Canad. Math. Bull. Vol. **52** (3), 2009 pp. 464–472

Two Volume Product Inequalities and Their Applications

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

Alina Stancu

Abstract. Let $K \subset \mathbb{R}^{n+1}$ be a convex body of class C^2 with everywhere positive Gauss curvature. We show that there exists a positive number $\delta(K)$ such that for any $\delta \in (0, \delta(K))$ we have $\mathrm{Vol}(K_\delta) \cdot \mathrm{Vol}((K_\delta)^*) \geq \mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*) \geq \mathrm{Vol}(K^\delta) \cdot \mathrm{Vol}((K^\delta)^*)$, where K_δ, K^δ and K^* stand for the convex floating body, the illumination body, and the polar of K, respectively. We derive a few consequences of these inequalities.

1 Introduction

Besides their intrinsic interest, convex floating bodies, respectively, illumination bodies, have been useful in convex geometry in a number of ways. These bodies provide geometric interpretations of affine surface area, they appear in volume estimates for approximations of convex bodies by polytopes, and, more importantly, they generalize the definition of affine surface area to arbitrary convex bodies consistent with the other existing generalizations, while they surface in other applications as well, see [2, 3, 20, 21, 27, 29, 30, 32, 33]. In what concerns the extension of affine surface area, recall that Blaschke's original definition in $\mathbb{R}^3$, extended by Leichtweiss to higher dimensions, is so that a convex body $K \subset \mathbb{R}^{n+1}$ with boundary of class C^2 has affine surface area

$$\Omega(K) = \int_{\partial K} \mathcal{K}(q)^{\frac{1}{n+2}} \, d\mu_{\partial K}(q),$$

where $\mathcal{K}$ denotes the Gauss curvature at $q \in \partial K$ and $d\mu_K$ stands for the surface area measure of the convex body, [5, 11].

In this paper, we show that by taking convex floating bodies, respectively illumination bodies, of small factors $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*)$, the product of the volume of a convex body of class C^2_+ containing the origin by the volume of its polar body increases, respectively decreases. This is not only interesting in itself, but it implies two characterizations of ellipsoids among convex bodies of class C^2_+ which were conjectured to be true among all convex bodies by Schütt and Werner.

The quantity $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*)$ may immediately bring to mind the volume product functional on convex bodies. Let K be a convex body in $\mathbb{R}^{n+1}$ containing the origin

Received by the editors November 29, 2007; revised May 22, 2008.
The author was partially supported by an NSERC grant and an FRDP grant.
AMS subject classification: Primary: 52A40; secondary: 52A38, 52A20.
Keywords: affine invariants, convex floating bodies, illumination bodies.

https://doi.org/10.4153/CMB-2009-049-0 Published online by Cambridge University Press

in its interior and denote by $K^* := \{x : x \cdot y \leq 1 \text{ for all } y \in K\}$ its polar body with respect to the origin. Taking the origin to coincide with the Santaló point of K, *i.e.*, the unique point of $\operatorname{int}(K)$ for which the volume of K^* is minimal, the volume product functional associates with the convex body K the value $\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^*)$, [26]. In this case, a standard notation for the aforementioned product is $\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^{s(K)})$ which we will use when the origin is the Santaló point of K. Throughout the paper, $\operatorname{Vol}(\,\cdot\,)$ refers to the top dimensional volume with the standard metric inherited from the ambient Euclidean space.

Recall that $\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^{s(K)})$ is the object of the famous Blaschke–Santaló inequality

$$\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^{s(K)}) \leq \sigma_{n+1}^2, \tag{1.1}$$

where equality holds if and only if K is an ellipsoid. Here σ_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$. The classical proof belongs to Blaschke [4] in $\mathbb{R}^3$ and to Santaló [24] in higher dimensions, including equality conditions under sufficient smoothness assumptions. These conditions without smoothness hypotheses are due to Saint Raymond [23] for centrally symmetric bodies and to Petty [22] for arbitrary convex bodies. In fact, simpler proofs were found for arbitrary convex bodies. In particular, Meyer and Pajor obtained an upper bound for $\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^z)$ for z satisfying a certain property. Their inequality is more general than Blaschke–Santaló's inequality, and from this they deduced that equality in the latter holds only for ellipsoids [19]. An alternate proof is due to Lutwak [16] who also relates Blaschke–Santaló inequality to several other affine inequalities emphasizing its central role [14, 17].

