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ON THE LOOMIS–SIKORSKI THEOREM FOR MV-ALGEBRAS WITH INTERNAL STATE

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Abstract

In Flaminio and Montagna [‘An algebraic approach to states on MV-algebras’, in: *Fuzzy Logic 2*, *Proc. 5th EUSFLAT Conference*, Ostrava, 11–14 September 2007 (ed. V. Novák) (Universitas Ostraviensis, Ostrava, 2007), Vol. II, pp. 201–206; ‘MV-algebras with internal states and probabilistic fuzzy logic’, *Internat. J. Approx. Reason.* **50** (2009), 138–152], the authors introduced MV-algebras with an internal state, called state MV-algebras. (The letters MV stand for multi-valued.) In Di Nola and Dvurečenskij [‘State-morphism MV-algebras’, *Ann. Pure Appl. Logic* **161** (2009), 161–173], a stronger version of state MV-algebras, called state-morphism MV-algebras, was defined. In this paper, we present the Loomis–Sikorski theorem for σ-complete MV-algebras with a σ-complete state-morphism-operator, showing that every such MV-algebra is a σ-homomorphic image of a tribe of functions with an internal state induced by a function where all the MV-operations are defined by points.

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1. Introduction

MV-algebras were introduced in the late fifties by Chang [3] as algebraic semantics for Łukasiewicz many-valued logic; the letters MV stand for multi-valued. Nowadays MV-algebras enter in many areas of mathematics and its applications, including quantum structures; see, for example, [12]. The seminal paper that is crucial for the theory of MV-algebras is that of Mundici [23], concerning the categorical equivalence of the variety of MV-algebras and the category of unital ℓ-groups; for an overview of MV-algebras see [4].

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The Loomis–Sikorski theorem was proved independently by Loomis [22] and Sikorski [29]; see, for example, [30]. It states that each σ-complete Boolean algebra is a σ-epimorphic image of a σ-algebra of subsets of some set Ω. This result was extended to σ-complete MV-algebras in [9, 25]; see also [1]. In this case, every σ-complete MV-algebra is a σ-epimorphic image of a tribe of $[0, 1]$-valued functions on a set Ω, where the MV-algebraic operations among functions are defined by points. This result was also extended to monotone σ-complete effect algebras in [2].

Forty years after the appearance of MV-algebras, Mundici [24] presented an analogue of probability measure for MV-algebras, called a *state*, as an averaging process for formulas in Łukasiewicz logic. In the last decade, the theory of states on MV-algebras and relative structures has been intensively studied; see, for example, [13, 16, 19, 20, 26–28]. We emphasize that a state is a proper notion for quantum structures; see [12].

Recently, Flaminio and Montagna in [14, 15] extended the language of MV-algebras, adding a unary operation τ, called an *internal state* or a *state-operator*. Such MV-algebras are called state MV-algebras. We recall that modality Pr (interpreted as *probably*) in many-valued logic has the following semantic interpretation: the probability of an event a is the truth value of $\Pr(a)$. Furthermore, if s is a state, then $s(a)$ is interpreted as an average of the appearances of the many-valued event a.

State MV-algebras have been intensively studied; see, for example, [5–7]. There is a special type of state-operators: *state-morphism-operators*, which are state-operators that are also MV-homomorphisms. In [5], we characterized the set of subdirectly irreducible state-morphism MV-algebras (we note that there is still no characterization of subdirectly irreducible state MV; see [14]). In [7, 8], we described different varieties of state MV-algebras; in particular, we showed that if A is an MV-algebra, τ is a state-operator and $\tau(A) \in \mathrm{V}(S_1, \ldots, S_n)$, then τ is a state-morphism-operator; we recall that $\mathrm{V}(S_1, \ldots, S_n)$ is the variety of MV-algebras generated by $S_1, \ldots, S_n$ and S_i is the MV-algebra of the form $S_i = \{0, 1/i, 2/i, \ldots, i/i\}$.

In this paper, we show that every σ-complete state-morphism MV-algebra (A, τ) is an epimorphic image of an appropriate tribe of functions on some set Ω with a state-morphism-operator induced by a function from Ω into itself. This gives a new variant of the Loomis–Sikorski theorem for σ-complete state-morphism MV-algebras with internal state.

The paper is organized as follows. In Section 2, we give elements of the theory of MV-algebras. We mention general comparability, which every σ-complete MV-algebra satisfies, and recall some basic representations of MV-algebras satisfying general comparability. Section 3 presents state MV-algebras and state-morphism MV-algebras. We give some characterizations of semisimple state-morphism MV-algebras and show that each state-morphism-operator is induced by some idempotent function g, which we may assume is continuous on some compact Hausdorff topological space. The main body of the article is Section 4, where the Loomis–Sikorski theorem and its variants, including a continuous variant, are proved. The last section gives some alternative proofs of Theorem 3.7 for special cases.

2. MV-algebras and general comparability—properties

We recall that an *MV-algebra* is an algebra $(A; \oplus, ^*, 0)$ of signature $\langle 2, 1, 0\rangle$, where $(A; \oplus, 0)$ is a commutative monoid with neutral element 0 and the following conditions hold for all $x, y \in A$:

- $(x^*)^* = x$;
- $x \oplus 1 = 1$, where $1 = 0^*$;
- $x \oplus (x \oplus y^*)^* = y \oplus (y \oplus x^*)^*$.

We define an additional total operation $\odot$ on A via $x \odot y := (x^* \oplus y^*)^*$.

Suppose that (G, u) is an Abelian ℓ-group with a strong unit $u \geq 0$, that is, G is a lattice-ordered group and for all $g \in G$, there is a positive integer n such that $g \leq nu$. Then a prototypical example of an MV-algebra is

$$A = (\Gamma(G, u); \oplus, ^*, 0)$$

where $\Gamma(G, u) := [0, u]$, Γ being the Mundici functor, $g_1 \oplus g_2 := (g_1 + g_2) \wedge u$ and $g^* := u - g$; indeed, by [23], every MV-algebra is isomorphic to some $\Gamma(G, u)$.

We recall that an *ideal* of an MV-algebra A is a nonempty subset I of A such that if $a \leq b$ and $b \in I$, then $a \in I$, and also if $a, b \in I$, then $a \oplus b \in I$. An ideal I is *maximal* if $I \neq A$, and also, if J is an ideal and $I \subseteq J \neq A$, then $I = J$. The dual notion to an ideal is a filter. We define the *radical* of A by $\mathrm{Rad}(A) := \bigcap\{I \in \mathcal{M}(A)\}$, where $\mathcal{M}(A)$ is the set of all maximal ideals of A.

A *state* on an MV-algebra A is a mapping $s : A \to [0, 1]$ such that $s(1) = 1$ and $s(a \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0$. The set of all states on A is denoted by $\mathcal{S}(A)$. The set $\mathcal{S}(A)$ is convex, that is, if s_1, s_2 are states on A and $\lambda \in [0, 1]$, then $\lambda s_1 + (1 - \lambda)s_2$ is a state on A. A state s is *extremal* if it cannot be written in the form $s = \lambda s_1 + (1 - \lambda)s_2$, where $s_1, s_2 \in \mathcal{S}(A)$ and $\lambda \in (0, 1)$. The set of extremal states is denoted by $\partial_e \mathcal{S}(A)$. We recall that a state s is extremal if and only if $\mathrm{Ker}(s)$, given by

$$\mathrm{Ker}(s) := \{a \in A : s(a) = 0\},$$

is a maximal ideal of A, or equivalently, $s(a \oplus b) = \min\{s(a) + s(b), 1\}$ for all $a, b \in A$ (such a mapping is also called a *state-morphism*). It is possible to show that both $\mathcal{S}(A)$ and $\partial_e \mathcal{S}(A)$ are nonempty. When we introduce the weak topology on the set of states, that is, a net $\{s_\alpha\}$ of states converges weakly to a state s if $\lim_\alpha s_\alpha(a) = s(a)$ for every $a \in A$, then $\mathcal{S}(A)$ and $\partial_e \mathcal{S}(A)$ are compact Hausdorff topological spaces. By the Krein–Mil'man theorem, [17, Theorem 5.17], every state on A is a weak limit of a net of convex combinations of extremal states. In addition, the topological space $\partial_e \mathcal{S}(A)$ is homeomorphic to the space of all maximal ideals $\mathcal{M}(A)$ (ultrafilters $\mathcal{F}(A)$) with the hull-kernel topology. This homeomorphism is given by $s \leftrightarrow \mathrm{Ker}(s)$, see [11], [17, Theorem 15.32], because every maximal ideal is the kernel of a unique extremal state, and a state s is extremal if and only if $\mathrm{Ker}(s)$ is a maximal ideal.

