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Discrepancy bounds for the distribution of *L*-functions near the critical line

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Abstract

We investigate the joint distribution of L-functions on the line $\sigma = 1/2 + 1/G(T)$ and $t \in [T, 2T]$, where $\log\log T \le G(T) \le \log T/(\log\log T)^2$. We obtain an upper bound on the discrepancy between the joint distribution of L-functions and that of their random models. As an application we prove an asymptotic expansion of a multi-dimensional version of Selberg's central limit theorem for L-functions on $\sigma = 1/2 + 1/G(T)$ and $t \in [T, 2T]$, where $(\log T)^{\varepsilon} \le G(T) \le \log T/(\log\log T)^{2+\varepsilon}$ for $\varepsilon > 0$.

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1. *Introduction*

We investigate the distribution of the Riemann zeta function $\zeta(s)$ for $\mathrm{Re}(s) > 1/2$ using its probabilistic model defined by the random Euler product

$$\zeta(\sigma, \mathbb{X}) = \prod_p \left(1 - \frac{\mathbb{X}(p)}{p^{\sigma}}\right)^{-1},$$

where the $\mathbb{X}(p)$ for primes p are the uniform, independent and identically distributed random variables on the unit circle in $\mathbb{C}$. The product converges almost surely for $\sigma > 1/2$ by Kolmogorov's three series theorem. Our main question is how well the distribution of $\zeta(\sigma, \mathbb{X})$ approximate that of the Riemann zeta function for $1/2 < \sigma < 1$.

Consider two measures

$$\Phi_{\zeta,T}(\sigma, \mathcal{B}) := \frac{1}{T}\mathrm{meas}\{t \in [T, 2T]\colon \log\zeta(\sigma + it) \in \mathcal{B}\}$$

and

$$\Phi_{\zeta}^{\mathrm{rand}}(\sigma, \mathcal{B}) := \mathbb{P}(\log\zeta(\sigma, \mathbb{X}) \in \mathcal{B})$$

for a Borel set $\mathcal{B}$ in $\mathbb{C}$. Define the discrepancy between the above two measures by

$$\mathbf{D}_{\zeta}(\sigma) := \sup_{\mathcal{R}} |\Phi_{\zeta,T}(\sigma, \mathcal{R}) - \Phi_{\zeta}^{\mathrm{rand}}(\sigma, \mathcal{R})|,$$

where $\mathcal{R}$ runs over all rectangular boxes in $\mathbb{C}$ with sides parallel to the coordinate axes and possibly unbounded. This quantity measures the amount to which the distribution of $\log\zeta(\sigma,\mathbb{X})$ approximates that of $\log\zeta(\sigma+it)$.

Harman and Matsumoto [2] showed that

$$\mathbf{D}_\zeta(\sigma) \ll (\log T)^{-\frac{4\sigma-2}{21+8\sigma}+\varepsilon}$$

for fixed $1/2<\sigma<1$ and any $\varepsilon>0$. See also Matsumoto's earlier results in [10–12]. Lamzouri, Lester and Radziwiłł [5] improved it to

$$\mathbf{D}_\zeta(\sigma) \ll (\log T)^{-\sigma}$$

for fixed $1/2<\sigma<1$. Define

$$\sigma_T := \frac{1}{2}+\frac{1}{G(T)} \tag{1·1}$$

with $4\le G(T)\le(\log T)^\theta$ and fixed $0<\theta<1/2$, then Ha and Lee [1] extended above results such that

$$\mathbf{D}_\zeta(\sigma_T) \ll (\log T)^{-\eta}$$

holds for some $0<\eta<(1-\theta)/4$. Here, we extend it to hold for σ_T closer to $1/2$.

THEOREM 1·1. *Assume that* $\log\log T\le G(T)\le \log T/(\log\log T)^2$, *then we have*

$$\mathbf{D}_\zeta(\sigma_T) \ll \frac{\sqrt{G(T)}\log\log T}{\sqrt{\log T}}.$$

Next we consider a multivariate extension. Let $L_1,\ldots,L_J$ be L-functions satisfying the following assumptions:

A1: (Euler product) For $j=1,\ldots,J$ and $\mathrm{Re}(s)>1$ we have

$$L_j(s)=\prod_p\prod_{i=1}^d\left(1-\frac{\alpha_{j,i}(p)}{p^s}\right)^{-1},$$

where $|\alpha_{j,i}(p)|\le p^\eta$ for some fixed $0\le\eta<1/2$ and for every $i=1,\ldots,d$.

A2: (Analytic continuation) Each $(s-1)^mL_j(s)$ is an entire function of finite order for some integer $m\ge 0$.

A3: (Functional equation) The functions $L_1,L_2,\ldots,L_J$ satisfy the same functional equation

$$\Lambda_j(s)=\omega\overline{\Lambda_j(1-\bar{s})},$$

where

$$\Lambda_j(s):=L_j(s)Q^s\prod_{\ell=1}^k\Gamma(\lambda_\ell s+\mu_\ell),$$

$|\omega|=1$, $Q>0$, $\lambda_\ell>0$ and $\mu_\ell\in\mathbb{C}$ with $\mathrm{Re}(\mu_\ell)\ge 0$.

A4: (Ramanujan hypothesis on average)

$$\sum_{p\le x}\sum_{i=1}^{d}|\alpha_{j,i}(p)|^2 = O(x^{1+\varepsilon})$$

holds for every $\varepsilon > 0$ and for every $j = 1, \ldots, J$ as $x \to \infty$.

A5: (Zero density hypothesis) Let $N_f(\sigma, T)$ be the number of zeros of $f(s)$ in $\mathrm{Re}(s) \ge \sigma$ and $0 \le \mathrm{Im}(s) \le T$. Then there exists a constant $\kappa > 0$ such that for every $j = 1, \ldots, J$ and all $\sigma \ge 1/2$ we have

$$N_{L_j}(\sigma, T) \ll T^{1-\kappa(\sigma-\frac{1}{2})}\log T.$$

A6: (Selberg orthogonality conjecture) By assumption A1 we can write

$$\log L_j(s) = \sum_p\sum_{r=1}^{\infty}\frac{\beta_{L_j}(p^r)}{p^{rs}}.$$

Then for all $1 \le j, k \le J$, there exist constants $\xi_j > 0$ and $c_{j,k}$ such that

$$\sum_{p\le x}\frac{\beta_{L_j}(p)\overline{\beta_{L_k}(p)}}{p} = \delta_{j,k}\xi_j\log\log x + c_{j,k} + O\left(\frac{1}{\log x}\right),$$

where $\delta_{j,k} = 0$ if $j \ne k$ and $\delta_{j,k} = 1$ if $j = k$.

The assumptions A1–A6 are standard and expected to hold for all L-functions arising from inequivalent automorphic representations of $GL(n)$. In particular, they are verified by $GL(1)$ and $GL(2)$ L-functions, which are the Riemann zeta function, Dirichlet L-functions, L-functions attached to Hecke holomorphic or Maass cusp forms.

Define

$$\mathbf{L}(s) := \Big(\log|L_1(s)|, \ldots, \log|L_J(s)|, \arg L_1(s), \ldots, \arg L_J(s)\Big)$$

and

$$\mathbf{L}(\sigma, \mathbb{X}) := \Big(\log|L_1(\sigma, \mathbb{X})|, \ldots, \log|L_J(\sigma, \mathbb{X})|, \arg L_1(\sigma, \mathbb{X}), \ldots, \arg L_J(\sigma, \mathbb{X})\Big)$$

for $\sigma > 1/2$, where

$$L_j(\sigma, \mathbb{X}) := \prod_p\prod_{i=1}^{d}\left(1 - \frac{\alpha_{j,i}(p)\mathbb{X}(p)}{p^{\sigma}}\right)^{-1} \tag{1·2}$$

converges almost surely for $\sigma > 1/2$ again by Kolmogorov's three series theorem. Then $\mathbf{L}(\sigma, \mathbb{X})$ is the random model of $\mathbf{L}(s)$. Define two measures

$$\Phi_T(\mathcal{B}) := \frac{1}{T}\mathrm{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma_T + it) \in \mathcal{B}\} \tag{1·3}$$

and

$$\Phi_T^{\mathrm{rand}}(\mathcal{B}) := \mathbb{P}(\mathbf{L}(\sigma_T, \mathbb{X}) \in \mathcal{B}) \tag{1·4}$$

for a Borel set $\mathcal{B}$ in $\mathbb{R}^{2J}$ and σ_T defined in (1·1). The discrepancy between the above two measures is defined by

$$\mathbf{D}(\sigma_T) := \sup_{\mathcal{R}} |\Phi_T(\mathcal{R}) - \Phi_T^{\mathrm{rand}}(\mathcal{R})|,$$

where $\mathcal{R}$ runs over all rectangular boxes of $\mathbb{R}^{2J}$ with sides parallel to the coordinate axes and possibly unbounded. Then Theorem 1·1 is a special case of the following theorem.