The quantity $\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^*)$ for K whose centroid is at the origin, denoted in this case by $\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^c)$, is the object of Mahler's conjecture

$$\operatorname{Vol}(K) \cdot \operatorname{Vol}(K^c) \geq \frac{(n+2)^{n+2}}{((n+1)!)^2}, \tag{1.2}$$

with equality if and only if K is a simplex. Note that our constants in both (1.1) and (1.2) are rescaled to reflect the fact that K lies in $\mathbb{R}^{n+1}$. For references and partial results on this outstanding problem, see [6, 7, 9, 10, 12].

Our main result shows that the minimum of the volume product functional cannot be reached for a centrally symmetric convex body belonging to C^2_+.

2 Preliminaries

A *convex body* $K \subset \mathbb{R}^{n+1}$ is a compact, convex set in $\mathbb{R}^{n+1}$ with non-empty interior. Throughout this paper we will use the following notations. Let $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and let $h_K : \mathbb{S}^n \to \mathbb{R}$, $h_K(u) = \max\{\langle x, u\rangle : x \in K\}$ be the support function of K. Often we will identify a convex body with its support function and vice versa. We may assume, without any loss of generality, that the origin is contained in the interior of K in order to have a support function strictly positive in all directions and facilitate the calculations. We will use $f_K(u)$ to denote the curvature function of K at the point of ∂K where the support hyperplane has normal u. If ∂K is of class C^2_+, then the support function of K is of class C^2, the boundary of K has strictly positive Gauss curvature everywhere, and the curvature function coincides with the reciprocal of the Gauss curvature $\mathcal{K}$ of ∂K at the point where the support hyperplane touches it. The Gauss curvature is viewed here as a function on the unit outer normals to ∂K, hence on $\mathbb{S}^n$.

We will consider illumination bodies and convex floating bodies associated with K as previously defined in the literature.

Definition 2.1 [33] Let $K \subset \mathbb{R}^{n+1}$ be a convex body and let $\delta > 0$ be a real number. The convex set $K^\delta = \{x \in \mathbb{R}^{n+1} : \mathrm{Vol}(\mathrm{co}[x, K] \setminus K) \le \delta\}$ is called the *δ-illumination body of K*, where $\mathrm{co}[x, K]$ is the convex hull of x and K.

The name illumination body could be motivated by the fact that, if at any point on the boundary of K^δ we place a source of light, the illuminated *cone* formed by the point and ∂K has volume δ. It is not hard to see that K^δ is itself a convex body in $\mathbb{R}^{n+1}$ containing K.

Werner has shown that

$$\int_{\partial K} \mathcal{K}^{\frac{1}{n+2}}(q)\, d\mu_K(q) = \frac{1}{c_n} \lim_{\delta \to 0} \frac{\mathrm{Vol}(K^\delta) - \mathrm{Vol}(K)}{\delta^{\frac{2}{n+2}}}, \tag{2.1}$$

where c_n is a normalization constant such that the affine area of B_2^{n+1}, the Euclidean ball in $\mathbb{R}^{n+1}$, is $\Omega(B_2^{n+1}) = \mathrm{Vol}(B_2^n)$. This constant depends solely on the dimension n and its precise value is known [33].

Since the right-hand side of (2.1) does not require any regularity assumptions on ∂K, it can be used to extend the definition of the affine surface area to arbitrary convex bodies as

$$\Omega(K) := \frac{1}{c_n} \lim_{\delta \to 0} \frac{\mathrm{Vol}(K^\delta) - \mathrm{Vol}(K)}{\delta^{\frac{2}{n+2}}}, \tag{2.2}$$

with c_n as above [33].