Let A be an MV-algebra. An element $a \in A$ is said to be *Boolean* if $a \oplus a = a$. Then a is Boolean if and only if any or all the following hold:

$$a \odot a = a; \quad a \wedge a^* = 0; \quad a \vee a^* = 1.$$

Let $B(A)$ be the set of all Boolean elements of A. Then $B(A)$ is a Boolean subalgebra of A. Let a be a fixed Boolean element of A. Then the interval $[0, a]$ can be endowed with the restriction of $\oplus$, $\odot$ to $[0, a]$ and with ^{*_a}, where $x^{*_a} := x^* \wedge a$ for all $x \in [0, a]$, and $([0, a], \oplus, \odot, {}^{*_a}, 0, a)$ is an MV-algebra. The mapping $p_a : a \to [0, a]$ defined by $p_a(x) = x \wedge a$ for all $x \in A$, is an MV-homomorphism. In addition, the mapping $\Phi_a : a \to [0, a] \times [0, a^*]$, defined by

$$\Phi_a(x) = (p_a(x), p_{a^*}(x)) = (x \wedge a, x \wedge a^*) \quad \forall x \in A,$$

is an MV-isomorphism.

We say that an MV-algebra A satisfies *general comparability* if, given $x, y \in A$, there is a Boolean element $a \in A$ such that $p_a(x) \leq p_a(y)$ and $p_{a^*}(x) \geq p_{a^*}(y)$. This means that the two coordinates of the elements $x = (p_a(x), p_{a^*}(x))$ and $y = (p_a(y), p_{a^*}(y))$ can be compared in $[0, a]$ and $[0, a^*]$, respectively.

For example, every linearly ordered MV-algebra satisfies general comparability (trivially); every Cartesian product of linearly ordered MV-algebras and every σ-complete MV-algebra satisfy general comparability. Further, if A satisfies the general comparability, so does A/I for each ideal I of A. However, there are examples of MV-algebras that do not satisfy the general comparability.

We recall that a topological space Ω is said to be *connected* if it cannot be expressed as a union of two nonempty disjoint open subsets, *totally disconnected* if it has a base consisting of clopen (closed and open) sets, and *basically disconnected* provided the closure of every open F_σ subset of Ω is open (an F_σ set is a countable union of closed sets). Totally disconnected spaces are also called *Stone spaces* or *Boolean spaces*. For example, if Ω is finite, or if Ω is a Cantor set in $[0, 1]$, then Ω is totally disconnected. Further, if A is a σ-complete MV-algebra, then $\partial_e \mathcal{S}(A)$ is a basically disconnected, compact, Hausdorff topological space [17].

Now let Ω be a compact Hausdorff topological space and let $\mathrm{C}(\Omega)$ be the set of all continuous real-valued functions on Ω. Then $\mathrm{C}(\Omega)$ is an Abelian ℓ-group with strong unit 1_Ω under the pointwise ordering of functions. Define the MV-algebra $C_1(\Omega) = \Gamma(\mathrm{C}(\Omega), 1_\Omega)$. Then $B(C_1(\Omega)) = \{\chi_A : A \text{ is clopen in } \Omega\}$. The system of all clopen subsets of Ω forms a Boolean algebra of a Stone space if and only if Ω is totally disconnected [17]. Therefore $C_1(\Omega)$ can satisfy general comparability only if Ω is totally disconnected.

For example, if $\Omega = [0, 1]$ with the usual topology, then $C_1([0, 1])$ is an MV-algebra which does not satisfy general comparability, while $B(C_1([0, 1])) = \{0_\Omega, 1_\Omega\}$. The same is true for all connected compact Hausdorff spaces X.

It is known that every extremal state on a Boolean algebra is two-valued. In what follows, we show every two-valued state on $B(A)$ can be uniquely extended to an extremal state on an MV-algebra A provided A satisfies general comparability.

The following results concerning MV-algebras satisfying general comparability can be found in [10].

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THEOREM 2.1. *Let A be an MV-algebra satisfying general comparability, and let K be a maximal ideal of $B(A)$. Then there is a unique state s on A such that $B(A) \cap \mathrm{Ker}(s) = K$. This state is extremal.*

We denote by $\mathcal{M}(B(A))$ the set of all maximal ideals of the Boolean algebra $B(A)$. With the hull-kernel topology, it is totally disconnected.

THEOREM 2.2. *Let A be an MV-algebra satisfying general comparability. Then the mapping,*

$$\phi(s) := B(A) \cap \mathrm{Ker}(s) \quad \forall s \in \partial_e \mathcal{S}(A), \tag{2.1}$$

defines a homeomorphism ϕ of $\partial_e \mathcal{S}(A)$ onto $\mathcal{M}(B(A))$.

THEOREM 2.3. *Let A be an MV-algebra satisfying general comparability. Then $\partial_e \mathcal{S}(A)$ and $\partial_e \mathcal{S}(B(A))$ are homeomorphic compact Hausdorff totally disconnected spaces. The mapping $\phi_A : s \in \partial_e \mathcal{S}(A) \mapsto s_{|B(A)}$ implements the homeomorphism.*

PROOF. This is a direct consequence of Theorems 2.2 and 2.1. □

We recall that an extremal state s is *discrete* if $s(A) = \{0, 1/n, \ldots, n/n\}$ for some positive integer n. An extremal state is discrete if and only if there exists a positive integer n such that $A/\mathrm{Ker}(s) = S_n =: \Gamma(n^{-1}\mathbb{Z}, 1)$.

Let A be an MV-algebra. Given an element $a \in A$, we define a continuous function $\hat{a} : \partial_e \mathcal{S}(A) \to [0, 1]$ by $\hat{a}(s) := s(a)$ for all $s \in \partial_e \mathcal{S}(A)$. Then $\hat{A} := \{\hat{a} : a \in A\}$ is an MV-algebra, and the mapping,

$$\psi(a) = \hat{a} \quad \forall a \in A, \tag{2.2}$$

is an MV-homomorphism from A onto $\hat{A}$. The mapping ψ is an isomorphism if and only if A is *semisimple*, that is, $\mathrm{Rad}(A) = \{0\}$.

The following representation of MV-algebras satisfying general comparability follows from [17, Theorem 8.20].

THEOREM 2.4. *Let A be an MV-algebra satisfying general comparability. Set*

$$M(A) := \{f \in C_1(\partial_e \mathcal{S}(A)) : f(s) \in s(A) \text{ for all discrete } s \in \partial_e \mathcal{S}(A)\}. \tag{2.3}$$

Then $\psi(A)$ is an MV-subalgebra of $M(A)$ that is dense in $M(A)$ in the supremum norm topology. If moreover A is semisimple, then A can be isomorphically embedded into $M(A)$.

If A is a σ-complete MV-algebra, then A is isomorphic to $M(A)$.