THEOREM 1·2. *Assume that* $\log\log T \le G(T) \le \log T/(\log\log T)^2$, *then we have*

$$\mathbf{D}(\sigma_T) \ll \frac{\sqrt{G(T)}\log\log T}{\sqrt{\log T}}.$$

The above theorem is an extension of [4, theorem 2·3], which shows the same estimate, but only for $\log\log T \le G(T) \le \sqrt{\log T}/\log\log T$. In the proof of [4, theorem 2·3] we have used an approximation of each $\log L_j(\sigma_T + it)$ by a Dirichlet polynomial

$$R_{j,Y}(\sigma_T + it) := \sum_{p^r \le Y} \frac{\beta_{L_j}(p^r)}{p^{r(\sigma_T+it)}} \tag{1·5}$$

for $t \in [T, 2T]$ with some exception. The exception essentially comes from possible nontrivial zeros of each $L_j(s)$ off the critical line and the set of exceptional t in $[T,2T]$ has a small measure by assumption A5. See [4, lemma 4·2] for details. However, this approximation is not useful if σ_T is closer to 1/2. We overcome such difficulty by means of the 2nd moment estimation of $\log L_j(\sigma_T + it)$ in Theorem 2·1.

As an application of Theorem 1·2 we consider Selberg's central limit theorem. Let $\psi_{j,T} := \xi_j \log G(T)$ for $j \le J$ and

$$\mathcal{R}_T := \prod_{j=1}^{J} [a_j\sqrt{\pi\psi_{j,T}}, b_j\sqrt{\pi\psi_{j,T}}] \times \prod_{j=1}^{J} [c_j\sqrt{\pi\psi_{j,T}}, d_j\sqrt{\pi\psi_{j,T}}]$$

for fixed real numbers a_j, b_j, c_j, d_j. Then an asymptotic formula for

$$\Phi_T(\mathcal{R}_T) = \frac{1}{T}\mathrm{meas}\{t \in [T, 2T]: \frac{\log L_j(\sigma_T + it)}{\sqrt{\pi\psi_{j,T}}} \in [a_j, b_j] \times [c_j, d_j] \text{ for } j = 1, \ldots, J\}$$

is called Selberg's central limit theorem. See [15, theorem 2] for Selberg's original idea. Let $0 < \theta < 1$. To find an asymptotic of $\Phi_T(\mathcal{R}_T)$ for

$$(\log T)^{\theta} \le G(T) \le \frac{\log T}{(\log\log T)^2}, \tag{1·6}$$

it is now enough to estimate $\Phi_T^{\mathrm{rand}}(\mathcal{R}_T)$ due to Theorem 1·2. One can easily check that the asymptotic formula of $\Phi_T^{\mathrm{rand}}(\mathcal{R}_T)$ in [9, theorem 2·1] holds also for $G(T)$ satisfying (1·6). Hence, we obtain the following corollary.

COROLLARY 1·3. *Assume (1·6) for some* $0 < \theta < 1$ *and assumptions* A1–A6 *for* $L_1, \ldots, L_J$. *Then there exist constants* $\varepsilon_1, \varepsilon_2 > 0$ *and a sequence* $\{b_{\mathbf{k},\mathbf{l}}\}$ *of real numbers such that*

$$\Phi_T(\mathcal{R}_T) = \sum_{\mathcal{K}(\mathbf{k}+\mathbf{l}) \le \varepsilon_1 \log\log T} b_{\mathbf{k},\mathbf{l}} \prod_{j=1}^{J} \frac{1}{\sqrt{\psi_{j,T}}^{k_j+\ell_j}} \times \prod_{j=1}^{J} \left(\int_{a_j}^{b_j} e^{-\pi u^2} \mathcal{H}_{k_j}(\sqrt{\pi} u) du \int_{c_j}^{d_j} e^{-\pi v^2} \mathcal{H}_{\ell_j}(\sqrt{\pi} v) dv \right) + O\left(\frac{1}{(\log T)^{\varepsilon_2}} + \frac{\sqrt{G(T)} \log\log T}{\sqrt{\log T}} \right), \tag{1·7}$$

where $\mathbf{k} = (k_1, \ldots, k_J)$ *and* $\mathbf{l} = (\ell_1, \ldots, \ell_J)$ *are vectors in* $(\mathbb{Z}_{\ge 0})^J$, $\mathcal{K}(\mathbf{k}) := k_1 + \cdots + k_J$ *and*

$$\mathcal{H}_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

is the nth Hermite polynomial. Moreover, $b_{0,0} = 1$, $b_{\mathbf{k},\mathbf{l}} = 0$ *if* $\mathcal{K}(\mathbf{k}+\mathbf{l}) = 1$ *and* $b_{\mathbf{k},\mathbf{l}} = O(\delta_0^{-\mathcal{K}(\mathbf{k}+\mathbf{l})})$ *for some* $\delta_0 > 0$ *and all* $\mathbf{k}, \mathbf{l} \in (\mathbb{Z}_{\ge 0})^J$.

Note that Corollary 1·3 extends the asymptotic expansion for $\zeta(s)$ in [8, theorem 1·2] and the asymptotic expansion for $\mathbf{L}(s)$ in [9, theorem 1·2]. If $G(T)$ is very close to $\log T / (\log\log T)^2$, the error term in (1·7) is large so that we have an approximation by a shorter sum as follows.

COROLLARY 1·4. *Under the same assumptions as in Corollary 1·3 except for*

$$G(T) = \frac{\log T}{(\log\log T)^{2+g}}$$

with a constant $g > 0$, *we have*

$$\Phi_T(\mathcal{R}_T) = \sum_{\mathcal{K}(\mathbf{k}+\mathbf{l}) < g} b_{\mathbf{k},\mathbf{l}} \prod_{j=1}^{J} \frac{1}{\sqrt{\psi_{j,T}}^{k_j+\ell_j}} \times \prod_{j=1}^{J} \left(\int_{a_j}^{b_j} e^{-\pi u^2} \mathcal{H}_{k_j}(\sqrt{\pi} u) du \int_{c_j}^{d_j} e^{-\pi v^2} \mathcal{H}_{\ell_j}(\sqrt{\pi} v) dv \right) + O\left(\frac{1}{(\log\log T)^{\frac{g}{2}}} \right).$$

Note that an asymptotic expansion similar to (1·7) was expected to hold in [3] without a proof.

2. *High moments of* $\log L$

Let L be an L-function satisfying assumptions A1–A6 in this section. Here, we use $\alpha_i(p)$ instead of $\alpha_{j,i}(p)$ in assumptions A1 and A4, and assumption A6 is simply

$$\sum_{p \le x} \frac{|\beta_L(p)|^2}{p} = \xi_L \log\log x + c_L + O\left(\frac{1}{\log x} \right)$$

for some constants $\xi_L > 0$ and $c_L \in \mathbb{R}$. Let σ_T be defined in (1·1) and assume that

$$(\log T)^{\frac{1}{3}} \le G(T) \le \frac{\log T}{(\log\log T)^2} \tag{2·1}$$

in this section. Then we need the following theorem to prove Theorem 1·2.

THEOREM 2·1. *Let κ be as in assumption A5 and $0<\varepsilon<\min\{1/48,\kappa/3\}$. Assume (2·1) and $e^{\frac{G(T)}{2}}\le Y\le T^{\varepsilon}$, then there exists $\kappa_0>0$ such that*

$$\frac{1}{T}\int_T^{2T}|\log L(\sigma_T+it)-R_Y(\sigma_T+it)|^2dt\ll e^{-\kappa_0\frac{\log T}{G(T)}}+e^{-2\frac{\log Y}{G(T)}}\frac{G(T)}{\log Y},$$

where

$$R_Y(s):=\sum_{p^r\le Y}\frac{\beta_L(p^r)}{p^{rs}}.$$

To prove above theorem, we modify high moments estimations of $\log\zeta$ in Tsang's thesis [**16**] and compute high moments of $\log L$. All these computations are based on Selberg [**13, 14**]. Since the Dirichlet coefficients of $L(s)$ are allowed to be larger than 1, Theorem 2·1 is not an immediate consequence of Tsang [**16**]. We need to bound various sums involving the Dirichlet coefficients of $\log L$ carefully using assumptions A4 and A6. As a result we obtain the following theorem.

