilon T.$$

As we let $\varepsilon \to 0$ sufficiently slowly, we conclude that (4·6) equals $o(T)$, as desired.

For the sum over $\boldsymbol{m}$ such that $x_{\boldsymbol{m}} = \langle x_{\boldsymbol{m}}\rangle$, which means $x_{\boldsymbol{m}} \in \mathbb{Z}$, the terms which are not prime powers will vanish due to the presence of the von Mangoldt function. For the other terms which are prime powers, we have $\boldsymbol{m}\cdot\boldsymbol{\alpha} = (k\log p)/(2\pi)$ for some $k \in \mathbb{N}$ by the definition of $x_{\boldsymbol{m}}$. This will only happen when $\boldsymbol{m}$ is a multiple of $q_j\boldsymbol{b}_j$ for some $j \in \{1, \dots, r\}$ due to our choice of the vector $\boldsymbol{\alpha}$ in (1·13), so

$$-\frac{2}{\pi}\Re \sum_{\substack{\boldsymbol{m}\cdot\boldsymbol{\alpha}>0\\ x_{\boldsymbol{m}}=\langle x_{\boldsymbol{m}}\rangle}} \frac{c_{\boldsymbol{m}}\Lambda_\pi(x_{\boldsymbol{m}})}{\sqrt{x_{\boldsymbol{m}}}} = -\frac{2}{\pi}\Re\sum_{j=1}^{r}\sum_{l=1}^{\infty}\frac{\Lambda_\pi(p_j^{a_jl})}{p_j^{a_jl/2}}c_{lq_j\boldsymbol{b}_j} = \int_{\mathbb{T}^n} h(\boldsymbol{t})g_{\pi,\boldsymbol{\alpha}}(\boldsymbol{t})\mathrm{d}\boldsymbol{t}.$$

The last equation holds because of (1·14) and (4·1).

5. *Proof of Corollary* 1·4

Let $\mathbb{B} \subseteq \mathbb{T}^n$ be a product of n subintervals of $\mathbb{T}$ for which $|\int_{\mathbb{B}} g_{f,\boldsymbol{\alpha}}(\boldsymbol{t})\mathrm{d}\boldsymbol{t}|$ attains its maximum. For $\varepsilon > 0$, let $\varphi_\varepsilon : \mathbb{T}^n \to \mathbb{R}$ satisfy the following conditions:

(i) φ_ε is nonnegative and infinitely differentiable;

(ii) φ_ε is supported on a compact subset of $U_\varepsilon := \{\boldsymbol{t} \in \mathbb{T}^n : \|\boldsymbol{t}\|_2 < \varepsilon\}$; and

(iii) $\int_{\mathbb{T}^n} \varphi_\varepsilon(\boldsymbol{t})\mathrm{d}\boldsymbol{t} = 1$.

Let $\mathbf{1}_{\mathbb{B}}$ be the indicator function of $\mathbb{B}$, and define $h_\varepsilon(\boldsymbol{t}) = \int_{\mathbb{T}^n}\varphi_\varepsilon(\boldsymbol{t})\mathbf{1}_{\mathbb{B}}(\boldsymbol{x}-\boldsymbol{t})\mathrm{d}\boldsymbol{t}$. Then h_ε is infinitely differentiable, and thus Theorem 1·3 holds with r arbitrarily large for $h = h_\varepsilon$. Consequently, for any fixed $r \ge n+2$, we have

$$\int_{\mathbb{T}^n} h_\varepsilon(\boldsymbol{y})\Big(\sum_{\substack{|\gamma|\le T\\ \{\gamma\boldsymbol{\alpha}\}\in\mathbb{B}+\boldsymbol{y}}} 1 - \mathrm{vol}(\mathbb{B})N_\pi(T)\Big)\mathrm{d}\boldsymbol{y} = T\int_{U_\varepsilon} h_\varepsilon(\boldsymbol{y})\int_{\mathbb{B}+\boldsymbol{y}} g_{f,\boldsymbol{\alpha}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} + o(T).$$