We note that if A is an MV-algebra, then $\hat{A}$ is a subalgebra of $M(A)$.

3. State-morphism-operators on semisimple MV-algebras

In this section, we define state MV-algebras and we characterize state-morphism-operators, defined mainly on semisimple MV-algebras. We show that, if A is representable as an MV-algebra of functions on some compact Hausdorff topological space, then each state-morphism-operator τ on A is of the form $\tau(f) = f \circ g$ for all $f \in A$, for some continuous function $g : \partial_e \mathcal{S}(A) \to \partial_e \mathcal{S}(A)$ with $g^2 = g$.

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According to [14, 15], a *state MV-algebra* $(A, \tau) := (A; \oplus, {}^*, 0, \tau)$ is an algebraic structure, where $(A; \oplus, {}^*, 0)$ is an MV-algebra [4] and τ is a unary operator on A, called an *internal state* or a *state-operator*, satisfying the following properties, for each $x, y \in A$:

(i) $\tau(0) = 0$;
(ii) $\tau(x^*) = (\tau(x))^*$;
(iii) $\tau(x \oplus y) = \tau(x) \oplus \tau(y \odot (x \odot y)^*)$;
(iv) $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.

In [15] it is shown that in each state MV-algebra the following hold:

- $\tau(\tau(x)) = \tau(x)$;
- $\tau(1) = 1$;
- if $x \leq y$, then $\tau(x) \leq \tau(y)$;
- $\tau(x \oplus y) \leq \tau(x) \oplus \tau(y)$;
- the image $\tau(A)$ is the domain of an MV-subalgebra of A and $(\tau(A), \tau)$ is a state MV-subalgebra of (A, τ).

In [5], the authors defined a stronger structure, a *state-morphism MV-algebra*, as a state MV-algebra (A, τ) (that is, an algebra satisfying (i)–(iv) above) with the following additional property.

(v) $\tau(x \oplus y) = \tau(x) \oplus \tau(y)$.

Equivalently, τ is an MV-endomorphism of A such that $\tau = \tau \circ \tau$. In this case, τ is called a *state-morphism-operator*.

Proposition 3.1. *Let τ be a state-morphism-operator on an MV-algebra A. Then $\tau(B(A)) \subseteq B(A)$ and τ restricted to $B(A)$ is a state-morphism.*

Proof. Because τ preserves $\odot$, for each Boolean element $a \in B(A)$, it is clear that $a \odot a = a$ so that $\tau(a) = \tau(a \odot a) = \tau(a) \odot \tau(a)$. □

Consider the following conditions on a system $\mathcal{T}$ of functions:

(i) $1 \in \mathcal{T}$;
(ii) If $f \in \mathcal{T}$, then $1 - f \in \mathcal{T}$;
(iii) if $f, g \in \mathcal{T}$, then $f \oplus g \in \mathcal{T}$, where $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}$ for all $\omega \in \Omega$;
(iv) if $\{f_n\}$ is a sequence of elements of $\mathcal{T}$, then $\bigoplus_n f_n \in \mathcal{T}$, where $(\bigoplus_n f_n)(\omega) := \min\{\sum_n f_n(\omega), 1\}$ for all $\omega \in \Omega$.

We say that a system $\mathcal{T}$ of functions from $[0, 1]^\Omega$ is a *Bold algebra* if (i)–(iii) hold, and a *tribe* if $\mathcal{T}$ also satisfies (iv). Hence, every Bold algebra is an MV-algebra whilst every tribe is a σ-complete MV-algebra, and in both cases, the MV-operations are defined pointwise.

We recall that if A is an MV-algebra, then $\hat{A} := \{\hat{a} : a \in A\}$ is a Bold algebra of continuous functions defined on the compact space $\partial_e \mathcal{S}(A)$. In addition, A is isomorphic to $\hat{A}$ under the mapping $a \mapsto \hat{a}$ if and only if A is semisimple. Then $B(A)$

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under this representation has the form

$$\hat{B}(A) = \{\hat{a} : a \in B(A)\} = \{\chi_A : A \text{ is a clopen subset of } \partial_e\mathcal{S}(A)\}.$$

Moreover, from Proposition 3.2, if τ is a state-operator on A, then we can define a state-operator $\hat{\tau}$ on $\hat{A}$ by $\hat{\tau}(\hat{a}) = (\tau(a))\hat{}$ for all $a \in A$.

Similarly, let τ_B be the restriction of τ to $B(A)$ and let $\hat{\tau}_B$ correspond to τ_B defined on $\hat{B}(A)$.

PROPOSITION 3.2. *Let A be an MV-algebra, $\hat{A}$ be the associated Bold algebra, and τ be a state-morphism-operator on A.*

(1) *The mapping g that assigns to each extremal state $s \in \partial_e\mathcal{S}(A)$ the extremal state $s \circ \tau$ is a continuous mapping from $\partial_e\mathcal{S}(A)$ into itself such that $g \circ g = g$ and $g(s)(A) \subseteq s(A)$ for all discrete extremal states $s \in \partial_e\mathcal{S}(A)$. Let*

$$M(A) = \{f \in C_1(\partial_e\mathcal{S}(A)) : f(s) \in s(A) \text{ for all discrete } s \in \partial_e\mathcal{S}(A)\}. \quad (3.1)$$

Define $\tau_g : M(A) \to M(A)$ by $\tau_g(f) = f \circ g$ for all $f \in M(A)$. Then τ_g is a state-morphism-operator on the Bold algebra $M(A)$.

(2) *Define $\hat{\tau} : \hat{A} \to \hat{A}$ by $\hat{\tau}(\hat{a}) := (\tau(a))\hat{}$ for all $a \in A$. Then $\hat{\tau}$ is a well-defined state-morphism-operator on $\hat{A}$ that is the restriction of τ_g.*

PROOF. First we prove (1). If s is a state on A, then $s \circ \tau$ is a state on A too. Further, if s is extremal, then $s \circ \tau$ is extremal by the characterization of extremal states and because τ is an endomorphism. Hence the mapping g on $\partial_e\mathcal{S}(A)$ is well defined.

Moreover, g is continuous because if $s_\alpha \to s$, then

$$\lim_\alpha g(s_\alpha)(a) = \lim_\alpha s_\alpha(\tau(a)) = s(\tau(a)) = g(s)(a) \quad \forall a \in A.$$

From the construction of g it follows that $g \circ g = g$ because

$$g(g(s)) = g(s \circ \tau) = s \circ \tau \circ \tau = s \circ \tau = g(s) \quad \forall s \in \partial_e\mathcal{S}(A).$$

Let s be a discrete state on A. Then $s(A) = \{0, 1/n, \ldots, n/n\}$ for some positive integer n. Then $s(\tau(A)) \subseteq \{0, 1/n, \ldots, n/n\}$, and because $g(s)$ is an extremal state, $s(\tau(A)) = \{0, 1/m, \ldots, m/m\}$ for some divisor m of n.

Now take $f \in M(A)$. Then f is a continuous function taking values in the interval $[0, 1]$. To verify that $\tau_g(f) \in M(A)$ we have to show that $\tau_g(f)(s) \in s(A)$ for all discrete extremal states s on A. We can check that

$$\tau_g(f)(s) = f(g(s)) = f(s \circ \tau) \in (s \circ \tau)(A) \subseteq s(A)$$

by the statement just proved. Hence, $\tau_g(f)$ is also an element of $M(A)$. It is now easy to verify that τ_g is a state-morphism-operator on the Bold algebra $M(A)$.