THEOREM 2·2. *Let κ be as in assumption A5 and $0<\varepsilon<\min\{1/48,\kappa/3\}$. Let k be a positive integer such that $k\le(\varepsilon/4)(\log\log T)^2$ Assume (2·1), then there exist $\kappa_0,c>0$ such that*

$$\frac{1}{T}\int_T^{2T}|\log L(\sigma_T+it)|^{2k}dt\ll c^kk^{4k}e^{-\kappa_0\frac{\log T}{G(T)}}+c^kk^k(\log G(T))^k \tag{2·2}$$

and

$$\mathbb{E}[|\log L(\sigma_T,\mathbb{X})|^{2k}]\ll c^kk^k(\log G(T))^k. \tag{2·3}$$

By Theorem 2·2 with $k=\log\log T$ one can easily derive the following corollary, which is necessary in Section 3.

COROLLARY 2·3 *Assume (2·1). Given constant $A_1>0$, there exists a constant $A_2>0$ such that*

$$\frac{1}{T}\text{meas}\{t\in[T,2T]:|\log L(\sigma_T+it)|\ge A_2\log\log T\}\ll(\log T)^{-A_1}$$

and

$$\mathbb{P}(|\log L(\sigma_T,\mathbb{X})|\ge A_2\log\log T)\ll(\log T)^{-A_1}.$$

We provide lemmas in Section 2·1 and then prove Theorems 2·1 and 2·2 in Section 2·2

2·1. *Lemmas.*

We adapt estimations in [**16**, chapter 5] for $\log L$. We begin with [**16**, lemma 5·1].

LEMMA 2·4. *Let κ be as in assumption A5, $0<\kappa'<\kappa$ and $\nu\ge0$. Then there is a constant $c>0$ such that*

$$\sum_{\substack{\beta>\sigma\\ T\le\gamma\le 2T}}(\beta-\sigma)^{\nu}X^{\beta-\sigma}=O\big(T^{1-\kappa(\sigma-\frac12)}(\log T)^{1-\nu}(c\nu)^{\nu}\big)$$

for $1/2\le\sigma\le1$ and $3\le X\le T^{\kappa-\kappa'}$, where $\beta+i\gamma$ denotes a zero of $L(s)$.

Proof. We only prove the case $\nu > 0$, since the case $\nu = 0$ is similar. First we see that

$$\sum_{\substack{\beta>\sigma\\ T\le\gamma\le 2T}} (\beta-\sigma)^\nu X^{\beta-\sigma} = \sum_{\substack{\beta>\sigma\\ T\le\gamma\le 2T}} \int_0^{\beta-\sigma} d(u^\nu X^u) = \int_0^{1-\sigma} \sum_{\substack{\beta>\sigma+u\\ T\le\gamma\le 2T}} d(u^\nu X^u)$$
$$\le \int_0^{1-\sigma} N_L(\sigma+u, 2T) d(u^\nu X^u).$$

By assumption A5, the above is

$$\begin{aligned}
&\ll T^{1-\kappa(\sigma-\frac{1}{2})} \log T \int_0^{1-\sigma} T^{-\kappa u}(\nu u^{\nu-1} X^u + u^\nu X^u \log X) du\\
&\le T^{1-\kappa(\sigma-\frac{1}{2})} \log T \int_0^{\infty} (\nu u^{\nu-1} + u^\nu \log X) T^{-\kappa' u} du\\
&\ll T^{1-\kappa(\sigma-\frac{1}{2})} (\log T)^{1-\nu} c^\nu \Gamma(\nu+1)
\end{aligned}$$

for some $c > 0$. Hence, the lemma follows.

Define

$$\sigma_{x,t} := \frac{1}{2} + 2\max\left\{\beta - \frac{1}{2}, \frac{2}{\log x}\right\}$$

for $t \in [T, 2T]$, where the maximum is taken over all zeros $\beta + i\gamma$ of $L(s)$ satisfying $|t-\gamma| \le x^{3(\beta-1/2)}/\log x$ and $\beta \ge 1/2$. Then the following lemma corresponds to [16, lemma 5·2].

LEMMA 2·5. *Let $\nu \ge 0$, $0 < \kappa' < \kappa$ and $x = T^{\varepsilon/k}$ for $\varepsilon, k > 0$. Suppose that $3 \le x^3 X^2 \le T^{\kappa-\kappa'}$. Then there is a constant $c > 0$ depending on κ, ε such that*

$$\int_{\substack{\sigma_{x,t}>\sigma\\ T\le t\le 2T}} (\sigma_{x,t}-\sigma)^\nu X^{\sigma_{x,t}-\sigma} dt \ll_\varepsilon \frac{(c\nu)^\nu k}{(\log T)^\nu} T^{1-\frac{\kappa}{2}(\sigma-\frac{1}{2})} x^{\frac{3}{2}(\sigma-\frac{1}{2})}$$

for $1/2 + 4/\log x \le \sigma \le 1$ and

$$\int_{\substack{\sigma_{x,t}>\sigma\\ T\le t\le 2T}} (\sigma_{x,t}-\sigma)^\nu X^{\sigma_{x,t}-\sigma} dt \ll_\varepsilon \frac{(c\nu)^\nu k}{(\log T)^\nu} T^{1-\frac{\kappa}{2}(\sigma-\frac{1}{2})} + T\frac{c^{k+\nu} k^\nu}{(\log T)^\nu}$$

for $1/2 \le \sigma \le 1/2 + 4/\log x$.

Proof. Define two sets

$$S_1 = \left\{t \in [T, 2T] : \sigma_{x,t} > \max\left(\sigma, \frac{1}{2} + \frac{4}{\log x}\right)\right\},$$
$$S_2 = \left\{t \in [T, 2T] : \sigma_{x,t} = \frac{1}{2} + \frac{4}{\log x} > \sigma\right\}.$$

Since $\sigma_{x,t} \ge 1/2 + \frac{4}{\log x}$, we see that

$$\int_{\substack{\sigma_{x,t}>\sigma\\ T\le t\le 2T}} (\sigma_{x,t}-\sigma)^\nu X^{\sigma_{x,t}-\sigma} dt = \int_{S_1} (\sigma_{x,t}-\sigma)^\nu X^{\sigma_{x,t}-\sigma} dt + \int_{S_2} (\sigma_{x,t}-\sigma)^\nu X^{\sigma_{x,t}-\sigma} dt.$$

For $t\in S_1$, by the definition of $\sigma_{x,t}$ and $\sigma_{x,t}>1/2+4/\log x$, there exists a zero $\beta+i\gamma$ such that $\sigma_{x,t}=2\beta-1/2$, $\beta-1/2>2/\log x$ and $|t-\gamma|\le x^{3(\beta-1/2)}/\log x$. Thus, we have

$$\begin{aligned}\int_{S_1}(\sigma_{x,t}-\sigma)^{\nu}X^{\sigma_{x,t}-\sigma}dt &\le \sum_{\substack{\beta>\frac12(\sigma+\frac12)\\ \frac{T}{2}\le\gamma\le 3T}}\int_{\gamma-\frac{x^{3(\beta-\frac12)}}{\log x}}^{\gamma+\frac{x^{3(\beta-\frac12)}}{\log x}}\left(2\beta-\frac12-\sigma\right)^{\nu}X^{2\beta-\frac12-\sigma}dt\\ &\le\frac{2^{1+\nu}x^{\frac32(\sigma-\frac12)}}{\log x}\sum_{\substack{\beta>\frac12(\sigma+\frac12)\\ \frac{T}{2}\le\gamma\le 3T}}\left(\beta-\frac12\left(\sigma+\frac12\right)\right)^{\nu}(x^3X^2)^{\beta-\frac12(\sigma+\frac12)}.\end{aligned}$$

By Lemma 2·4 the above is

$$\ll\frac{k}{\varepsilon}\frac{(c\nu)^{\nu}}{(\log T)^{\nu}}T^{1-\frac{\kappa}{2}(\sigma-\frac12)}x^{\frac32(\sigma-\frac12)} \tag{2·4}$$

for some $c>0$.

We see that $S_2=\emptyset$ for $\sigma\ge 1/2+4/\log x$. If $1/2\le\sigma\le 1/2+4/\log x$, then

$$\int_{S_2}(\sigma_{x,t}-\sigma)^{\nu}X^{\sigma_{x,t}-\sigma}dt\le T\left(\frac{4}{\log x}\right)^{\nu}X^{\frac{4}{\log x}}\le T\frac{c^{k+\nu}k^{\nu}}{(\log T)^{\nu}}$$

for some $c>0$.