It follows from our definition of $g_{f,\boldsymbol{\alpha}}(\boldsymbol{t})$ in (1·14) that $g_{f,\boldsymbol{\alpha}}(\boldsymbol{t}) \ll 1$, hence

$$\Big|\int_{\mathbb{B}+\boldsymbol{y}} g_{f,\boldsymbol{\alpha}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{\mathbb{B}} g_{f,\boldsymbol{\alpha}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\Big| \ll \varepsilon$$

for all $\boldsymbol{y} \in U_\varepsilon$. Thus, we have

$$\int_{\mathbb{T}^n} h_\varepsilon(\boldsymbol{y})\Big(\sum_{\substack{|\gamma|\le T\\ \{\gamma\boldsymbol{\alpha}\}\in\mathbb{B}+\boldsymbol{y}}} 1 - \mathrm{vol}(\mathbb{B})N_\pi(T)\Big)\mathrm{d}\boldsymbol{y} = T\int_{\mathbb{B}} g_{f,\boldsymbol{\alpha}}(\boldsymbol{t})\mathrm{d}\boldsymbol{t} + O(\varepsilon T) + o(T).$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

By the mean value theorem, there exists $\boldsymbol{y} \in U_\varepsilon$ such that

$$\Big| \sum_{\substack{|\gamma|\leq T \\ \{\gamma\boldsymbol{\alpha}\}\in\mathbb{B}+\boldsymbol{y}}} 1 - \mathrm{vol}(\mathbb{B})N_\pi(T)\Big| \geq T\Big|\int_{\mathbb{B}} g_{f,\boldsymbol{\alpha}}(\boldsymbol{t})\mathrm{d}\boldsymbol{t}\Big| + O(\varepsilon T) + o(T).$$

The proof follows once we let $\varepsilon \to 0$ sufficiently slowly as a function of T.

6. *Proofs of Corollaries* 1·5-1·8

Throughout Sections 6·3-6·5, all levels are assumed to be squarefree.

6·1. *An estimate for the density function*

We begin with a useful estimate for the density function $g_{f,\boldsymbol{\alpha}}$ associated to a holomorphic cuspidal newform $f \in S_k^{\mathrm{new}}(\Gamma_0(q))$ as in Section 2.2.

LEMMA 6·1. *Let $f \in S_k(\Gamma_0(q))$ be a newform and let $\alpha = (a\log p)/(2\pi q)$. Then we have*

$$\Big|g_{f,\alpha}(t) + \frac{2}{\pi}\frac{\Lambda_\pi(p^a)}{p^{a/2}}\cos(2\pi qt)\Big| \leq \frac{4\log p}{\pi p^a(1-p^{-a/2})}.$$

Proof. In this case, Deligne's bound implies that

$$\begin{aligned}&\Big|g_{f,\alpha}(t) + \frac{2}{\pi}\frac{\Lambda_\pi(p^a)}{p^{a/2}}\cos(2\pi qt)\Big| \\ &= \Big|\frac{2}{\pi}\sum_{\ell=2}^{\infty}\frac{\Lambda_\pi(p^{a\ell})}{p^{\frac{a\ell}{2}}}\cos(2\pi q\ell t)\Big| \leqslant \sum_{\ell=2}^{\infty}\frac{2\log p}{p^{a\ell/2}} = \frac{2\log p}{p^a(1-p^{-a/2})}.\end{aligned}$$

6·2. *Proof of Corollary* 1·5

First, we prove a simple criterion for f_1 winning the $((\log p)/(2\pi), h)$-race. From Lemma 6·1, we obtain

$$\begin{aligned}\int_{\mathbb{T}} h(t)(g_{f_1,\alpha}(t) - g_{f_2,\alpha}(t))dt &> \frac{(\lambda_{f_2}(p) - \lambda_{f_1}(p))2\log p}{\pi\sqrt{p}}\int_{\mathbb{T}} h(t)\cos(2\pi t)dt \\ &- \frac{\int_{\mathbb{T}} h(t)dt \cdot 8\log p}{\pi p(1-p^{-1/2})}.\end{aligned}$$

Consequently, the inequality

$$(\lambda_{f_2}(p) - \lambda_{f_1}(p))\int_{\mathbb{T}} h(t)\cos(2\pi t)dt > \frac{4\int_{\mathbb{T}} h(t)dt}{\sqrt{p}(1-p^{-1/2})}, \tag{6·1}$$

implies that

$$\int_{\mathbb{T}} h(t)(g_{f_1,\alpha}(t) - g_{f_2,\alpha}(t))dt > 0,$$

which by Theorem 1·3 tells us that f_1 wins the $((\log p)/(2\pi), h)$-race.