Now we prove (2). We are going to show that $\hat{\tau}$ is a well-defined operator on $\hat{A}$. Assume that $\hat{a} = \hat{b}$. This means that $s(a) = s(b)$ for all $s \in \partial_e\mathcal{S}(A)$. Hence

$$s(\tau(a)) = g(s)(a) = g(s)(b) = s(\tau(b)),$$

so that $(\tau(a))\hat{} = (\tau(b))\hat{}$ and then $\hat{\tau}(\hat{a}) = \hat{a} \circ g = \hat{b} \circ g = \hat{\tau}(\hat{b})$. Now $\hat{A}$ is a subalgebra of $M(A)$, so $\hat{\tau}$ is the restriction of τ_g. □

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REMARK 3.3. We summarize Proposition 3.2: if A is a semisimple MV-algebra and τ is a state-morphism-operator on A, then τ is uniquely determined by an appropriate continuous function g.

THEOREM 3.4. *Let τ be a state-morphism-operator on an MV-algebra A. Then there is a continuous function $g : \partial_e\mathcal{S}(A) \to \partial_e\mathcal{S}(A)$ such that $g \circ g = g$, $g(s)(A) \subseteq s(A)$ for all discrete extremal states s and $(\tau(a))\hat{} = \hat{\tau}(\hat{a}) = \hat{a} \circ g$ for all $a \in A$.*

PROOF. Take the continuous function g defined in Proposition 3.2. Then $g \circ g = g$ and $g(s)(A) \subseteq s(A)$ for all discrete extremal states s on A. Define $\hat{\tau}$ on $\hat{A}$ by $\hat{\tau}(\hat{a}) = (\tau(a))\hat{}$ for all $a \in A$, as in Proposition 3.2. Then $\hat{\tau}$ is a state-morphism-operator on $\hat{A}$. Now let $s \in \partial_e\mathcal{S}(A)$ and $a \in A$. Then

$$\hat{\tau}(\hat{a})(s) = \tau(\hat{a})(s) = s(\tau(a)) = (s \circ g)(a) = g(s)(a) = \hat{a}(g(s)) = (\hat{a} \circ g)(s),$$

as required. □

Let B be a Boolean algebra and let $\partial_e\mathcal{S}(B)$ be the system of all extremal states on B. Then each such state is two-valued on B. For each $b \in B$, define the continuous function $\hat{b}$ on $\partial_e\mathcal{S}(B)$ by $\hat{b}(s) = s(b)$ for all $s \in \partial_e\mathcal{S}(B)$, and let $\hat{B} := \{\hat{b} : b \in B\}$. Each $\hat{b}$ is in fact the characteristic function of some clopen set $E \subseteq \partial_e\mathcal{S}(B)$. Let τ_B be a state-operator on B, and let $g = g_B$ be the continuous function on $\partial_e\mathcal{S}(B)$ whose existence is guaranteed by Proposition 3.2.

PROPOSITION 3.5. *Let B be a Boolean algebra and τ_B be a state-operator on B. Then τ_B is a state-morphism-operator. Define the mapping τ_{g_B} on $\hat{B}$ by $\tau_{g_B}(\hat{b}) = \hat{b} \circ g_B$ for all $\hat{b} \in \hat{B}$. Then τ_{g_B} is a state-morphism on $\hat{B}$ and $\tau_{g_B} = \hat{\tau}_B$, where $\hat{\tau}_B$ is the state-morphism-operator on $\hat{B}$ defined by $\hat{\tau}_B(\hat{b}) = \tau_B(b)\hat{}$.*

PROOF. Let $s \in \partial_e\mathcal{S}(B)$. Then we have $\tau_{g_B}(\hat{b})(s) = \hat{b}(g(s)) = \hat{b}(s \circ \tau_B) = s(\tau_B(b)) = \hat{\tau}_B(\hat{b})(s)$. □

PROPOSITION 3.6. *Suppose that A is a σ-complete MV-algebra. Then the mapping $\psi : A \to C_1(\partial_e\mathcal{S}(A))$, given by $\psi(a) = \hat{a}$ for all $a \in A$, preserves all countable suprema and infima that exist in A.*

PROOF. If $A = \Gamma(G, u)$, where (G, u) is an Abelian ℓ-group with strong unit u, then, by [18], A is σ-complete if and only if G is Dedekind σ-complete, that is, if $g_n, g \in G$ and $g_n \leq g$ for all $n \geq 1$ imply that $\bigvee_n g_n$ exists in G. Applying the Mundici functor [4] and [17, Lemma 9.12], we have the desired statement. □

Let A be an MV-algebra. We introduce a partial binary operation $+$ as follows: $a + b$ is defined in A if and only if $a \leq b^*$ and, when it is defined, $a + b := a \oplus b$. Then the operation $+$ is commutative and associative. Further, if $A = \Gamma(G, u)$, then $a + b$ corresponds to the group addition $+$ in the Abelian ℓ-group G.

We define $0 \cdot a := 0$ and $1 \cdot a := a$. Inductively, if $n \cdot a$ is defined in A and $n \cdot a \leq a^*$, then we set $(n + 1) \cdot a := (n \cdot a) + a$. Now $\mathrm{Rad}(A)$ consists of all elements $a \in A$ such that $n \cdot a$ exists in A for each integer $n \geq 1$. Such elements are said to be *infinitesimal*.

We say that a state-operator τ on an MV-algebra A is *monotone σ-complete* if $a_n \nearrow a$ (that is, $a_n \le a_{n+1}$ for all $n \ge 1$ and $a = \bigvee_n a_n$) implies that $\tau(a) = \bigvee_n \tau(a_n)$. We recall that if τ is monotone σ-complete, then it preserves all countable suprema and infima that exist in A, and we call it a *σ-complete state-morphism-operator.*

For any function $f : \partial_e\mathcal{S}(A) \to [0, 1]$, we set $N(f) := \{s \in \partial_e\mathcal{S}(A) : f(s) \neq 0\}$.

THEOREM 3.7. *Let τ be a σ-complete state-morphism-operator on a σ-complete MV-algebra A. Then there is a continuous function g defined on $\partial_e\mathcal{S}(A)$ such that $g \circ g = g$, $g(s)(A) \subseteq s(A)$ for all discrete extremal states s on E and $\hat{\tau}(\hat{a}) = \hat{a} \circ g$ for all $a \in A$.*

Conversely, let $g : \partial_e\mathcal{S}(A) \to \partial_e\mathcal{S}(A)$ be a continuous function such that $g \circ g = g$ and $g(s)(A) \subseteq s(A)$ for all discrete extremal states s. Define the mapping τ_g on $\hat{A}$ by $\tau_g(\hat{a}) := \hat{a} \circ g$ for all $a \in A$. Then τ_g is a σ-complete state-morphism-operator on $\hat{A}$.

In addition, if $\tilde{\tau}_g$ is defined on A via $\tilde{\tau}_g(a) = \tau_g(\hat{a})$ for all $a \in A$, then $\tilde{\tau}_g$ is a σ-complete state-morphism-operator on A, and $g(s) = s \circ \tilde{\tau}_g$ for all $s \in \partial_e\mathcal{S}(A)$.

PROOF. Since A is necessarily semisimple, because A is σ-complete, it follows from Theorem 3.4 that $(\tau(a))\hat{} = \hat{\tau}(\hat{a}) = \hat{a} \circ g$ for all $a \in A$.

By Proposition 3.2(2), the mapping τ_g, defined on $\hat{A}$ by $\tau_g(\hat{a}) := \hat{a} \circ g$ for all $a \in A$, is a state-morphism-operator on $\hat{A}$.

Assume that $a_n \nearrow a$. Then $\hat{a}_n \circ g \le \hat{a}_{n+1} \circ g \le \hat{a} \circ g$. Further, $a = \bigvee_n a_n$, whence $\hat{a} = \bigvee_n \hat{a}_n$.