Next we consider [16, lemma 5·3] and observe that the condition (ii) therein does not hold in our setting. To adapt its proof to our setting, it requires several inequalities regarding β_L. By assumptions A1 and A6 we have

$$\beta_L(p^r)=\frac1r\sum_{i=1}^{d}\alpha_i(p)^r. \tag{2·5}$$

From (2·5) and assumption A1 it is easy to derive that

$$|\beta_L(p^r)|\le\frac{d}{r}p^{r\eta}\quad\text{for } r\ge 1, \tag{2·6}$$

$$|\beta_L(p^r)|\le\frac1r\sum_{i=1}^{d}|\alpha_i(p)|^r\le\frac{p^{(r-2)\eta}}{r}\sum_{i=1}^{d}|\alpha_i(p)|^2\quad\text{for } r\ge 2 \tag{2·7}$$

and

$$|\beta_L(p)|^2\le\left(\sum_{i=1}^{d}|\alpha_i(p)|\right)^2\le d\sum_{i=1}^{d}|\alpha_i(p)|^2. \tag{2·8}$$

For convenience we extend β_L by letting $\beta_L(n)=0$ if n is not a power of a prime. Then we see that

$$\log L(s)=\sum_n\frac{\beta_L(n)}{n^s}.$$

Define

$$\lambda_t := \lambda(\sigma, x, t) := \max\{\sigma_{x,t}, \sigma\}$$

for $\sigma \in [1/2, 1]$ and

$$g_x(n) := \begin{cases} 1 & \text{for } 1 \le n \le x, \\ \frac{\log^2(x^3/n) - 2\log^2(x^2/n)}{2\log^2 x} & \text{for } x \le n \le x^2, \\ \frac{\log^2(x^3/n)}{2\log^2 x} & \text{for } x^2 \le n \le x^3, \\ 0 & \text{for } x^3 \le n, \end{cases}$$

then we have the following lemma.

LEMMA 2·6. *Let k and m be positive integers such that $k \le m \le 16k$, κ as in assumption A5 and $x = T^{\frac{\varepsilon}{k}}$. Assume that $\varepsilon/k < \kappa/3$ and $0 < \varepsilon \le 1/48$. Then there exists a constant $c > 0$ such that*

$$\int_T^{2T} \left| \sum_n \frac{\beta_L(n) g_x(n)}{n^{\lambda_t + it}} \right|^{2m} dt \ll T c^k k^m \left(\min\left\{ \log\log x, \log \frac{1}{\sigma - \frac{1}{2}} \right\} \right)^m$$

and

$$\int_T^{2T} \left| \sum_n \frac{\beta_L(n) g_x(n) \log n}{n^{\lambda_t + it}} \right|^{2m} dt \ll T c^k k^m \left(\min\left\{ \log x, \frac{1}{\sigma - \frac{1}{2}} \right\} \right)^{2m}$$

for $1/2 \le \sigma \le 1$.

Proof. Let ℓ be a nonnegative integer, then we see that

$$\sum_n \frac{\beta_L(n) g_x(n)(\log n)^\ell}{n^{\lambda_t + it}} = \sum_n \frac{\beta_L(n) g_x(n)(\log n)^\ell}{n^{\sigma + it}} + \sum_n \frac{\beta_L(n) g_x(n)(\log n)^\ell}{n^{it}} (n^{-\lambda_t} - n^{-\sigma}).$$

We split the first sum on the right-hand side as

$$\sum_n \frac{\beta_L(n) g_x(n)(\log n)^\ell}{n^{\sigma + it}} = \sum_p \frac{\beta_L(p) g_x(p)(\log p)^\ell}{p^{\sigma + it}} + \sum_p \frac{\beta_L(p^2) g_x(p^2)(2\log p)^\ell}{p^{2\sigma + 2it}} + \sum_p \sum_{r \ge 3} \frac{\beta_L(p^r) g_x(p^r)(r \log p)^\ell}{p^{r\sigma + irt}}.$$

By (2·7) and assumption A4 we have

$$\left| \sum_p \sum_{r \ge 3} \frac{\beta_L(p^r) g_x(p^r)(r\log p)^\ell}{p^{r\sigma + irt}} \right| \le \sum_p \sum_{3 \le r \le \frac{3\log x}{\log p}} \frac{p^{(r-2)\eta} \sum_{i=1}^d |\alpha_i(p)|^2 (r \log p)^\ell}{r p^{r\sigma}} \ll \sum_p \frac{\sum_{i=1}^d |\alpha_i(p)|^2 (\log p)^\ell}{p^{\frac{3}{2} - \eta}} \ll 1.$$

By [16, lemma 3·3] we have

$$\int_T^{2T}\Big|\sum_p\frac{\beta_L(p)g_x(p)(\log p)^\ell}{p^{\sigma+it}}\Big|^{2m}dt\ll Tm!\Big(\sum_p\frac{|\beta_L(p)g_x(p)|^2(\log p)^{2\ell}}{p^{2\sigma}}\Big)^m$$
$$\int_T^{2T}\Big|\sum_p\frac{\beta_L(p^2)g_x(p^2)(\log p)^\ell}{p^{2\sigma+2it}}\Big|^{2m}dt\ll Tm!\Big(\sum_p\frac{|\beta_L(p^2)g_x(p^2)|^2(\log p)^{2\ell}}{p^{4\sigma}}\Big)^m$$

provided that $x^{3m}\ll T$, which holds for $0<\varepsilon\le 1/48$. By assumption A6 we have

$$\sum_p\frac{|\beta_L(p)g_x(p)|^2(\log p)^{2\ell}}{p^{2\sigma}}\le\sum_{p\le x^3}\frac{|\beta_L(p)|^2(\log p)^{2\ell}}{p}\ll\begin{cases}\log\log x & \text{if } \ell=0,\\ (\log x)^{2\ell} & \text{if } \ell\ge 1\end{cases}$$

for $1/2\le\sigma\le 1/2+4/\log x$,

$$\sum_p\frac{|\beta_L(p)g_x(p)|^2(\log p)^{2\ell}}{p^{2\sigma}}\le\sum_p\frac{|\beta_L(p)|^2(\log p)^{2\ell}}{p^{2\sigma}}\ll\int_2^\infty u^{-2\sigma}(\log u)^{2\ell-1}du$$
$$\ll\begin{cases}\log\frac{1}{\sigma-\frac12} & \text{if } \ell=0,\\ \frac{1}{(\sigma-\frac12)^{2\ell}} & \text{if } \ell\ge 1\end{cases}$$

for $1/2+4/\log x\le\sigma\le 1$. By (2·7) and assumption A4 we have

$$\sum_p\frac{|\beta_L(p^2)g_x(p^2)|^2(\log p)^{2\ell}}{p^{4\sigma}}\ll\sum_p\frac{\sum_{i=1}^d|\alpha_i(p)|^2(\log p)^{2\ell}}{p^{2-2\eta}}\ll 1$$

for $\sigma\ge 1/2$. Since

$$\Big|\sum_n\frac{\beta_L(n)g_x(n)(\log n)^\ell}{n^{\sigma+it}}\Big|^{2m}$$
$$\le 3^m\Big(\Big|\sum_p\frac{\beta_L(p)g_x(p)(\log p)^\ell}{p^{\sigma+it}}\Big|^{2m}+\Big|\sum_p\frac{\beta_L(p^2)g_x(p^2)(2\log p)^\ell}{p^{2\sigma+2it}}\Big|^{2m}+c^m\Big)$$

for some $c>0$, by collecting above equations we find that

$$\int_T^{2T}\Big|\sum_n\frac{\beta_L(n)g_x(n)(\log n)^\ell}{n^{\sigma+it}}\Big|^{2m}dt$$
$$\ll\begin{cases}Tc^kk^m\Big(\min\Big\{\log\log x,\log\frac{1}{\sigma-\frac12}\Big\}\Big)^m & \text{if } \ell=0,\\ Tc^kk^m\Big(\min\Big\{\log x,\frac{1}{\sigma-\frac12}\Big\}\Big)^{2\ell m} & \text{if } \ell\ge 1\end{cases}\tag{2·9}$$

for some constant $c>0$ and for $1/2\le\sigma\le 1$.