Throughout the proof, we let $k_h := \int_{\mathbb{T}} h(t)\cos(2\pi t)dt$. The number of p for which f_1 wins the race is equal to $T_1 - T_2 + T_3$, where

$$T_1 := \#\{p \leqslant X : k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\},$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

$$T_2 := \#\{p \leqslant X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\},$$
$$T_3 := \#\{p \leqslant X : f_1 \text{ wins the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \leqslant 0\}.$$

From the symmetry of $\mu_{ST,2}$, we see that $\nu(\{(x,y) \in [-2,2]^2 : x - y > 0\}) = 1/2$. By Corollary 2·2 with $\mathcal{I} = (0,4)$, we have

$$T_1 = \frac{1}{2}\pi(X) + O\Big(\pi(X)\frac{\sqrt{\log\log\log X}}{(\log\log X)^{1/4}}\Big).$$

We have to show that T_2 and T_3 are both $O(\pi(X)(\sqrt{\log\log\log X}/(\log\log X)^{1/4}))$. Note that

$$T_2 = \#\{\sqrt{X} \leqslant p \leqslant X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\} + O(\pi(\sqrt{X})).$$

If $p \geqslant \sqrt{X}$ and f_1 loses the (α, h)-race and $k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0$, then by (6·1), we have

$$k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \Big(0, \frac{4\int_{\mathbb{T}} h(t)dt}{\sqrt{p}(1-p^{-1/2})}\Big] \subseteq \Big(0, \frac{4\int_{\mathbb{T}} h(t)dt}{X^{\frac{1}{4}}(1-X^{-1/2})}\Big].$$

Denoting by J_X the rightmost interval in the preceding display, we have

$$\begin{aligned}&\#\{p \leqslant X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\}\\ &\quad\leqslant \#\{p \leqslant X : \lambda_{f_2}(p) - \lambda_{f_1}(p) \in J_X\} + O(\pi(\sqrt{X})).\end{aligned}$$

By Corollary 2·2, this is at most

$$\begin{aligned}&\#\{p \leqslant X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\}\\ &\qquad\qquad \leq \nu\Big(|k_h|^{-1}J_X\Big)\pi(X) + O\Big(\pi(X)\frac{\sqrt{\log\log\log X}}{(\log\log X)^{1/4}}\Big).\end{aligned}$$

Since $\nu(|k_h|^{-1}J_X) = O(|J_X|) = O(X^{-1/4})$, it follows that

$$\#\{p \leqslant X : f_1 \text{ loses the } (\alpha, h)\text{-race, and } k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) > 0\} = O\Big(\pi(X)\frac{\sqrt{\log\log\log X}}{(\log\log X)^{1/4}}\Big).$$

If the conditions for T_3 are true, then

$$k_h(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \Big[-\frac{4\int_{\mathbb{T}} h(t)dt}{\sqrt{p}(1-p^{-1/2})}, 0\Big].$$

Therefore, $T_3 \ll \pi(X)(\sqrt{\log\log\log X}/(\log\log X)^{1/4})$ by the argument used to bound T_2.

The result follows from the estimate shown for T_1 and the bounds for T_2 and T_3.