If $a_0(s) = \lim_n \hat{a}_n(s)$ for all $s \in \partial_e\mathcal{S}(A)$, that is, a_0 is a pointwise limit of a sequence of continuous functions on a compact Hausdorff space, then by [21, pp. 86, 405–406], the set $N(a_0 - \hat{a})$ is meager. Similarly, $N(\hat{a} \circ g - a_0 \circ g)$ is a meager set. If $h = \bigvee_n \hat{a}_n \circ g$, then $h \le \hat{a} \circ g$. Since

$$N(h - \hat{a} \circ g) \subseteq N(h - a_0 \circ g) \cup N(a_0 \circ g - \hat{a} \circ g),$$

it follows that $N(h - \hat{a} \circ g)$ is a meager set. By the Baire category theorem, no nonempty open subset of a compact Hausdorff space can be meager, and consequently $N(h - \hat{a} \circ g) = \emptyset$, that is, $h = \hat{a} \circ g$.

Finally, let $a \in A$ and $s \in \partial_e\mathcal{S}(A)$. Then

$$(s \circ \tilde{\tau}_g)(a) = s(\tilde{\tau}_g(a)) = s(\tau_g(\hat{a})) = \hat{a}(g(s)) = g(s)(a),$$

that is, $g(s) = s \circ \tilde{\tau}_g$ for all $s \in \partial_e\mathcal{S}(A)$. □

Two alternative proofs for special cases of Theorem 3.7 are presented in Section 5.

4. The Loomis–Sikorski theorem

We now present the main result of this paper: a generalization of the Loomis–Sikorski theorem for σ-complete state-morphism MV-algebras. We show that each such algebra is a σ-epimorphic image of some tribe, that is, a σ-complete MV-algebra of functions on some nonempty set Ω, where the MV-operations are defined pointwise,

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and the state-morphism-operator is induced by an idempotent function g. In addition, we present a continuous version of the Loomis–Sikorski theorem.

Let A be a σ-complete MV-algebra. Then $\hat{A} = M(A)$, but $\hat{A}$ is not necessarily a tribe. Let $\mathcal{T}(A)$ be the tribe of functions on $[0, 1]^{\partial_e \mathcal{S}(A)}$ generated by $\hat{A} = M(A)$.

PROPOSITION 4.1. *Let A be a σ-complete MV-algebra and let g be a continuous function on $\partial_e \mathcal{S}(A)$ such that $g \circ g = g$ and $g(s)(A) \subseteq s(A)$ for all discrete $s \in \partial_e \mathcal{S}(A)$. Then the operator $\mathcal{T}_g$, defined on $\mathcal{T}(A)$ by $\mathcal{T}_g(f) = f \circ g$ for all $f \in \mathcal{T}(A)$, is a σ-complete state-morphism-operator that is the unique extension of the σ-complete state-morphism-operator τ_g on $M(A)$ defined by $\tau_g(f) = f \circ g$ for all $f \in M(A)$.*

PROOF. First, we show that $\mathcal{T}_g$ is a well-defined operator on $\mathcal{T}(A)$, that is, if $f \in \mathcal{T}(A)$, then $f \circ g \in \mathcal{T}(A)$. Let $\mathcal{T}'$ be the set of all $f \in \mathcal{T}(A)$ such that $f \circ g \in \mathcal{T}(A)$. Then $\mathcal{T}'$ contains $M(A) = \hat{A}$ and if $f \in \mathcal{T}'$, then $1 - f \in \mathcal{T}'$. Now let $f_1, f_2 \in \mathcal{T}'$, then $f_1 \oplus f_2$ and $f_1 \vee f_2$ belong to $\mathcal{T}'$. Hence, if $\{f_n\}$ is a sequence of monotone functions from $\mathcal{T}'$, then $f \circ g = \lim_n f_n \circ g \in \mathcal{T}'$, where $f = \lim_n f_n$. This implies that $\mathcal{T}'$ is the tribe generated by $M(A)$, and consequently, $\mathcal{T}' = \mathcal{T}(A)$ and $\mathcal{T}_g$ is a σ-complete state-morphism-operator on $\mathcal{T}(A)$ that is an extension of τ_g.

Now, if τ is any σ-complete state-morphism-operator on $\mathcal{T}(A)$ that is an extension of τ_g, then the set of elements $f \in \mathcal{T}(A)$ such that $\tau(f) = \mathcal{T}_g(f)$ is a tribe containing $M(A)$, and so has to be $\mathcal{T}(A)$, whence $\tau = \mathcal{T}_g$. □

We now characterize the tribe generated by $C_1(\Omega) = \Gamma(\mathrm{C}(\Omega), 1_\Omega)$, where $\mathrm{C}(\Omega)$ is the space of all continuous fuzzy functions on a compact Hausdorff space Ω. We recall that $\mathcal{B}(\Omega)$ denotes the Baire σ-algebra generated by compact G_δ sets on Ω (a G_δ set is a countable intersection of open sets), or equivalently, by the collection $\{f^{-1}([a, \infty)) : f \in \mathrm{C}(\Omega), a \in \mathbb{R}\}$.

The following result can be found, for example, in [12, Proposition 7.1.11].

PROPOSITION 4.2. *Let Ω be a compact Hausdorff space. Then $\mathcal{T}(C_1(\Omega)) = \mathcal{M}(\Omega)$, where $\mathcal{T}(C_1(\Omega))$ is the tribe generated by $C_1(\Omega)$, and $\mathcal{M}(\Omega)$ is the set of all Baire measurable functions on* $[0, 1]^\Omega$.

PROPOSITION 4.3. *Let $\mathcal{T}$ be a tribe of functions defined on a nonempty set Ω. Let g be a function on Ω such that $g \circ g = g$ and $f \circ g \in \mathcal{T}$ for all $f \in \mathcal{T}$. Then the operator $\tau_g : \mathcal{T} \to \mathcal{T}$, defined by $\tau_g(f) = f \circ g$ for all $f \in \mathcal{T}$, is a σ-complete state-morphism-operator.*

PROOF. Clearly τ_g is a state-morphism-operator on $\mathcal{T}$. Suppose that $f_n(\omega) \nearrow f(\omega)$ for all $\omega \in \Omega$. Then $f_n(g(\omega)) \nearrow f(g(\omega))$ for all $\omega \in \Omega$, so that τ_g is monotone σ-complete; consequently, it is a σ-complete state-morphism-operator. □

Suppose that (A_1, τ_1) and (A_2, τ_2) are state MV-algebras. An MV-homomorphism $h : A_1 \to A_2$ is said to be a *state MV-homomorphism* if $h \circ \tau_1 = \tau_2 \circ h$. Similarly we define both a *state-morphism MV-homomorphism* if τ_1 and τ_2 are state-morphisms, and a *σ-state-morphism MV-homomorphism* if (A_1, τ_1) and (A_2, τ_2) are σ-complete state-morphism MV-algebras and h is a state-morphism σ-MV-homomorphism.

We now present a variant of the Loomis–Sikorski theorem for σ-complete state-morphism MV-algebras.