This extension is equivalent with the others given in the last decade by Leichtweiss [11], Lutwak [15], Meyer and Werner [18], Schmuckenschläger [25], Schütt and Werner [30]. Note also [8, 13, 28] for related references.

However, for our paper, the regularity of ∂K plays an important role.

Lemma 2.2 *Let $K \subset \mathbb{R}^{n+1}$ be a convex body of class C^2_+. There exists a positive number δ_K such that for any $\delta \in (0, \delta_K)$, we have in all unitary directions $u \in \mathbb{S}^n$*

$$h_{K^\delta}(u) \ge h_K(u) + \delta^{\frac{2}{n+2}} c_n f_K^{-\frac{1}{n+2}}(u) + o(\delta^{\frac{2}{n+2}}), \tag{2.3}$$

where $c_n = \frac{1}{2}\Big[\frac{(n+1)(n+2)}{\sigma_n}\Big]^{\frac{2}{n+2}}$, with $\sigma_n = \mathrm{Vol}(B_2^n)$, is the same constant as in (2.1)*, and $f = o(s)$ means $f/s \to 0$ as $s \to 0$.*

Proof The inequality (2.3) is meaningful as long as $\delta < 1$. However, in this paper we focus on illumination bodies of *small* factor δ, *i.e.,* δ close to zero, and in this context, (2.3) is extremely useful.

Let $u \in \mathbb{S}^n$ be a fixed, arbitrary, unitary direction. As K is convex, there exists a unique hyperplane of normal u supporting the boundary of K,

$$\mathcal{H}_u = \{y \in \mathbb{R}^{n+1} \mid \langle u, y\rangle = h_K(u)\},$$

and, similarly, there exists a unique hyperplane of normal u supporting the boundary of K^δ, $\mathcal{H}_u^\delta = \{y \in \mathbb{R}^{n+1} \mid \langle u, y\rangle = h_K^\delta(u)\}$. The distance d between the two parallel hyperplanes described above is precisely the difference $h_{K^\delta}(u) - h_K(u)$.

Choose coordinates $x_1, x_2, \dots, x_{n+1}$ in $\mathbb{R}^{n+1}$ such that $\{e_1, \dots, e_n, u\}$ is a basis of $\mathbb{R}^{n+1}$ and the supporting point $\{q\} := \mathcal{H}_u \cap \partial K$ (unique due to the strict convexity of K), lies at the origin. Then ∂K is locally a graph in these coordinates,

$$x_{n+1} = -\frac{1}{2}\sum_{i,j=1}^{n} h_{ij}x_i x_j + o(|x|^2),$$

where h_{ij} is the second fundamental form of ∂K at the supporting point and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^{n+1}$.

Moreover, there exists a volume preserving linear transformation that fixes u and brings ∂K locally to the form

$$x_{n+1} = -\frac{1}{2}\mathcal{K}^{1/n}(q)\sum_{i=1}^{n} x_i^2 + o(|x|^2),$$

where $\mathcal{K}(q) = \det[(h_{ij})_{ij}](u) = f_K^{-1}(u)$ is the Gauss curvature of ∂K at the point q.

Following the line of direction u for distance $d' \leq d$ from q, one reaches the point $x \in K^\delta$ for which the cone of light $\operatorname{co}[x, K] \setminus K$ has volume δ. In particular, we will have $d' = d$ if and only if the hyperplane supporting K^δ at x has normal u.

Moreover, one has a description of the cone's volume as

$$\operatorname{Vol}\big(\operatorname{co}[x, K] \setminus K\big) = \frac{(2d')^{\frac{n}{2}+1}}{\mathcal{K}^{\frac{1}{2}}}\frac{\sigma_n}{(n+1)(n+2)} + o\big(d'^{\frac{n+2}{2}}\big),$$

where σ_n is the volume of the unit ball in $\mathbb{R}^n$.