We next estimate

$$\int_T^{2T}\Big|\sum_n\frac{\beta_L(n)g_x(n)(\log n)^\ell}{n^{it}}(n^{-\lambda_t}-n^{-\sigma})\Big|^{2m}dt.$$

By equations in [**16**, p. 67] the above integral is bounded by

$$\ll\left(\int_T^{2T}(\lambda_t-\sigma)^{4m}X_1^{4m(\lambda_t-\sigma)}dt\right)^{\frac{1}{2}}\left(\int_\sigma^\infty X_1^{\sigma-\nu}d\nu\right)^{2m-\frac{1}{2}}$$
$$\times\left(\int_\sigma^\infty X_1^{\sigma-\nu}\int_T^{2T}\left|\sum_n\frac{\beta_L(n)g_x(n)(\log n)^{\ell+1}\log(X_1n)}{n^{\nu+it}}\right|^{4m}dtd\nu\right)^{\frac{1}{2}}$$

with $X_1=T^{\frac{\varepsilon_1}{m}}$ for some $\varepsilon_1>0$. Let $\nu=4m$ and $X=X_1^{4m}=T^{4\varepsilon_1}$ in Lemma 2·5. One can easily check that the assumptions in Lemma 2·5 follow from the assumptions in Lemma 2·6. Thus, by Lemma 2·5 there exists $c>0$ such that

$$\int_T^{2T}(\lambda_t-\sigma)^{4m}X_1^{4m(\lambda_t-\sigma)}dt\ll c^kk^{4m}T^{1-\frac{1}{2}(\kappa-\frac{3\varepsilon}{k})(\sigma-\frac{1}{2})}(\log T)^{-4m}$$

for $1/2\le\sigma\le 1$. By (2·9) we have

$$\int_\sigma^\infty X_1^{\sigma-\nu}\int_T^{2T}\left|\sum_n\frac{\beta_L(n)g_x(n)(\log n)^{\ell+1}\log(X_1n)}{n^{\nu+it}}\right|^{4m}dtd\nu$$
$$\ll Tc^kk^{2m}\left(\frac{\log T}{k}\right)^{2m(2\ell+3)-1}\left(\min\left\{\log x,\frac{1}{\sigma-\frac{1}{2}}\right\}\right)^{2m}.$$

Therefore, by combining above results we obtain

$$\int_T^{2T}\left|\sum_n\frac{\beta_L(n)g_x(n)(\log n)^\ell}{n^{it}}(n^{-\lambda_t}-n^{-\sigma})\right|^{2m}dt$$
$$\ll c^kk^{2m-2m\ell}T^{1-\frac{1}{4}(\kappa-\frac{3\varepsilon}{k})(\sigma-\frac{1}{2})}(\log T)^{2m\ell-m}\left(\min\left\{\log x,\frac{1}{\sigma-\frac{1}{2}}\right\}\right)^m \quad (2{\cdot}10)$$

for $1/2\le\sigma\le 1$. The lemma follows from (2·9) and (2·10).

The following lemma is an analogy of [**16**, lemma 5·4]. The proof of [**7**, lemma 8] is for Hecke L-functions of number fields, but it works also for our L-functions. So we state the lemma without a proof.

LEMMA 2·7 *Let $t\in[T,2T]$, $1/2\le\sigma\le 1$ and $t\ne\mathrm{Im}(\rho)$ for any zeros ρ of $L(s)$. Then we have:*

$$\log L(s)=\sum_n\frac{\beta_L(n)g_x(n)}{n^{\lambda_t+it}}+\tilde{L}(s)$$
$$+O\left(\left(\frac{x^{\frac{1}{4}-\frac{1}{2}\lambda_t}}{\log x}+(\lambda_t-\sigma)\right)\left(\left|\sum_n\frac{\beta_L(n)g_x(n)\log n}{n^{\sigma_{x,t}+it}}\right|+\log T\right)\right),$$

where

$$\tilde{L}(s)=\sum_\rho\int_\sigma^{\lambda_t}\frac{u-\lambda_t}{(u+it-\rho)(\lambda_t+it-\rho)}du. \quad (2{\cdot}11)$$

The following lemma is proved for the Riemann zeta function in the proof of [16, lemma 5·5]. We rewrite its proof for convenience.

LEMMA 2·8 *Let* $\tilde{L}(s)$ *be as in (2·11) and* $x = T^{\frac{\varepsilon}{k}}$. *Assume that* $\varepsilon/k < \kappa/3$ *and* $0 < \varepsilon \leq 1/48$. *Then we have*

$$|\mathrm{Im}(\tilde{L}(s))| \ll (\lambda_t - \sigma)\left(\left|\sum_n \frac{\beta_L(n) g_x(n)\log n}{n^{\lambda_t+it}}\right| + \log T\right),$$
$$|\mathrm{Re}(\tilde{L}(s))| \ll (\lambda_t - \sigma)\big(1 + (\lambda_t-\sigma)\log x + \log^+ \frac{1}{\eta_t \log x}\big) \times \left(\left|\sum_n \frac{\beta_L(n) g_x(n)\log n}{n^{\lambda_t+it}}\right| + \log T\right),$$

where $\log^+ w := \max\{\log w, 0\}$ *and* $\eta_t = \min|t-\gamma|$ *with the minimum taken over all zeros* $\beta + i\gamma$ *of* $L(s)$ *with* $\beta \geq 1/2$. *Moreover, we have*

$$\int_T^{2T}\left(\log^+ \frac{1}{\eta_t \log x}\right)^{2k} dt \ll T(ck)^{2k}$$

for some $c > 0$.

Proof. If $\sigma \geq \sigma_{x,t}$, then $\lambda_t = \sigma$, $\tilde{L}(s) = 0$ and the lemma holds trivially. Thus, we assume that $\sigma < \sigma_{x,t}$, then $\lambda_t = \sigma_{x,t}$. By (2·11) we find that

$$\mathrm{Im}(\tilde{L}(s)) = \sum_\rho \int_\sigma^{\sigma_{x,t}} \frac{(\sigma_{x,t}-u)(t-\gamma)(u-\beta+\sigma_{x,t}-\beta)}{|u+it-\rho|^2|\sigma_{x,t}+it-\rho|^2} du \tag{2·12}$$

and

$$\mathrm{Re}(\tilde{L}(s)) = \sum_\rho \int_\sigma^{\sigma_{x,t}} \frac{(u-\sigma_{x,t})\big((u-\beta)(\sigma_{x,t}-\beta)-(t-\gamma)^2\big)}{|u+it-\rho|^2|\sigma_{x,t}+it-\rho|^2} du. \tag{2·13}$$

First we find an upper bound of $\mathrm{Im}(\tilde{L}(s))$. By (2·12) and $|\sigma_{x,t}-u| \leq |\sigma_{x,t}-\sigma|$, we have

$$\begin{aligned}|\mathrm{Im}(\tilde{L}(s))| &\leq \sum_\rho \int_\sigma^{\sigma_{x,t}} \frac{|\sigma_{x,t}-u||t-\gamma|(|\sigma_{x,t}-u|+2|u-\beta|)}{|u+it-\rho|^2|\sigma_{x,t}+it-\rho|^2} du \\ &\leq \sum_\rho \frac{|\sigma_{x,t}-\sigma|^2}{|\sigma_{x,t}+it-\rho|^2}\int_\sigma^{\sigma_{x,t}} \frac{|t-\gamma|}{(u-\beta)^2+(t-\gamma)^2} du \\ &\quad + 2\sum_\rho \frac{|\sigma_{x,t}-\sigma|}{|\sigma_{x,t}+it-\rho|^2}\int_\sigma^{\sigma_{x,t}} \frac{|t-\gamma||u-\beta|}{(u-\beta)^2+(t-\gamma)^2} du.\end{aligned}$$

The integrals on the right-hand side are

$$\int_\sigma^{\sigma_{x,t}} \frac{|t-\gamma|}{(u-\beta)^2+(t-\gamma)^2} du \leq \int_{-\infty}^{\infty} \frac{|t-\gamma|}{(u-\beta)^2+(t-\gamma)^2} du = \int_{-\infty}^{\infty} \frac{du}{u^2+1} = \pi,$$
$$\int_\sigma^{\sigma_{x,t}} \frac{|t-\gamma||u-\beta|}{(u-\beta)^2+(t-\gamma)^2} du \leq (\sigma_{x,t}-\sigma),$$

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so that

$$|\mathrm{Im}(\tilde{L}(s))| \le (\pi+2)\sum_{\rho}\frac{|\sigma_{x,t}-\sigma|^2}{|\sigma_{x,t}+it-\rho|^2}. \tag{2·14}$$

Selberg in (4·8) of [13] proved that

$$\sum_{\rho}\frac{1}{|\sigma_{x,t}+it-\rho|^2} \ll \frac{1}{\sigma_{x,t}-\frac{1}{2}}\left(\left|\sum_{n}\frac{\beta_L(n)g_x(n)\log n}{n^{\sigma_{x,t}+it}}\right|+\log T\right) \tag{2·15}$$

for the Riemann zeta function, and it also holds for our L-functions. We may prove (2·15) by (4·4) and (4·6) of [7] in the proof of [7, lemma 8]. By (2·14) and (2·15) the first inequality in Lemma 2·8 holds.