THEOREM 4.4 (Loomis–Sikorski theorem). *Let (A, τ) be a σ-complete state-morphism MV-algebra. Then there are a σ-complete state-morphism MV-algebra $(\mathcal{T}, \mathcal{T}_g)$, where $\mathcal{T}$ is a tribe of functions from $[0, 1]^\Omega$ and a function $g : \Omega \to \Omega$ such that $g \circ g = g$ and $f \circ g \in \mathcal{T}$ for all $f \in \mathcal{T}$, such that $\mathcal{T}_g$, defined by $\mathcal{T}_g(f) := f \circ g$ for all $f \in \mathcal{T}$, is a σ-complete state-morphism-operator on $\mathcal{T}$. Moreover, there is a σ-state-morphism MV-homomorphism h from $\mathcal{T}$ onto A such that $h \circ \mathcal{T}_g = \tau \circ h$.*

PROOF. Let A be a σ-complete MV-algebra with a σ-complete state-morphism-operator τ. We isomorphically embed A onto $\hat{A}$. We set $\Omega = \partial_e\mathcal{S}(A)$; then Ω is a basically disconnected compact Hausdorff topological space and $\hat{A} = M(A)$. Let $\mathcal{T} = \mathcal{T}(A)$ be the tribe of functions from $[0, 1]^\Omega$ that is generated by $\hat{A}$. According to Proposition 3.2, the function $g : \partial_e\mathcal{S}(A) \to \partial_e\mathcal{S}(A)$, defined by $g(s) = s \circ g$ for all $s \in \partial_e\mathcal{S}(A)$, is continuous and $g \circ g = g$. The mapping $\mathcal{T}_g : \mathcal{T} \to \mathcal{T}$, defined by $\mathcal{T}_g(f) = f \circ g$ for all $f \in \mathcal{T}(A)$, is a σ-complete state-morphism-operator on $\mathcal{T}$ by Theorem 3.7, and by Proposition 4.1, it is a unique extension of the σ-complete state-morphism-operator τ_g on $\mathcal{T}$, defined by $\tau_g(\hat{a}) = \hat{a} \circ g$ for all $a \in A$.

Let $f \in \mathcal{T}$ and $a \in A$. We will say that $f \sim a$ if $N(f - \hat{a}) := \{s \in \partial_e\mathcal{S}(A) : f(s) \neq \hat{a}(s)\}$ is a meager set. Let us denote by $\mathcal{T}'$ the set of all functions $f \in \mathcal{T}$ such that there is $a \in A$ with $f \sim a$.

If a_1 and a_2 are two elements of A such that $f \sim a_1$ and $f \sim a_2$, then

$$N(\hat{a}_1 - \hat{a}_2) \subseteq N(f - \hat{a}_1) \cup N(f - \hat{a}_2),$$

so $N(\hat{a}_1 - \hat{a}_2)$ is a meager set. The functions $\hat{a}_1$ and $\hat{a}_2$ are continuous, and it follows from the Baire category theorem that $\hat{a}_1 = \hat{a}_2$.

Therefore the mapping $h : \mathcal{T}' \to A$ defined by $h(f) = a$ when $f \sim a$ is well defined. In [9], it was proved that $\mathcal{T}'$ is a tribe containing $\hat{A}$, so $\mathcal{T}' = \mathcal{T}$, and h is in fact a σ-homomorphism from $\mathcal{T}$ onto A.

Finally, we now let $f \in \mathcal{T}$ and $a \in A$ be such $h(f) = a$. Then $f \sim a$ so that $N(f - \hat{a})$ is a meager set. Then $N(f \circ g - \hat{a} \circ g) = g^{-1}(N(f - \hat{a}))$ is also meager. It follows from Theorem 3.4 that $h(\mathcal{T}_g f) = \tau(a) = \tau(h(f))$. □

Theorem 4.4 can also be reformulated using topological language.

THEOREM 4.5. *Let (A, τ) be a σ-complete state-morphism MV-algebra. Then there is a nonempty basically disconnected compact Hausdorff topological space Ω, a tribe $\mathcal{T}$ of functions on $[0, 1]^\Omega$, and a continuous function $g : \Omega \to \Omega$ such that $g \circ g = g$ and $f \circ g \in \mathcal{T}$ for all $f \in \mathcal{T}$, such that $\mathcal{T}_g$, given by $\mathcal{T}_g(f) := f \circ g$ for all $f \in \mathcal{T}$, is a σ-complete state-morphism-operator on $\mathcal{T}$. Moreover, there is a σ-homomorphism h from $\mathcal{T}$ onto A such that $h \circ \mathcal{T}_g = \tau \circ h$.*

PROOF. Set $\Omega = \partial_e\mathcal{S}(A)$; then the result follows from the proof of Theorem 4.4. □

Let (A, τ) be a σ-complete state-morphism MV-algebra. We define a quintuple $(\Omega, \mathcal{T}, g, \mathcal{T}_g, h)$, where $\Omega = \partial_e\mathcal{S}(A)$, $\mathcal{T} = \mathcal{T}(A)$, g is the continuous function on Ω

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such that $g \circ g = g$ defined by Proposition 3.2, $\mathcal{T}_g(f)$, given by $\mathcal{T}_g(f) = f \circ g$ for all $f \in \mathcal{T}$, is a σ-complete state-morphism on $\mathcal{T}$, and h is a σ-MV-homomorphism from $\mathcal{T}$ onto A such that $h \circ \mathcal{T}_g = \tau \circ h$. Then $(\Omega, \mathcal{T}, g, \mathcal{T}_g, h)$ is said to be a *canonical representation* of the σ-complete state-morphism MV-algebra (A, τ).

5. Alternative proofs for special cases

In this final section, we give alternative proofs to Theorem 3.7 for two special cases: for tribes, in Theorem 5.2, and for weakly divisible σ-complete MV-algebras, in Theorem 5.4.

First, the following result can be found, for example, in [12, Theorem 7.1.7].

THEOREM 5.1. *Let $\mathcal{T}$ be a tribe of* $[0, 1]$*-valued functions on the nonempty set* Ω*, and define*

$$\mathcal{S}_0(\mathcal{T}) := \{A \subseteq \Omega : \chi_A \in \mathcal{T}\}. \tag{5.1}$$

Then the following results hold.

(1) *$\mathcal{S}_0(\mathcal{T})$ is a σ-algebra of subsets of Ω.*
(2) *If $f \in \mathcal{T}$, then f is $\mathcal{S}_0(\mathcal{T})$-measurable.*
(3) *$\mathcal{T}$ contains all $\mathcal{S}_0(\mathcal{T})$-measurable functions from Ω into the real interval* $[0, 1]$ *if and only if $\mathcal{T}$ contains all constant functions with values in* $[0, 1]$.

THEOREM 5.2. *Let $\mathcal{T}$ be a tribe of functions from $[0, 1]^\Omega$ containing all constant functions and let τ be a σ-complete state-morphism-operator on $\mathcal{T}$ such that the tribe $\mathcal{T}$ is countably generated and such that $\chi_{\{\omega\}} \in \mathcal{T}$ for all $\omega \in \Omega$. Then there is a unique $\mathcal{S}_0(\mathcal{T})$-measurable function g from Ω into itself such that $g \circ g = g$ and $\tau_g = \tau$, where $\tau_g(f) := f \circ g$ for all $f \in \mathcal{T}$.*

PROOF. There are three steps in the proof.

Step 1. Note that $\mathcal{T}$ is countably generated if and only if $\mathcal{S}_0(\mathcal{T})$ is countably generated, where $\mathcal{S}_0(\mathcal{T})$ is defined by (5.1). Indeed, if $\{f_n\}$ is a countable generator of $\mathcal{T}$, then $\{f_n^{-1}(B) : B \in \mathcal{B}_0\}$, where $\mathcal{B}_0$ is a countable generator of the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, is a countable generator of $\mathcal{S}_0(\mathcal{T})$.