6·3. *Proof of Corollary* 1·6

Proof. From Theorem 1·3, Lemma 6·1, and (2·8) we have

$$H\Big(f_1, f_2, h, \frac{\log p}{2\pi}\Big) = \frac{2k_h}{\pi} \cdot \frac{\log p \cdot (\lambda_{f_2}(p) - \lambda_{f_1}(p))}{p^{1/2}} + O_h\Big(\frac{\log p}{p}\Big). \tag{6·2}$$

Consider the statements

$$\frac{2k_h}{\pi}(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \mathcal{I} \tag{6·3}$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

and

$$\frac{\sqrt{X}}{\log X} H(f_1, f_2, h, \frac{\log p}{2\pi}) \in \mathcal{I}. \tag{6·4}$$

Defining

$$\begin{aligned} T_1 &:= \#\{p \in [(1-\varepsilon_X)X, X] : \text{(6·3) holds}\}, \\ T_2 &:= \#\{p \in [(1-\varepsilon_X)X, X] : \text{(6·3) holds and (6·4) fails}\}, \\ T_3 &:= \#\{p \in [(1-\varepsilon_X)X, X] : \text{(6·3) fails and (6·4) holds}\}, \end{aligned}$$

we have

$$\#\left\{p \in [(1-\varepsilon_X)X, X] : \frac{\sqrt{X}}{\log X} H\left(f_1, f_2, h, \frac{\log p}{2\pi}\right) \in \mathcal{I}\right\} = T_1 - T_2 + T_3.$$

By Corollary 2·2 we have:

$$\begin{aligned} T_1 &= \#\{p \leqslant X : \text{(6·3) holds}\} - \#\{p \leqslant (1-\varepsilon_X)X : \text{(6·3) holds}\} \\ &= \nu\left(\frac{\pi}{2k_h}\mathcal{I}\right)\pi(X) + O(\pi(X)\varepsilon_X^2) \\ &\quad - \nu\left(\frac{\pi}{2k_h}\mathcal{I}\right)\pi((1-\varepsilon_X)X) + O\left(\pi(X-\varepsilon_X X)\varepsilon_X^2\right) \\ &= \nu\left(\frac{\pi}{2k_h}\mathcal{I}\right) \cdot \left(\pi(X) - \pi((1-\varepsilon_X)X)\right) + O\left(\pi(X)\varepsilon_X^2\right). \end{aligned}$$

We proceed to show that T_2 and T_3 are $O\left(\varepsilon_X^2 \pi(X)\right)$ as $X \to \infty$. We first examine T_2. Set $\mathcal{I} = (\delta_1, \delta_2)$. If the condition in (6·4) is false, then

$$\frac{\sqrt{X}}{\log X} H\left(f_1, f_2, h, \frac{\log p}{2\pi}\right) \notin [\delta_1, \delta_2].$$

Applying (6·2), we deduce that

$$\frac{\sqrt{X}}{\log X} \frac{2k_h}{\pi} \cdot \frac{\log p \cdot (\lambda_{f_2}(p) - \lambda_{f_1}(p))}{p^{1/2}} \notin \left[\delta_1 + \frac{C \log p}{p}, \delta_2 - \frac{C \log p}{p}\right],$$

where C is an implied constant in (6·2). Then (6·3) gives us

$$\frac{2k_h}{\pi}(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \left(\delta_1, \delta_1\left(\frac{p^{1/2}\log X}{X^{1/2}\log p}\right) + \frac{C}{\sqrt{p}}\right) \cup \left(\delta_2\left(\frac{p^{1/2}\log X}{X^{1/2}\log p}\right) - \frac{C}{\sqrt{p}}, \delta_2\right). \tag{6·5}$$

Since $p \in [(1-\varepsilon_X)X, X]$, it follows that $(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in I_X$, where I_X is

$$\frac{\pi}{2k_h}\left(\delta_1, \frac{\delta_1 \log X}{\log X(1-\varepsilon_X)} + \frac{C}{\sqrt{X(1-\varepsilon_X)}}\right) \cup \frac{\pi}{2k_h}\left(\delta_2\sqrt{1-\varepsilon_X} - \frac{C}{\sqrt{X(1-\varepsilon_X)}}, \delta_2\right).$$

By Corollary 2·2, we have

$$\begin{aligned} &\#\{(1-\varepsilon_X)X \leqslant p \leqslant X : (\lambda_{f_1}(p) - \lambda_{f_2}(p)) \in I_X\} \\ &= \nu(I_X)(\pi(X) - \pi((1-\varepsilon_X)X)) + O\left(\pi(X)\varepsilon_X^2\right). \end{aligned}$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