Recalling that the above volume is equal to δ, we obtain

$$d' = \frac{1}{2}\Big[\frac{(n+1)(n+2)}{\sigma_n}\Big]^{\frac{2}{n+2}}\mathcal{K}^{\frac{1}{n+2}}\delta^{\frac{2}{n+2}} + o(\delta^{\frac{2}{n+2}}),$$

which concludes the proof of the lemma. ∎

We recall now the definition of the convex floating body belonging to Schütt and Werner [30].

Definition 2.3 Suppose that K is a convex body in $\mathbb{R}^{n+1}$ with support function $h_K\colon \mathbb{S}^n \to \mathbb{R}$. From the convexity of K, we have that for each unitary direction $u \in \mathbb{S}^n$, there exists a unique hyperplane of normal u supporting the boundary of K, $H_u = \{y \in \mathbb{R}^{n+1} \mid u \cdot y = h_K(u)\}$.

If $H_{u,\delta} = \{y \in \mathbb{R}^{n+1} \mid u \cdot y = h_{K_\delta}(u)\}$ denotes the hyperplane parallel to H_u such that the $(n+1)$-dimensional volume of the cap cut from K by $H_{\mathbf{u},\delta}$ is precisely δ,

$$\mathrm{Vol}(\{y \in K \mid h_{K_\delta}(u) \le u \cdot y \le h_K(u)\}) = \delta,$$

for some positive $\delta < \mathrm{Vol}(K)/2$, then

$$K_\delta = \bigcap_{u \in \mathbb{S}^n} \{y \in \mathbb{R}^{n+1} \mid u \cdot y \le h_{K_\delta}(u)\}$$

is said to be the *convex floating body* of K of factor δ.

The convex floating body of a convex body always exists as long as $\delta \le \mathrm{Vol}(K)/2$, reducing to a point in the upper limiting case.

An equivalent formula to (2.2) was shown by Schütt and Werner [30]; namely, for any convex body $K \subset \mathbb{R}^{n+1}$ one has

$$\lim_{\delta \to 0} \frac{\mathrm{Vol}(K) - \mathrm{Vol}(K_\delta)}{\delta^{\frac{2}{n+2}}} = d_n \, \Omega(K), \tag{2.4}$$

where $d_n = \frac{1}{2}\left(\frac{n+2}{\sigma_n}\right)^{\frac{2}{n+2}}$ once again depends only on the dimension.

Convex floating bodies have been the object of an earlier paper whose main theorem, conjectured by Schütt and Werner in [30] for arbitrary convex bodies, is the following.

Theorem 2.4 [31] *Let $K \subset \mathbf{R}^{n+1}$ be a convex body with boundary of class $C^{\ge 4}$. There exists a positive number $\delta(K)$ such that K_δ is homothetic to K with respect to the same center of homothety, for some $\delta < \delta(K)$, if and only if K is an ellipsoid.*

In a manner analogous with the calculations of Lemma 2.2, we have the following asymptotic expansion of the support function of a convex floating body K_δ in terms of the support function of the original body K.

Lemma 2.5 [31] *Let $K \subset \mathbb{R}^{n+1}$ be a convex body of class C^2_+. There exists a positive number δ_K such that for any $\delta \in (0, \delta_K)$*

$$h_{K_\delta}(u) = h_K(u) - \delta^{\frac{2}{n+2}} d_n f_K^{-\frac{1}{n+2}}(u) + o(\delta^{\frac{2}{n+2}}), \ \forall u \in \mathbb{S}^n, \tag{2.5}$$

where d_n is as above.

To conclude the preliminaries, we recall an existing characterization of ellipsoids which in many instances, including the present paper, is used in technical arguments.