Conversely, let $\{A_n\}$ be a countable generator of $\mathcal{S}_0(\mathcal{T})$. We assert that the system $\{r_n \chi_{A_n}\}$, where each r_n is a rational number in $[0, 1]$, is a countable generator of $\mathcal{T}$. Let $\mathcal{T}'$ be the tribe generated by $\{r_n \chi_{A_n}\}$. Then $\chi_A \in \mathcal{T}'$ for all $A \in \mathcal{S}_0(\mathcal{T})$, so $t\chi_A \in \mathcal{T}'$ for all $t \in [0, 1]$ and $A \in \mathcal{S}_0(\mathcal{T})$. Therefore every step function $f = \sum_{i=1}^k t_i \chi_{B_i}$, where $t_i \in [0, 1]$ and $B_1, \dots, B_k$ are mutually disjoint sets from $\mathcal{S}_0(\mathcal{T})$, is in $\mathcal{T}'$. It is well known that if $f \in \mathcal{T}$, then there is a sequence of step functions $\{f_n\}$ in $\mathcal{T}'$ such that $f_n \nearrow f$, and this implies that $f \in \mathcal{T}'$. Hence, $\mathcal{T}' = \mathcal{T}$.

Step 2. Given $\omega \in \Omega$, let $I_\omega := \{f \in \mathcal{T} : f(\omega) = 0\}$. This is a σ-ideal of $\mathcal{T}$, that is, if $f_n \in I_\omega$ then $\sup_n f_n \in I_\omega$. If $f \in \mathcal{T} \setminus I_\omega$, then $f(\omega) > 0$, so there is a positive integer n such that $nf(\omega) \wedge 1 = 1$, whence $(nf)^* \in I_\omega$, and this says that I_ω is a maximal ideal.

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Conversely, let I be any maximal ideal of $\mathcal{T}$ that is a σ-ideal. We claim that there is a unique $\omega \in \Omega$ such that $I = I_\omega$. Let $\hat{I} := \{A \in \mathcal{S}_0(\mathcal{T}) : \chi_A \in I\}$. Then $\hat{I}$ is a maximal ideal of $\mathcal{S}_0(\mathcal{T})$ that is also a σ-ideal, that is, if $C_n \in \hat{I}$ when $n \geq 1$, then $\bigcup_n C_n \in \hat{I}$. Since if $\{A_n\}$ is a generator of $\mathcal{S}_0(\mathcal{T})$, then $\{B_n\}$, where $B_n = A_n$ if $\omega \notin A_n$ and $B_n = \Omega \setminus A_n$ otherwise, is also a generator of $\mathcal{S}_0(\mathcal{T})$. Set $B_0 = \bigcap_n B_n$, then $B_0 \in \mathcal{S}_0(\mathcal{T})$. Let $\mathcal{S}_0 := \{A \in \mathcal{S}_0(\mathcal{T}) : a \cap B_0 = \emptyset \text{ or } A \supseteq B_0\}$. Then $\mathcal{S}_0$ is a σ-algebra containing the generator $\{B_n\}$ so that $\mathcal{S}_0 = \mathcal{S}_0(\mathcal{T})$. Since each singleton $\{\omega\}$ belongs to $\mathcal{S}_0$, then there is a unique $\omega \in \Omega$ such that $B_0 = \{\omega\}$. Now let $I'_\omega := \{f \in I : f(\omega) = 0\}$. Then each $\chi_{B_n} \in I'_\omega$ as well as $\chi_A \in I'_\omega$ whenever $A \in \hat{I}$. Because $t\chi_A \leq \chi_A$, we have $t\chi_A \in I'_\omega$ and therefore each step function $f = \sum_{i=1}^k t_i \chi_{C_i} \in I'_\omega$ with $t_i \in [0, 1]$ and with mutually disjoint sets $C_1, \ldots, C_k \in \hat{I}$. Hence, by approximating any function $f \in I$ from below by step functions from I'_ω, we see that $f \in I'_\omega$ and $I = I_\omega$.

Step 3. Let I_ω be given. Then $\tau^{-1}(I_\omega) := \{f \in \mathcal{T} : \tau(f) \in I_\omega\}$ is also a maximal ideal that is a σ-ideal. By Step 2, there is a unique $\omega' \in \Omega$ such that $\tau^{-1}(I_\omega) = I_{\omega'}$, so we can define a function $g : \Omega \to \Omega$ such that $g(\omega) = \omega'$ if and only if $\tau^{-1}(I_\omega) = I_{\omega'}$. It is clear that $g \circ g = g$.

Given $\omega \in \Omega$, define $s_\omega : \mathcal{T} \to [0, 1]$ by $s_\omega(f) := f(\omega)$ for all $f \in \mathcal{T}$. Then s_ω is an extremal state that is σ-continuous, that is, if $f_n \nearrow f$, then $s_\omega(f) = \lim_n s_\omega(f_n)$. Then $\mathrm{Ker}(s_\omega) = I_\omega$ and $f = g$ if and only if $s_\omega(f) = s_\omega(g)$ for all $\omega \in \Omega$. Moreover,

$$\mathrm{Ker}(s_\omega \circ \tau) = (s_\omega \circ \tau)^{-1}(\{0\}) = \tau^{-1}(I_\omega) = I_{g(\omega)}.$$

Then

$$(\tau(f))(\omega) = s_\omega(\tau(f)) = s_\omega \circ \tau \circ f = s_{g(\omega)} \circ f = f(g(\omega)),$$

and so $\tau(f) = f \circ g \in \mathcal{T}$ for all $f \in \mathcal{T}$. We show that g is $\mathcal{S}_0(\mathcal{T})$-measurable. For all $B \in \mathcal{B}_0(\mathbb{R})$,

$$(\tau(f))^{-1}(B) = (f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)) \in \mathcal{S}_0(\mathcal{T}).$$

Hence, if $A \in \mathcal{S}_0(\mathcal{T})$ and $B = \{1\}$, then $g^{-1}(A) = g^{-1}(\chi_A^{-1}(\{1\})) \in \mathcal{S}_0(\mathcal{T})$, so g is $\mathcal{S}_0(\mathcal{T})$-measurable.

Hence, the mapping τ_g, defined by $\tau_g(f) := f \circ g$ for all $f \in \mathcal{T}$, is a σ-complete state-morphism-operator on $\mathcal{T}$ such that $\tau = \tau_g$. Now let $g' : \Omega \to \Omega$ be an $\mathcal{S}_0(T)$-measurable function such that $g' \circ g' = g'$ and $f \circ g' = f \circ g$ for all $f \in \mathcal{T}$. Then for all $A \in \mathcal{S}_0(T)$, we have $\chi_A \circ g' = \chi_A \circ g$, that is, $g'^{-1}(A) = g^{-1}(A)$. If ω_0 is an element of Ω, then $\{\omega \in \Omega : g'(\omega) = \omega_0\} = \{\omega \in \Omega : g(\omega) = \omega_0\}$. As ω_0 is arbitrary, this yields $g' = g$. □

The second case depends on the notions of divisibility and the following lemma.

LEMMA 5.3. *Let Ω be a basically disconnected compact Hausdorff topological space. For each continuous function $f : \Omega \to [0, 1]$, there is a monotone sequence $\{f_n\}$ of continuous step functions defined on Ω and with values in the interval $[0, 1]$ such that $f_n \nearrow f$ uniformly.*

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PROOF. There are three steps in the proof.

Step 1. Let X be a clopen subset of Ω and $f : X \to [\alpha, \beta]$ be a continuous function, where $0 \le \alpha < \beta \le 1$. Then there are two mutually disjoint clopen sets X_1 and X_2 such that $X = X_1 \cup X_2$ and $f(X_1) \subseteq [\alpha, (\alpha+\beta)/2]$ and $f(X_2) \subseteq [(\alpha+\beta)/2, \beta]$. Indeed, the set $f^{-1}(((\alpha+\beta)/2, \beta])$ is an open F_σ set. Its closure X_2 is both open and closed. Then $X_1 = X \setminus X_2$ is also a clopen set, $f(X_1) \subseteq [\alpha, (\alpha+\beta)/2]$ and $f(X_2) \subseteq [(\alpha+\beta)/2, \beta]$.