From the prime number theorem, we obtain

$$\pi(X) - \pi((1-\varepsilon_X)X) \sim \varepsilon_X \pi(X).$$

Combining this with the fact that $\nu(I_X) = O(\varepsilon_X)$, we conclude the following:

$$\#\{(1-\varepsilon_X)X \leqslant p \leqslant X : (\lambda_{f_1}(p) - \lambda_{f_2}(p)) \in I_X\} = O(\varepsilon_X^2 \pi(X)).$$

Therefore, $T_2 = O(\varepsilon_X^2 \pi(X))$. A very similar argument can be used to bound T_3. More specifically, if (6·4) holds, then we have

$$\frac{\sqrt{X}}{\log X}\frac{2k_h}{\pi} \cdot \frac{\log p \cdot (\lambda_{f_2}(p) - \lambda_{f_1}(p))}{p^{1/2}} \in (\delta_1 - \frac{C\log p}{p}, \delta_2 + \frac{C\log p}{p}),$$

If (6·3) fails and (6·4) holds, then, much like (6·5), we obtain

$$\frac{2k_h}{\pi}(\lambda_{f_2}(p) - \lambda_{f_1}(p)) \in \left(\delta_1\left(\frac{p^{1/2}\log X}{X^{1/2}\log p}\right) - \frac{C}{\sqrt{p}}, \delta_1\right) \cup \left(\delta_2, \delta_2\left(\frac{p^{1/2}\log X}{X^{1/2}\log p}\right) + \frac{C}{\sqrt{p}}\right).$$

By Corollary 2·2, the number of such $p \in ((1-\varepsilon_X)X, X)$ is $O(\varepsilon_X^2\pi(X))$.

6·4. *Proof of Corollary* 1·7

By Theorem 1·3, if f_2 wins the (α, h)-race, where $\alpha = \log p/2\pi$, then we must have

$$0 < \frac{\log(q_1/q_2)}{\pi}\int_{\mathbb{T}} h(t)dt < \int_{\mathbb{T}} h(t)(g_{f_2,\alpha}(t) - g_{f_1,\alpha}(t))dt.$$

By Lemma 6·1, we have

$$\left|\int_{\mathbb{T}} (g_{f_2,\alpha}(t) - g_{f_1,\alpha}(t))h(t)dt\right| \leq \int_{\mathbb{T}} h(t)dt \cdot \frac{2\log p}{\pi}\left(p^{-\frac{1}{2}} + \frac{1}{(p^{\frac{1}{2}}-1)p^{\frac{1}{2}}}\right).$$

It follows that $\log(q_1/q_2) \leqslant 2p^{-1/2}(1 + (p^{\frac{1}{2}} - 1)^{-1})\log p$. The left hand side is independent of α and positive, while the right hand side tends to zero as p grows. Thus, this inequality holds for only finitely many primes p.

6·5. *Proof of Corollary* 1·8

Fix $t_0 \in [0, 1)$, and $k_1, k_2 \in \mathbb{Z}$. By Theorem 1·3 and the same reasoning as in [**LZ21**, theorem 1·2], that f_1 wins the local (α, t_0)-race against f_2 if

$$g_{f_1,\alpha}(t_0) + \frac{\log q_1}{2\pi} > g_{f_2,\alpha}(t_0) + \frac{\log q_2}{2\pi},$$

or, equivalently, if $(1/2\pi)\log(q_1/q_2) > g_{f_2,\alpha}(t_0) - g_{f_1,\alpha}(t_0)$.

We first assume $t_0 \neq 1/4, 3/4$. For $\alpha = \log p/2\pi$ and $q_2 \in (q_1, q_1 + \sqrt{q_1})$, by Lemma 6·1, the following is sufficient to guarantee that f_1 wins the (α, t_0) race:

$$\frac{\sqrt{p}}{2\log p}\log(1 + q_1^{-1/2}) < \cos(2\pi t_0)(\lambda_{f_2}(p) - \lambda_{f_1}(p)) - \frac{4}{p^{1/2}(1-p^{-1/2})}. \tag{6·6}$$

This inequality is automatically true if both

$$(\lambda_{f_1}(p) - \lambda_{f_2}(p))\cos(2\pi t_0) \geqslant \frac{|\cos(2\pi t_0)|}{2} \tag{6·7}$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

and

$$\frac{|\cos(2\pi t_0)|}{2} > \frac{p^{1/2}}{2\log p}\log(1+q_1^{-1/2}) + \frac{4}{p^{1/2}(1-p^{-1/2})} \tag{6·8}$$

are satisfied.