Next we find an upper bound of $\mathrm{Re}(\tilde{L}(s))$. By (2·13), we have

$$\begin{aligned}|\mathrm{Re}(\tilde{L}(s))| &\le \sum_{\rho}\int_{\sigma}^{\sigma_{x,t}}\frac{|\sigma_{x,t}-u|\big(|u-\beta|(|\sigma_{x,t}-u|+|u-\beta|)+|t-\gamma|^2\big)}{|u+it-\rho|^2|\sigma_{x,t}+it-\rho|^2}du\\ &\le \sum_{\rho}\frac{|\sigma_{x,t}-\sigma|^2}{|\sigma_{x,t}+it-\rho|^2}\int_{\sigma}^{\sigma_{x,t}}\frac{|u-\beta|}{(u-\beta)^2+(t-\gamma)^2}du+\sum_{\rho}\frac{|\sigma_{x,t}-\sigma|}{|\sigma_{x,t}+it-\rho|^2}.\end{aligned}$$

The integral on the right-hand side is

$$\int_{\sigma}^{\sigma_{x,t}}\frac{|u-\beta|}{(u-\beta)^2+(t-\gamma)^2}du \le 2\int_{\sigma}^{\sigma_{x,t}}\frac{1}{|u-\beta|+|t-\gamma|}du \le 4\log\left(1+\frac{\sigma_{x,t}-\sigma}{|t-\gamma|}\right).$$

Define $\log^+ w=\max\{\log w, 0\}$ for $w>0$, then for any $v, w>0$, it is easy to verify $\log(1+w)\le 1+\log^+ w$, $\log^+(w/v)\le \log^+ w+\log^+(1/v)$ and $\log^+ w\le w$. Then we have

$$\begin{aligned}\log\left(1+\frac{\sigma_{x,t}-\sigma}{|t-\gamma|}\right) &\le \log\left(1+\frac{(\sigma_{x,t}-\sigma)\log x}{\eta_t\log x}\right)\\ &\le 1+\log^+\big((\sigma_{x,t}-\sigma)\log x\big)+\log^+\frac{1}{\eta_t\log x}\\ &\le 1+(\sigma_{x,t}-\sigma)\log x+\log^+\frac{1}{\eta_t\log x}.\end{aligned}$$

Thus, we find that

$$|\mathrm{Re}(\tilde{L}(s))| \le \left(1+4(\sigma_{x,t}-\sigma)\left(1+(\sigma_{x,t}-\sigma)\log x+\log^+\frac{1}{\eta_t\log x}\right)\right)\sum_{\rho}\frac{|\sigma_{x,t}-\sigma|}{|\sigma_{x,t}+it-\rho|^2}.$$

Now, the second inequality of Lemma 2·8 follows from the above inequality and (2·15).

By the definition of $\log^+$ and η_t we find that

$$\int_T^{2T}\left(\log^+\frac{1}{\eta_t\log x}\right)^{2k}dt \le \sum_{\substack{\beta\ge\frac{1}{2}\\ T-\frac{1}{\log x}\le\gamma\le 2T+\frac{1}{\log x}}}\int_0^{\frac{1}{\log x}}\left(\log^+\frac{1}{w\log x}\right)^{2k}dw.$$

The number of zeros in the above sum is $O(T\log T)$. By substituting $w\log x=e^{-v}$, the last integral equals to $\Gamma(2k+1)/\log x=(2k)!/\log x$. Hence, the last inequality of Lemma 2·8 follows.

2·2. *Proof of Theorems 2·1 and 2·2*

To prove Theorems 2·1 and 2·2, we need to find an upper bound of the $2k$th moment

$$\int_T^{2T} \left| \log L(\sigma_T + it) - \sum_n \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}} \right|^{2k} dt,$$

where $x = T^{(\varepsilon/k)}$, $k \le \varepsilon/4(\log\log T)^2$ and $0 < \varepsilon < \min\{1/48, \kappa/3\}$. Let $\sigma = 1/2$ and $k = m$ in Lemma 2·6, then we get

$$\int_T^{2T} \left| \sum_n \frac{\beta_L(n) g_x(n) \log n}{n^{\sigma_{x,t} + it}} \right|^{2k} dt \ll c^k k^k T (\log x)^{2k}. \tag{2·16}$$

By Lemmas 2·7 and 2·8 and (2·16), we have

$$\begin{aligned}
&\int_T^{2T} \left| \log L(\sigma_T + it) - \sum_n \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}} \right|^{2k} dt \\
&\ll c^k \int_T^{2T} \left| \sum_n \frac{\beta_L(n) g_x(n)}{n^{\lambda_t + it}} - \sum_n \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}} \right|^{2k} dt \\
&+ c^k \int_T^{2T} (\lambda_t - \sigma_T)^{2k} \left(1 + (\lambda_t - \sigma_T) \log x + \log^+ \frac{1}{\eta_t \log x} \right)^{2k} \\
&\times \left| \sum_n \frac{\beta_L(n) g_x(n) \log n}{n^{\lambda_t + it}} \right|^{2k} dt \\
&+ c^k (\log T)^{2k} \int_T^{2T} (\lambda_t - \sigma_T)^{2k} \left(1 + (\lambda_t - \sigma_T) \log x + \log^+ \frac{1}{\eta_t \log x} \right)^{2k} dt \\
&+ c^k k^{2k} T e^{-\varepsilon \frac{\log T}{G(T)}}
\end{aligned} \tag{2·17}$$

for some $c > 0$. It remains to bound the integrals on the right-hand side.

Since $k \le \varepsilon/4(\log\log T)^2$, we see that

$$\sigma_T - \frac{1}{2} = \frac{1}{G(T)} \ge \frac{(\log\log T)^2}{\log T} \ge \frac{4}{\log x}.$$

By (2·10) we have

$$\int_T^{2T} \left| \sum_n \frac{\beta_L(n) g_x(n)}{n^{\lambda_t + it}} - \sum_n \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}} \right|^{2k} dt \ll c^k k^{2k} T e^{-\frac{1}{4}(\kappa - \frac{3\varepsilon}{k}) \frac{\log T}{G(T)}} \frac{G(T)^k}{(\log T)^k} \tag{2·18}$$

for some $c > 0$. By Lemmas 2·5 and 2·8 we have

$$\int_T^{2T} (\lambda_t - \sigma_T)^{2m} dt \ll \frac{c^k m^{2m}}{(\log T)^{2m}} T e^{-\frac{1}{2}(\kappa - \frac{3\varepsilon}{k}) \frac{\log T}{G(T)}}$$

and

$$\begin{aligned}
&\int_T^{2T} (\lambda_t - \sigma_T)^{2m} \left(\log^+ \frac{1}{\eta_t \log x} \right)^{2m} dt \\
&\le \left(\int_T^{2T} (\lambda_t - \sigma_T)^{4m} dt \right)^{\frac{1}{2}} \left(\int_T^{2T} \left(\log^+ \frac{1}{\eta_t \log x} \right)^{4m} dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$\ll \frac{c^k m^{4m}}{(\log T)^{2m}} T e^{-\frac{1}{4}(\kappa - \frac{3\varepsilon}{k})\frac{\log T}{G(T)}}$$

for $k \le m \le 4k$. Thus, we obtain

$$\int_T^{2T} (\lambda_t - \sigma_T)^{2m} \left(1 + (\lambda_t - \sigma_T)\log x + \log^+ \frac{1}{\eta_t \log x}\right)^{2m} dt \\ \ll \frac{c^k m^{4m}}{(\log T)^{2m}} T e^{-\frac{1}{4}(\kappa - \frac{3\varepsilon}{k})\frac{\log T}{G(T)}} \tag{2·19}$$

for $k \le m \le 2k$. By Lemma 2·6, the Cauchy–Schwarz inequality and the above inequality we have

$$\int_T^{2T} (\lambda_t - \sigma_T)^{2k} \left(1 + (\lambda_t - \sigma_T)\log x + \log^+ \frac{1}{\eta_t \log x}\right)^{2k} \left|\sum_n \frac{\beta_L(n) g_x(n) \log n}{n^{\lambda_t + it}}\right|^{2k} dt \\ \ll \frac{c^k k^{5k} G(T)^{2k}}{(\log T)^{2k}} T e^{-\frac{1}{8}(\kappa - \frac{3\varepsilon}{k})\frac{\log T}{G(T)}}. \tag{2·20}$$

Therefore, by (2·17) – (2·20) there exist $\kappa_0 > 0$ such that

$$\int_T^{2T} \left|\log L(\sigma_T + it) - \sum_n \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}}\right|^{2k} dt \ll c^k k^{4k} T e^{-\kappa_0 \frac{\log T}{G(T)}}. \tag{2·21}$$

Let $k = 1$ in (2·21), then we see that

$$\int_T^{2T} \left|\log L(\sigma_T + it) - \sum_n \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}}\right|^{2} dt \ll T e^{-\kappa_0 \frac{\log T}{G(T)}}, \tag{2·22}$$

where $x = T^\varepsilon$ and $0 < \varepsilon < \min\{1/48, \kappa/3\}$. Let $e^{\frac{G(T)}{2}} \le Y \le x$, then we have

$$\int_T^{2T} \left|\sum_{n>Y} \frac{\beta_L(n) g_x(n)}{n^{\sigma_T + it}}\right|^2 dt \ll T \sum_{n>Y} \frac{|\beta_L(n)|^2}{n^{2\sigma_T}} \ll T \frac{Y^{1-2\sigma_T}}{(2\sigma_T - 1)\log Y} \tag{2·23}$$

by [**4**, lemma 4·1]. Thus, Theorem 2·1 follows from (2·22) and (2·23).