***Lemma 2.6* (Petty's Lemma [22])** *Let $K \subset \mathbb{R}^{n+1}$ be a convex body with boundary of class C^2 for which h_K and f_K denote the support function and the curvature function, respectively, as functions on the unit sphere $\mathbb{S}^n$. If there exists a non-zero constant c such that for all $u \in \mathbb{S}^n$, $h_K(u)\, f_K^{1/(n+2)}(u) = c$, then K is an ellipsoid.*

3 Results

Theorem 3.1 *Let $K \subset \mathbb{R}^{n+1}$ be a convex body of class C^2_+ containing the origin in its interior. There exists a positive constant $\delta(K)$ such that for any $\delta \in (0, \delta(K))$,*

$$\mathrm{Vol}(K_\delta) \cdot \mathrm{Vol}((K_\delta)^*) \geq \mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*) \geq \mathrm{Vol}(K^\delta) \cdot \mathrm{Vol}((K^\delta)^*).$$

Furthermore, each equality is reached if and only if K is an ellipsoid.

Proof For the first inequality, we start with the asymptotic description of the support function of the floating body (2.5) as $h_t(u) = h(u) - d_n f_K^{-1/(n+2)}(u)t + o(t)$, where t stands for $\delta^{\frac{2}{n+2}}$. To simplify the calculations, on what follows, we will often consider t instead of $\delta^{\frac{2}{n+2}}$ and, based on this correspondence between t and δ, we will also use the notations h_t, h^t for the support function h_{K_δ}, h^{K^δ} respectively.

From here, we may estimate the volume of the dual polar of the floating body

$$\begin{aligned}\mathrm{Vol}((K_t)^*) &= \frac{1}{n+1}\int_{\mathbb{S}^n} h_t^{-(n+1)}\, d\mu_{\mathbb{S}^n}\\ &= \mathrm{Vol}(K^*) + t\, d_n \int_{\mathbb{S}^n} h^{-(n+2)} f_K^{-\frac{1}{n+2}}\, d\mu_{\mathbb{S}^n} + o(t).\end{aligned}$$

On the other hand, due to (2.4), one also has $\mathrm{Vol}(K_t) = \mathrm{Vol}(K) - d_n\, \Omega(K)\, t + o(t)$. Consequently,

$$\begin{aligned}&\frac{1}{d_n}\cdot \lim_{t \searrow 0} \frac{\mathrm{Vol}(K_t)\cdot \mathrm{Vol}((K_t)^*) - \mathrm{Vol}(K)\cdot\mathrm{Vol}(K^*)}{t}\\ &\quad= \mathrm{Vol}(K)\cdot \int_{\mathbb{S}^n} h^{-(n+2)} f_K^{-\frac{1}{n+2}}\, d\mu_{\mathbb{S}^n} - \Omega(K)\cdot \mathrm{Vol}(K^*)\\ &\quad= \mathrm{Vol}(K^*)\cdot \int_{\mathbb{S}^n} h^{-(n+2)} f_K^{-\frac{1}{n+2}}\, d\mu_{\mathbb{S}^n}\\ &\qquad\times \left[\frac{\int_{\mathbb{S}^n} h f_K\, d\mu_{\mathbb{S}^n}}{\int_{\mathbb{S}^n} h^{-(n+1)}\, d\mu_{\mathbb{S}^n}} - \frac{\int_{\mathbb{S}^n} f_K^{\frac{n+1}{n+2}}\, d\mu_{\mathbb{S}^n}}{\int_{\mathbb{S}^n} h^{-(n+2)} f_K^{-\frac{1}{n+2}}\, d\mu_{\mathbb{S}^n}}\right].\end{aligned}$$

Recall now a generalized Hölder inequality due to Andrews, which we will use to conclude that the above limit is non-negative. If M is a compact manifold with a volume form $d\omega$, g is a continuous function on M and F is a decreasing real, positive function, then

$$\frac{\int_M gF(g)\, d\omega}{\int_M F(g)\, d\omega} \leq \frac{\int_M g\, d\omega}{\int_M d\omega}. \tag{3.1}$$

If F is strictly decreasing, then equality occurs if and only if g is constant, see [1, Lemma I3.3]. Similarly, if F is an increasing real, positive function, the conclusion

https://doi.org/10.4153/CMB-2009-049-0 Published online by Cambridge University Press

holds with $\geq$ in (3.1) and, similarly, if F is strictly increasing, then equality occurs if and only if g is a constant function.