Now set $X = q_1^{1/4}$ and $Y = q_1^{1/6}$. For $p \in [Y, X)$, if q_1 is sufficiently large, then (6·8) is satisfied. Choose $I_1, I_2 \subseteq [-2, 2]$ such that $(\lambda_{f_1}(p), \lambda_{f_2}(p)) \in I_1 \times I_2$ implies (6·7). Suppose $f_1 \in S_{k_1}^{new}(\Gamma_0(q_1)), f_2 \in S_{k_2}^{new}(\Gamma_0(q_2))$ are non-CM newforms, where $q_2 \in [q_1, q_1 + \sqrt{q_1}]$, and assume, as in the statement of the Corollary, that $f_2 \neq f_1 \otimes \chi$ for any primitive Dirichlet character χ. Let

$$\pi_{f_1,f_2,I_1,I_2}(X) := \#\{p \leqslant X : (\lambda_{f_1}(p), \lambda_{f_2}(p)) \in I_1 \times I_2\}.$$

Then by Theorem 2·1 we have

$$\begin{aligned}\pi_{f_1,f_2,I_1,I_2}(X) - \pi_{f_1,f_2,I_1,I_2}(Y) &\geqslant \mu_{ST}(I_1)\mu_{ST}(I_2)(\pi(X) - \pi(Y)) \\ &\quad - c\pi(X)\frac{\log\log\log(k_1k_2q_1^{\frac{1}{4}+\frac{1}{2}+1})}{(\log\log q_1^{1/4})^{1/2}} - c\pi(Y)\frac{\log\log\log(k_1k_2q_1^{\frac{1}{6}+\frac{1}{2}+1})}{(\log\log q_1^{1/6})^{1/2}}.\end{aligned}$$

So if q_1 is sufficiently large, then $\pi_{f_1,f_2,I_1,I_2}(X) > \pi_{f_1,f_2,I_1,I_2}(Y)$. So there exists p between Y and X such that (6·6) is satisfied and therefore f_1 wins the local $(t_0, (\log p)/(2\pi))$ race.

Finally, if $t_0 = 1/4, 3/4$, then instead of (6·6), we wish to find p such that

$$\frac{p}{2\log p}\log(1+q_1^{-1/2}) < (\lambda_{f_1}(p)^2 - \lambda_{f_2}(p)^2) - \frac{4}{p^{3/2}(1-p^{-1/2})}.$$

We obtain this the same way as the first case.

7. *Example*

We conclude with a numerical example to illustrate (1·15). For our example, we consider the L-function $L(s, \Delta)$ associated to the discriminant modular form

$$\Delta(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2\pi inz})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz} \in S_{12}^{\text{new}}(\Gamma_0(1)),$$

where $\tau(n)$ denotes the Ramanujan tau function. We use Rubinstein's lcalc package [**Rub14**] to calculate the $2 \cdot 10^5$ nontrivial zeros $L(s, \Delta)$ up to height $T = 74920.77$.

Let M and $\boldsymbol{\alpha}$ satisfy the following relation for (1·13),

$$M\boldsymbol{\alpha}^{\mathsf{T}} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \frac{1}{2\pi}\begin{pmatrix}\log 2\\ \log 3\end{pmatrix},$$

so that (1·14) will define our density function $g_{\Delta,\boldsymbol{\alpha}}(x, y)$. We graph $g_{\Delta,\boldsymbol{\alpha}}(x, y)$ in Figure 1(a) below. Next, we partition the unit square $[0, 1) \times [0, 1)$ as

$$[0,1)\times[0,1) = \bigcup_{a=0}^{29}\bigcup_{b=0}^{29} S_{a,b}, \qquad S_{a,b} := \left[\frac{a}{30}, \frac{a+1}{30}\right) \times \left[\frac{b}{30}, \frac{b+1}{30}\right).$$

https://doi.org/10.1017/S0305004122000445 Published online by Cambridge University Press