Next we prove Theorem 2·2. We see that (2·2) holds by (2·9) and (2·21). The proof of (2·3) is similar, but simpler than the proof of Lemma 2·6. Since

$$\log L(\sigma_T, \mathbb{X}) = \sum_p \frac{\beta_L(p)\mathbb{X}(p)}{p^{\sigma_T}} + \sum_p \frac{\beta_L(p^2)\mathbb{X}(p^2)}{p^{2\sigma_T}} + O(1),$$

by [**16**, lemma 3·3] we have

$$\mathbb{E}[|\log L(\sigma_T, \mathbb{X})|^{2k}] \le c^k \left(k! \left(\sum_p \frac{|\beta_L(p)|^2}{p^{2\sigma_T}}\right)^k + k! \left(\sum_p \frac{|\beta_L(p^2)|^2}{p^{4\sigma_T}}\right)^k + 1\right)$$

for some $c > 0$. By (2·7) and assumption A4 we have

$$\sum_p \frac{|\beta_L(p^2)|^2}{p^{4\sigma_T}} \ll \sum_p \frac{\sum_{i=1}^d |\alpha_i(p)|^2}{p^{2-2\eta}} \ll 1.$$

By assumption A6 we have

$$\sum_p \frac{|\beta_L(p)|^2}{p^{2\sigma_T}} \ll \int_2^\infty \frac{du}{u^{1+\frac{2}{G(T)}} \log u} \ll \log G(T).$$

Thus, we have

$$\mathbb{E}[|\log L(\sigma_T, \mathbb{X})|^{2k}] \ll c^k k!(\log G(T))^k$$

for some $c > 0$.

3. *Discrepancy*

In this section we will prove Theorem 1·2 for $G(T)$ satisfying (2·1). First we need to extend [4, proposition 5·1]. Define the Fourier transforms of Φ_T and Φ_T^{rand} by

$$\widehat{\Phi}_T(\mathbf{x}, \mathbf{y}) := \int_{\mathbb{R}^{2J}} e^{2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})} d\Phi_T(\mathbf{u}, \mathbf{v})$$

and

$$\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x}, \mathbf{y}) := \int_{\mathbb{R}^{2J}} e^{2\pi i(\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v})} d\Phi_T^{\mathrm{rand}}(\mathbf{u}, \mathbf{v}),$$

where $\mathbf{x} = (x_1, \ldots, x_J)$ and similarly $\mathbf{y}, \mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^J$ and $\mathbf{x}\cdot\mathbf{u} := \sum_{j\le J} x_j u_j$ is the dot product. Then we obtain the following proposition.

PROPOSITION 3·1. *Assume (2·1). Given constant $A_4 > 0$, there exists a constant $A_5 > 0$ such that*

$$\widehat{\Phi}_T(\mathbf{x}, \mathbf{y}) = \widehat{\Phi}_T^{rand}(\mathbf{x}, \mathbf{y}) + O\left(\frac{1}{(\log T)^{A_4}}\right)$$

for $\max_{j\le J}\{|x_j|, |y_j|\} \le \sqrt{\log T}/A_5\sqrt{G(T)} \log\log T$.

Proof. By definition we get

$$\widehat{\Phi}_T(\mathbf{x}, \mathbf{y}) = \frac{1}{T}\int_T^{2T} \exp\left[2\pi i \sum_{j\le J} \big(x_j \log|L_j(\sigma_T + it)| + y_j \arg L_j(\sigma_T + it)\big)\right] dt,$$

$$\widehat{\Phi}_T^{\mathrm{rand}}(\mathbf{x}, \mathbf{y}) = \mathbb{E}\left[\exp\left[2\pi i \sum_{j\le J} \big(x_j \log|L_j(\sigma_T, X)| + y_j \arg L_j(\sigma_T, \mathbb{X})\big)\right]\right].$$

Since the inequality

$$|e^{ix} - e^{iy}|^2 = 4\sin^2\left(\frac{x-y}{2}\right) \le |x-y|^2$$

holds for any $x, y \in \mathbb{R}$, by the Cauchy–Schwarz inequality and Theorem 2·1 with

$$\log Y = A_6 G(T) \log\log T$$

we have

$$\begin{aligned}
&\widehat{\Phi}_T(\mathbf{x},\mathbf{y}) - \frac{1}{T}\int_T^{2T} \exp\left[2\pi i \sum_{j\le J}\left(x_j \mathrm{Re}(R_{j,Y}(\sigma_T+it)) + y_j \mathrm{Im}(R_{j,Y}(\sigma_T+it))\right)\right]dt\\
&\quad = O\left(\frac{1}{T}\int_T^{2T}\sum_{j\le J}(|x_j|+|y_j|)|\log L_j(\sigma_T+it) - R_{j,Y}(\sigma_T+it)|dt\right)\\
&\quad = O\left(\sum_{j\le J}(|x_j|+|y_j|)\left(\frac{1}{T}\int_T^{2T}|\log L_j(\sigma_T+it) - R_{j,Y}(\sigma_T+it)|^2 dt\right)^{\frac{1}{2}}\right)\\
&\quad = O\left(\frac{M}{(\log T)^{A_6}}\right)
\end{aligned}$$

for all $|x_j|, |y_j| \le M$. Let

$$N = \left[\frac{\log T}{10A_6 G(T)\log\log T}\right],$$

then by the Taylor theorem and [4, lemma 4·5] we have

$$\begin{aligned}
&\widehat{\Phi}_T(\mathbf{x},\mathbf{y}) - \sum_{n=0}^{2N-1}\frac{(2\pi i)^n}{n!T}\int_T^{2T}\left(\sum_{j\le J}\left(x_j \mathrm{Re}(R_{j,Y}(\sigma_T+it)) + y_j \mathrm{Im}(R_{j,Y}(\sigma_T+it))\right)\right)^n dt\\
&\quad = O\left(\frac{c^N M^{2N}}{(2N)!}\frac{1}{T}\int_T^{2T}\sum_{j\le J}\left|R_{j,Y}(\sigma_T+it)\right|^{2N}dt + \frac{M}{(\log T)^{A_6}}\right)\\
&\quad = O\left(\left(\frac{cM^2\log\log T}{N}\right)^N + \frac{M}{(\log T)^{A_6}}\right)
\end{aligned}$$

for some $c>0$. Let

$$M = \frac{\sqrt{\log T}}{A_5\sqrt{G(T)}\log\log T}$$

with a constant $A_5 \ge \sqrt{10cA_6}e^{5A_6^2}$, then we have

$$\begin{aligned}
\widehat{\Phi}_T(\mathbf{x},\mathbf{y}) = &\sum_{n=0}^{2N-1}\frac{(2\pi i)^n}{n!T}\int_T^{2T}\left(\sum_{j\le J}\left(x_j \mathrm{Re}(R_{j,Y}(\sigma_T+it)) + y_j \mathrm{Im}(R_{j,Y}(\sigma_T+it))\right)\right)^n dt\\
&+ O\left(\frac{1}{(\log T)^{A_6-\frac{1}{2}}}\right).
\end{aligned}$$

By following the second half of the proof of [4, proposition 5·1] one can conclude that the proposition holds.

We next need to introduce Beurling–Selberg functions. Define

$$F_{[a,b],\Delta}(z) = \frac{1}{2}(H(\Delta(z-a)) - K(\Delta(z-a)) + H(\Delta(b-z)) - K(\Delta(b-z)))$$

for $z \in \mathbb{C}$ and $\Delta > 0$, where

$$H(z) = \frac{\sin^2(\pi z)}{\pi^2}\left(\sum_{n=-\infty}^{\infty} \frac{\mathrm{sgn}(n)}{(z-n)^2} + \frac{2}{z}\right) \quad \text{and} \quad K(z) = \frac{\sin^2(\pi z)}{(\pi z)^2}.$$

Then we summarise some results in [6, section 7] as a lemma.