The function $g : X \to [\alpha, \beta]$, defined by $g(x) = \alpha$ if $x \in X_1$ and $g(x) = (\alpha+\beta)/2$ if $x \in X_2$, is continuous, $g \le f$ and $f(x) - g(x) \le (\beta - \alpha)/2$ for all $x \in X$.

Step 2. Let $f : \Omega \to [0, 1]$ be a continuous function. Setting $X_0 = \Omega$ and applying Step 1, we can find two disjoint clopen sets X_1^1 and X_2^1 such that $X_0 = X_1^1 \cup X_2^1$ and $f(X_1^1) \subseteq [0, 1/2]$ and $f(X_1^2) \subseteq [1/2, 1]$.

Suppose inductively that we have partitioned X into mutually disjoint clopen sets $X_n^0, X_n^1, \ldots, X_n^{2^n-1}$ such that $f(X_n^i) \subseteq [i/2^n, (i+1)/2^n]$ when $i = 0, 1, \ldots, 2^n - 1$.

Using Step 1, we decompose each of the sets $X_n^0, X_n^1, \ldots, X_n^{2^n-1}$ into two mutually disjoint clopen sets, to obtain a partition of X into clopen sets $X_{n+1}^0, X_{n+1}^1, \ldots, X_{n+1}^{2^{n+1}-1}$ such that $f(X_{n+1}^i) \subseteq [i/2^{n+1}, (i+1)/2^{n+1}]$ when $i = 0, 1, \ldots, 2^{n+1} - 1$.

Step 3. Given a sequence of refining partitions into clopen sets $X_n^0, X_n^1, \ldots, X_n^{2^n-1}$, where $n \ge 1$, we can define the function $f_n : X \to [0, 1]$ by $f_n(x) = i/2^n$ when $x \in X_n^i$ and $i = 0, 1, \ldots, 2^n - 1$. Then f_n is a continuous step function and $f_n(x) \le f_{n+1}(x) \le f(x)$ for all $x \in X$ and all $n \ge 1$. Moreover, $f(x) - f_n(x) \le 1/2^n$ for all $x \in X$. Hence the sequence of step functions $\{f_n\}$ converges uniformly to f. □

We say that an MV-algebra A is *weakly divisible* if, given a positive integer n, there is an element $v \in A$ such that $n \cdot v = 1$, and *divisible* if, given any $a \in A$ and positive integer n, there is an element $v \in A$ such that $n \cdot v = a$. In any case, A has no extremal discrete state. According to (2.3), for σ-complete MV-algebras, the notions of weak divisibility and divisibility, as well as the property that A admits no discrete (extremal) state, coincide.

THEOREM 5.4. *Let τ be a σ-complete state-morphism-operator on a weakly divisible σ-complete MV-algebra A. If g is the mapping defined in Proposition 3.2, then the operator $\tau_g : \hat{A} \to \hat{A}$, defined by $\tau_g(\hat{a})(s) = \hat{a}(g(s))$ for all $a \in A$ and $s \in \partial_e\mathcal{S}(A)$, is a σ-complete state-morphism-operator on $\hat{A}$ such that*

$$\tau_g(\hat{a}) = (\tau(a))^\wedge \quad \forall a \in A.$$

PROOF. By Theorem 2.4, $\hat{A} = M(A)$, where $M(A)$ is defined by (2.3).

Define the operator τ_g on $M(A)$ by $\tau_g(f) := f \circ g$ for all $f \in M(A)$. Then τ_g is a state-morphism-operator on $M(A)$, by Proposition 3.5. We will now show that $\tau_g = \hat{\tau}$.

Since A is σ-complete, A satisfies general comparability. Let $B(A)$ be the set of all Boolean elements of A; then $B(A)$ is a Boolean σ-algebra. In view of Proposition 3.5, the restriction, τ_B of τ onto B is a state-morphism-operator on B. We set $B := B(A)$. By Theorem 2.3, the state spaces $\partial_e\mathcal{S}(A)$ and $\partial_e\mathcal{S}(B)$ are homeomorphic basically disconnected compact spaces. Therefore the functions g on $\hat{A}$ and g_B on $\hat{B}$ determined by Proposition 3.2 are practically the same, that is, if ϕ_A is the homeomorphism from Theorem 2.3, then $g_B \circ \phi_A = \phi_A \circ g$. Using Proposition 3.5, we see that $\tau_{g_B} = \hat{\tau}_B$.

First, let f be a Boolean element in $\hat{A}$. Then $\tilde{f} := f \circ \phi_A^{-1}$ is a Boolean element in $\hat{B}$, and *vice versa*. Moreover, if $s \in \partial_e\mathcal{S}(A)$, then $\tilde{s} = \phi_A \circ s = s_{|B(A)}$. Consequently,

$$\hat{\tau} \circ f \circ s = \hat{\tau}_B \circ \tilde{f} \circ \tilde{s} = \tilde{f} \circ g_B \circ \tilde{s} = f \circ \phi_A^{-1} \circ g_B \circ \phi_A \circ s = f \circ g \circ s,$$

and so $\hat{\tau}(f) = \tau_g(f)$ whenever f is a Boolean element.

Second, since $M(A)$ consists of all continuous functions defined on $\partial_e\mathcal{S}(A)$ taking values in the interval [0, 1], then if $f \in M(A)$, then $n^{-1}f \in M(A)$ for all positive integers n. Suppose that f is a Boolean element from $M(A)$, then $f = n \cdot n^{-1}f$, so $\tau_g(f) = \hat{\tau}(f) = n \cdot \hat{\tau}(n^{-1}f)$. Hence

$$\hat{\tau}(n^{-1}f) = n^{-1}\tau_g(f) = \tau_g(n^{-1}f) = n^{-1}\hat{\tau}(f).$$

Therefore $\tau_g((m/n)f) = \hat{\tau}((m/n)f)$ for all integers m between 0 and n. Let t be an irrational number in [0, 1], and take sequences of rational numbers $r_n \nearrow t$ and $s_n \searrow t$. Hence

$$r_n\tau_g(f) = \tau_g(r_nf) = \hat{\tau}(r_nf) \le \hat{\tau}(tf) \le \hat{\tau}(s_nf) = \tau_g(s_nf) = s_n\tau_g(f),$$

so $\tau_g(tf) = t\tau_g(f) = \hat{\tau}(tf) = t\hat{\tau}(f)$.

Third, let $f \in M(A)$ be a step function, that is, $f = \sum_{i=1}^n t_i f_i$, where each f_i is a characteristic function of some clopen set E_i and $t_i \in [0, 1]$ for all i. Without loss of generality, we can assume that $E_1, \ldots, E_n$ are pairwise disjoint. Consequently, $f = t_1f_1 + \cdots + t_nf_n$, where $+$ is the partial addition in the MV-algebra $M(A)$, which coincides with addition of functions. Hence,

$$\begin{aligned}\hat{\tau}(f) &= \hat{\tau}(t_1f_1) + \cdots + \hat{\tau}(t_nf_n)\\ &= \tau_g(t_1f_1) + \cdots + \tau_g(t_nf_n)\\ &= \tau_g(t_1f_1 + \cdots + t_nf_n) = \tau_g(f).\end{aligned}$$

Finally, let f be a continuous function from $M(A)$. By Lemma 5.3, there is a sequence $\{f_n\}$ of continuous step functions from $M(A)$ such that $\{f_n\} \nearrow f$ uniformly. Then $f = \bigvee_n f_n$. In view of Proposition 3.6, $\hat{\tau}$ is also a σ-complete state-morphism-operator, so that

$$\hat{\tau}(f) = \bigvee_n \hat{\tau}(f_n) = \bigvee_n \tau_g(f) = \tau_g\Big(\bigvee_n f_n\Big) = \tau_g(f),$$

by the argument of the previous paragraph. □

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