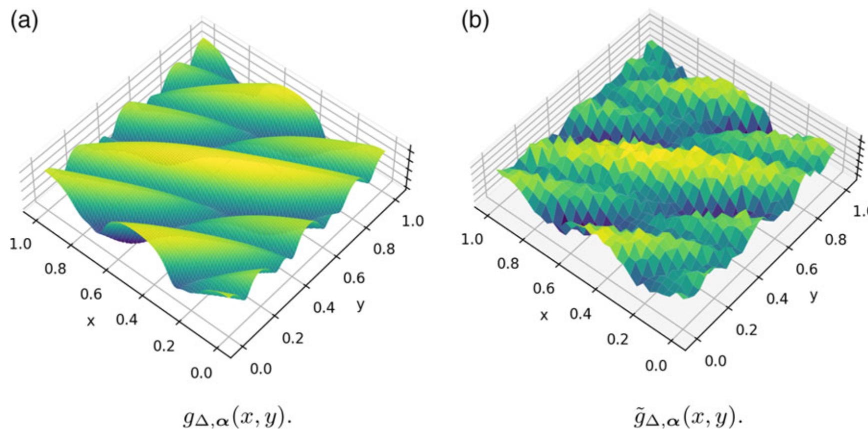

Fig. 1. Example.

Given $(x, y) \in [0, 1) \times [0, 1)$, there exists a unique pair of integers a and b with $0 \leq a, b \leq 29$ such that $(x, y) \in S_{a,b}$. Denoting this unique square as $S(x,y)$, we define

$$\tilde{g}_{\Delta,\boldsymbol{\alpha}}(x, y) := \#\{\rho = \beta + i\gamma : L(\rho, \Delta) = 0 \text{ and } (\{\alpha_1\gamma\}, \{\alpha_2\gamma\}) \in S(x, y)\}.$$

This gives us a discretised approximation to $g_{\Delta,\boldsymbol{\alpha}}(x, y)$, which we plot in Figure 1(b) above.

REFERENCES

[AT21] N. ANDERSEN and J. THORNER. Zeros of GL_2 L-functions on the critical line. *Forum Math.* **33** (2021), no. 2, 477–491.

[Bal17] S. ALAN C. BALUYOT. On the Zeros of Riemann's Zeta-function. (ProQuest LLC, Ann Arbor, MI, 2017). PhD. thesis. University of Rochester.

[BB11] V. BLOMER and F. BRUMLEY. On the Ramanujan conjecture over number fields. *Ann. of Math.* (2) **174** (2011), no. 1, 581–605.

[Ber15] D. BERNARD. Modular case of Levinson's theorem. *Acta Arith.* **167** (2015), no. 3, 201–237.

[BH95] E. BOMBIERI and D. A. HEJHAL. On the distribution of zeros of linear combinations of Euler products. *Duke Math. J.* **80** (1995), no. 3, 821–862.

[BLGHT11] T. BARNET-LAMB, D. GERAGHTY, M. HARRIS and R. TAYLOR. A family of Calabi-Yau varieties and potential automorphy II. *Publ. Res. Inst. Math. Sci.* **47** (2011), no. 1, 29–98.

[Das20] M. DAS. Selberg's Central Limit Theorem for L-functions of level aspect. (2020), arXiv:2012.10766.

[FHL19] K. FORD, A. J. HARPER and Y. LAMZOURI. Extreme biases in prime number races with many contestants. *Math. Ann.* **374** (2019), no. 1-2, 517–551.

[FK02] K. FORD and S. KONYAGIN. The prime number race and zeros of L-functions off the critical line. *Duke Math. J.* **113** (2002), no. 2, 313–330.

[FLK13] K. FORD, Y. LAMZOURI and S. KONYAGIN. The prime number race and zeros of Dirichlet L-functions off the critical line: Part III. *Q. J. Math.* **64** (2013), no. 4, 1091–1098.

[FM13] D. FIORILLI and G. MARTIN. Inequities in the Shanks–Rényi prime number race: an asymptotic formula for the densities. *J. Reine Angew. Math.* **676** (2013), 121–212.