LEMMA 3·2. *For all $x \in \mathbb{R}$ we have $|F_{[a,b],\Delta}(x)| \le 1$ and*

$$0 \le \mathbf{1}_{[a,b]}(x) - F_{[a,b],\Delta}(x) \le K(\Delta(x-a)) + K(\Delta(b-x)).$$

Moreover, the Fourier transform $\widehat{F}_{[a,b],\Delta}$ satisfies

$$\widehat{F}_{[a,b],\Delta} = \begin{cases} \widehat{\mathbf{1}}_{[a,b]}(y) + O(\Delta^{-1}) & \text{if } |y| \le \Delta, \\ 0 & \text{if } |y| \ge \Delta. \end{cases}$$

We are ready to prove Theorem 1·2 for $G(T)$ satisfying (2·1). By Corollary 2·3 there exists a constant $A_3 > 0$ such that

$$\frac{1}{T}\mathrm{meas}\{t \in [T, 2T] : \mathbf{L}(\sigma_T + it) \notin I_T\} \ll \frac{1}{(\log T)^{10}},$$
$$\mathbb{P}\{\mathbf{L}(\sigma_T, \mathbb{X}) \notin I_T\} \ll \frac{1}{(\log T)^{10}},$$

where

$$I_T := [-A_3 \log\log T, A_3 \log\log T]^{2J}.$$

Then we see that

$$\Phi_T(\mathcal{R}) = \Phi_T(\mathcal{R} \cap I_T) + O\left(\frac{1}{(\log T)^{10}}\right),$$
$$\Phi_T^{\mathrm{rand}}(\mathcal{R}) = \Phi_T^{\mathrm{rand}}(\mathcal{R} \cap I_T) + O\left(\frac{1}{(\log T)^{10}}\right)$$

for any $\mathcal{R} \in \mathbb{R}^{2J}$. Thus, we have

$$\mathbf{D}(\sigma_T) = \sup_{\mathcal{R} \subset I_T} |\Phi_T(\mathcal{R}) - \Phi_T^{\mathrm{rand}}(\mathcal{R})| + O\left(\frac{1}{(\log T)^{10}}\right), \tag{3·1}$$

where $\mathcal{R} \subset I_T$ runs over all rectangular boxes of $\mathbb{R}^{2J}$ with sides parallel to the coordinate axes. By (3·1) it is enough to show that

$$\Phi_T(\mathcal{R}) - \Phi_T^{\mathrm{rand}}(\mathcal{R}) = O(M^{-1}) \tag{3·2}$$

for

$$\mathcal{R} = \prod_{j=1}^{J} I_{1,j} \times \prod_{j=1}^{J} I_{2,j} \subset I_T,$$

where $I_{1,j} = [a_j, b_j]$ and $I_{2,j} = [c_j, d_j]$ for $j = 1, \ldots, J$.

By definition we see that

$$\Phi_T(\mathcal{R}) = \frac{1}{T}\int_T^{2T}\prod_{j=1}^{J}\mathbf{1}_{I_{1,j}}(\log|L_j(\sigma_T+it)|)\mathbf{1}_{I_{2,j}}(\arg L_j(\sigma_T+it))dt,$$

$$\Phi_T^{\text{rand}}(\mathcal{R}) = \mathbb{E}\bigg[\prod_{j=1}^{J}\mathbf{1}_{I_{1,j}}(\log|L_j(\sigma_T,\mathbb{X})|)\mathbf{1}_{I_{2,j}}(\arg L_j(\sigma_T,\mathbb{X}))\bigg].$$

By Lemma 3·2 with $\Delta = M$ we have

$$\Phi_T(\mathcal{R}) = \frac{1}{T}\int_T^{2T}\prod_{j=1}^{J}F_{I_{1,j},M}(\log|L_j(\sigma_T+it)|)F_{I_{2,j},M}(\arg L_j(\sigma_T+it))dt + O(M^{-1}),$$

$$\Phi_T^{\text{rand}}(\mathcal{R}) = \mathbb{E}\bigg[\prod_{j=1}^{J}F_{I_{1,j},M}(\log|L_j(\sigma_T,\mathbb{X})|)F_{I_{2,j},M}(\arg L_j(\sigma_T,\mathbb{X}))\bigg] + O(M^{-1}). \tag{3·3}$$

To confirm the above O-terms, it requires inequalities similar to

$$\frac{1}{T}\int_T^{2T}K(M(\log|L_1(\sigma_T+it)|-\alpha))dt$$
$$= \frac{1}{M}\int_{-M}^{M}\left(1-\frac{|u|}{M}\right)e^{-2\pi i\alpha u}\widehat{\Phi}_T(u,0,\ldots,0)du \ll \frac{1}{M},$$

which holds by Fourier inversion, Proposition 3·1, [**4**, lemma 7·1] and

$$\hat{K}(x) = \max(0, 1-|x|).$$

By Fourier inversion, Lemma 3·2 and Proposition 3·1 we obtain

$$\frac{1}{T}\int_T^{2T}\prod_{j=1}^{J}F_{I_{1,j},M}(\log|L_j(\sigma_T+it)|)F_{I_{2,j},M}(\arg L_j(\sigma_T+it))dt$$
$$= \int_{\mathbb{R}^{2J}}\left(\prod_{j=1}^{J}\widehat{F}_{I_{1,j},M}(x_j)\widehat{F}_{I_{2,j},M}(y_j)\right)\widehat{\Phi}_T(-\mathbf{x},-\mathbf{y})d\mathbf{x}d\mathbf{y}$$
$$= \int_{\substack{|x_j|,|y_j|\le M\\ j=1,\ldots,J}}\left(\prod_{j=1}^{J}\widehat{F}_{I_{1,j},M}(x_j)\widehat{F}_{I_{2,j},M}(y_j)\right)\widehat{\Phi}_T^{\text{rand}}(-\mathbf{x},-\mathbf{y})d\mathbf{x}d\mathbf{y} + O\left(\frac{(M\log\log T)^{2J}}{(\log T)^{A_4}}\right)$$
$$= \mathbb{E}\bigg[\prod_{j=1}^{J}F_{I_{1,j},M}(\log|L_j(\sigma_T,\mathbb{X})|)F_{I_{2,j},M}(\arg L_j(\sigma_T,\mathbb{X}))\bigg] + O\left(\frac{(M\log\log T)^{2J}}{(\log T)^{A_4}}\right). \tag{3·4}$$

Here, we also have used that

$$|\hat{F}_{[a,b],M}(y)| \le |\hat{\mathbf{1}}_{[a,b]}(y)| + O(M^{-1}) \ll \log\log T$$

for $|y| \le M$ and $|b-a| \ll \log\log T$. We choose A_4 sufficiently large so that

$$\frac{(M\log\log T)^{2J}}{(\log T)^{A_4}} \le \frac{1}{M},$$

then (3·2) holds by (3·3) and (3·4). This completes the proof of Theorem 1·2.

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REFERENCES

[1] J. HA and Y. LEE. *The a-values of the Riemann zeta function near the critical line*, *J. Math. Anal. Appl.* **464** (2018), 838–863.

[2] G. HARMAN and K. MATSUMOTO. *Discrepancy estimates for the value-distribution of the Riemann zeta-function, IV*, J. London Math. Soc. (2) 50(1) (1994), 17–24.

[3] D. HEJHAL. *On Euler products and multi-variate Gaussians*, C. R. Acad. Sci. Paris, Ser. I 337 (2003), 223–226.

[4] Y. LAMZOURI and Y. LEE. The number of zeros of linear combinations of L-functions near the critical line, *J. Anal. Math.* **152**(2) (2024), 669–727.

[5] Y. LAMZOURI, S. LESTER and M. RADZIWIŁŁ. Discrepancy bounds for the distribution of the Riemann zeta-function and applications, *J. Anal. Math.* **139**(2) (2019), 453–494.

[6] Y. LAMZOURI, S. LESTER and M. RADZIWIŁŁ. An effective universality theorem for the Riemann zeta function, *Comment. Math. Helv.* **93**(4) (2018), 709–736.

[7] Y. LEE. The universality theorem for Hecke L-functions, *Math. Z.* **271** (2012), 893–909.

[8] Y. LEE. An asymptotic expansion of Selberg's central limit theorem near the critical line, *J. Number Theory* **236** (2022), 323–333.

[9] Y. LEE. Selberg's central limit theorem of L-functions near the critical line. *J. Math. Anal. Appl.* **527**(1) (2023), 17, Paper No. 127380.

[10] K. MATSUMOTO. Discrepancy estimates for the value-distribution of the Riemann zeta-function, I, *Acta Arith.* **48** (1987), 167–190.

[11] K. MATSUMOTO. *Discrepancy estimates for the value-distribution of the Riemann zeta-function, II*. Number Theory and Combinatorics, Japan 1984 (World Scientific, Singapore, 1985), 265–278.

[12] K. MATSUMOTO. Discrepancy estimates for the value-distribution of the Riemann zeta-function, III, *Acta Arith.* **50** (1988), 315–337.

[13] A. SELBERG. Contributions to the theory of the Riemann zeta function, *Arch. Math. Naturvid.* **48**(5) (1946), 89–155.

[14] A. SELBERG. Contributions to the theory of Dirichlet's L-functions, *Skr. Norske Vid. Akad. Oslo.* I. **3** (1946), 1–62.

[15] A. SELBERG. *Old and new conjectures and results about a class of Dirichlet series*. Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 367–385. Università di Salerno, Salerno (1992).

[16] K.M. TSANG. The distribution of the values of the Riemann zeta-function, ProQuest LLC, Ann Arbor, MI, 1984. Ph.D. thesis. Princeton University (1984).