Taking $g = h^{n+2} f_K$, $F(x) = x^{-1/(n+2)}$, $x > 0$, and $d\omega = h^{-(n+1)}\, d\mu_{\mathbb{S}_n}$ in (3.1), one deduces that $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*)$ increases as we *pass* to convex floating bodies, unless $g =$ constant.

Similarly, one uses (2.2) and (2.3) to obtain

$$\mathrm{Vol}(K^t) = \mathrm{Vol}(K) + c_n\, \Omega(K)\, t + o(t),$$

respectively,

$$\begin{aligned}
\mathrm{Vol}((K^t)^*) &= \frac{1}{n+1} \int_{\mathbb{S}^n} (h^t)^{-(n+1)}\, d\mu_{\mathbb{S}^n} \\
&\leq \frac{1}{n+1} \int_{\mathbb{S}^n} \Big(h + t c_n f_K^{-\frac{1}{n+2}} + o(t) \Big)^{-(n+1)}\, d\mu_{\mathbb{S}^n} \\
&\leq \mathrm{Vol}(K^*) - t\, c_n \int_{\mathbb{S}^n} h^{-(n+2)} f_K^{-\frac{1}{n+2}}\, d\mu_{\mathbb{S}^n} + o(t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{c_n} \cdot \lim_{t \searrow 0} \frac{\mathrm{Vol}(K^t) \cdot \mathrm{Vol}((K^t)^*) - \mathrm{Vol}(K) \cdot \mathrm{Vol}(K)}{t} \\
&\qquad\qquad \leq -\mathrm{Vol}(K) \cdot \int_{\mathbb{S}^n} h^{-(n+2)} f_K^{-\frac{1}{n+2}}\, d\mu_{\mathbb{S}^n} + \Omega(K) \cdot \mathrm{Vol}(K^*),
\end{aligned}$$

concluding the proof of the second inequality by the same argument as before.

Note that in each case, equality occurs when $h^{n+2} f_K$ is constant in all directions, hence, by Petty's lemma, when K is an ellipsoid. (In that case, equality is reached in (2.3) in all unitary directions as well.) ■

As a corollary, we validate Schütt-Werner conjecture for convex bodies of class C^2_+.

Corollary 3.2 *Let $K \subset \mathbf{R}^{n+1}$ be a convex body of class C^2_+. There exists a positive number $\delta(K)$ such that K_δ is homothetic to K with respect to the same center of homothety, for some $\delta < \delta(K)$, if and only if K is an ellipsoid.*

Proof If K is an ellipsoid, the implication is trivial. In the other direction, the proof is immediate too as K homothetic to K_δ with respect to the same center of homothety implies the equality $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^*) = \mathrm{Vol}(K_\delta) \cdot \mathrm{Vol}((K_\delta)^*)$. ■

The proof of Theorem 2.4 for convex bodies K of class C^4_+ can be slightly adjusted to extend it to the class C^4. However, in Corollary 3.2 the assumption of an everywhere positive Gauss curvature cannot be dropped.

Similarly, we have a second characterization of ellipsoids.

Corollary 3.3 *Let $K \subset \mathbf{R}^{n+1}$ be a convex body of class C^2_+. There exists a positive number $\delta(K)$ such that K^δ is homothetic to K with respect to the same center of homothety, for some $\delta < \delta(K)$ if and only if K is an ellipsoid.*

Moreover, in connection to (1.2), the right inequality of Theorem 3.1 implies the following two consequences.

Corollary 3.4 **(On Mahler's conjecture)** *Denote by* $K^{s(K)}$ *the polar of a convex body* $K \subset \mathbb{R}^{n+1}$ *with respect to its Santaló point and by* $K^{c(K)}$ *the polar of a convex body* $K \subset \mathbb{R}^{n+1}$ *with respect to its centroid.*

(i) *If* K *is centrally symmetric and is of class* C^2_+, *then* $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^{s(K)})$ *(also equal to* $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^{c(K)}))$*) is not minimal among volume products of centrally symmetric convex bodies.*

(ii) $\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^{s(K)})$ *is not minimal among volume products of convex bodies in* $\mathbb{R}^{n+1}$ *if* $K \in C^2_+$.