[FMZ17] K. FORD, X. MENG and A. ZAHARESCU. Simultaneous distribution of the fractional parts of Riemann zeta zeros. *Bull. Lond. Math. Soc.* **49** (2017), no. 1, 1–9.

[FSZ09] K. FORD, K. SOUNDARARAJAN and A. ZAHARESCU. On the distribution of imaginary parts of zeros of the Riemann zeta function. II. *Math. Ann.* **343** (2009), no. 3, 487–505.

[FZ05] K. FORD and A. ZAHARESCU. On the distribution of imaginary parts of zeros of the Riemann zeta function. *J. Reine Angew. Math.* **579** (2005), 145–158.

[FZ15] K. FORD and A. ZAHARESCU. Unnormalized differences between zeros of L-functions. *Compositio. Math.* **151** (2015), no. 2, 230–252.

[Har09] M. HARRIS, *Potential automorphy of odd-dimensional symmetric powers of elliptic curves and applications*, Algebra, arithmetic and geometry: in honor of Yu. I. Manin. Vol. II. Progr. *Math.*, vol. **270** (Birkhäuser Boston, Boston, MA, 2009), pp. 1–21.

[Hla75] E. HLAWKA. Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen. *Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B.* II **184** (1975), no. 8-10, 459–471.

[IK04] H. IWANIEC and E. KOWALSKI. *Analytic Number Theory*. (*American Mathematical Society Colloquium Publications*, American Mathematical Society, 2004).

[JLW21] Y. JIANG, G. LÜ and Z. WANG. Exponential sums with multiplicative coefficients without the Ramanujan conjecture. *Math. Ann.* **379** (2021), no. 1-2, 589–632.

[Kem05] R. S. KEMBLE. A Groshev theorem for small linear forms. *Mathematika* **52** (2005), no. 1-2, 79–85 (2006).

[Khi24] A. KHINTCHINE. Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.* **92** (1924), no. 1-2, 115–125.

[Kim03] H. H. KIM. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. *J. Amer. Math. Soc.* **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[Luo95] W. Z. LUO. Zeros of Hecke L-functions associated with cusp forms. *Acta Arith.* **71** (1995), no. 2, 139–158.

[LZ21] D. LIU and A. ZAHARESCU. Races with imaginary parts of zeros of the Riemann zeta function and Dirichlet L-functions. *J. Math. Anal. Appl.* **494** (2021), no. 1, 124591.

[MV74] H. L. MONTGOMERY and R. C. VAUGHAN. Hilbert's inequality. *J. London Math. Soc.* (2) **8** (1974), 73–82.

[Ono04] K. ONO, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102. Published for the Conference Board of the Mathematical Sciences, Washington, DC (American Mathematical Society, Providence, RI, 2004).

[RS94] M. RUBINSTEIN and P. SARNAK. Chebyshev's bias. *Experiment. Math.* **3** (1994), no. 3, 173–197.

[RS17] M. RADZIWIŁŁ and K. SOUNDARARAJAN. Selberg's central limit theorem for $\log|\zeta(1/2+it)|$. *Enseign. Math.* **63** (2017), no. 1-2, 1–19.

[Rub14] M. RUBINSTEIN. *L-function c++ class library and the command line program lcalc*. https://github.com/agrawroh/l-calc (2014).

[Sel92] A. SELBERG, *Old and new conjectures and results about a class of Dirichlet series*. Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), University of Salerno (1992), pp. 367–385.

[Tho21] J. THORNER, *Effective forms of the Sato-Tate conjecture*, *Res. Math. Sci.* **8** (2021), no. 1, 4.

[Tit58] E. C. TITCHMARSH, *The Theory of Functions* (Oxford University Press, Oxford, 1958). Reprint of the second (1939) edition.

[Tit86] E. C. TITCHMARSH, *The Theory of the Riemann Zeta-function*, second ed. (The Clarendon Press, Oxford University Press, New York, 1986). Edited and with a preface by D. R. Heath–Brown.

[Won19] P.-J. WONG, *On the Chebotarev–Sato–Tate phenomenon*, *J. Number Theory* **196** (2019), 272–290.