Proof (i) is immediate, while for (ii) note that

$$\begin{aligned}\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^{s(K)}) &\geq \mathrm{Vol}(K^{\delta}) \cdot \mathrm{Vol}((K^{\delta})^{s(K)}) \\ &\geq \mathrm{Vol}(K^{\delta}) \cdot \mathrm{Vol}((K^{\delta})^{s(K^{\delta})}),\end{aligned}$$

where the first inequality is strict if K is not an ellipsoid. ■

We are inclined to believe that the minimum of the volume product functional $K \mapsto \mathrm{Vol}(K) \cdot \mathrm{Vol}(K^c)$ does not occur for a centrally symmetric convex body.

In a separate paper, we will introduce weighted illumination bodies which are used to give a new geometric interpretation of the p-affine surface area. Analogous to Theorem 3.1, we derive two inequalities for the volume product for weighted illumination bodies and weighted convex floating bodies, the latter being defined by Werner in [34].

Acknowledgement The author is indebted to the referee for comments and suggestions which improved the original manuscript.

References

[1] B. Andrews, *Evolving convex curves.* Calc. Var. Partial Differential Equations **7**(1998), 315–371.

[2] I. Bárány and R. A. Vitale, *Random convex hulls: floating bodies and expectations.* J. Approx. Theory **75**(1993), 130–135.

[3] C. Buchta and M. Reitzner, *Equiaffine inner parallel curves of a plane convex body and the convex hulls of randomly chosen points.* Probab. Theory Related Fields **108**(1997), 385–415.

[4] W. Blaschke, *Über Affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid.* Leipziger Ber. **69**(1917), 306—318.

[5] ———, *Vorlesungen über Differentialgeometrie II.* Springer-Verlag, Berlin, 1923.

[6] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$. Invent. Math. **88**(1987), 319–340.

[7] Y. Gordon, M. Meyer, and S. Reisner, *Zonoids with minimal volume-product — a new proof.* Proc. Amer. Math. Soc. **104**(1988), 273–276.

[8] D. Hug, *Contributions to affine surface area.* Manuscripta Math. **91**(1996), 283–301.

[9] G. Kuperberg, *A low-technology estimate in convex geometry.* Internat. Math. Res. Notices **9**(1992), 181–183.

[10] ———, *From the Mahler conjecture to Gauss linking integrals.* Geom. Funct. Anal. **18**(2008), 870–892.

[11] K. Leichtweiss, *Über ein Formel Blaschkes zur Affinoberfläche.* Studia Sci. Math. Hungar. **21**(1986), 453–474.

[12] M. A. Lopez and S. Reisner, *A special case of Mahler's conjecture.* Discrete Comput. Geom. **20**(1998), 163–177.
[13] M. Ludwig and M. Reitzner, *A characterization of affine surface area.* Adv. Math. **147**(1999), 138–172.
[14] E. Lutwak, *Selected affine isoperimetric inequalities.* In: Handbook of Convex Geometry, North-Holland, Amsterdam, 1993, pp. 151–176.
[15] ———, *Extended affine surface area.* Adv. Math. **85**(1991), 39–68.
[16] ———, *On the Blaschke-Santaló inequality.* Ann. N.Y. Acad. Sci. **440**(1985), 106–112.
[17] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities.* J. Differential Geom. **47**(1997), 1–16.
[18] M. Meyer and E. Werner, *The Santaló regions of a convex body.* Trans. Amer. Math. Soc. **350**(1998), 4569–4591.
[19] M. Meyer and A. Pajor, *On Santaló's inequality.* In: Geometric Aspects of Functional Analysis, Lecture Notes in Math. 1476, Springer, Berlin, 1989, pp. 261–263.
[20] M. Meyer and S. Reisner, *Characterization of ellipsoids by section-centroid location.* Geom. Dedicata **31**(1989), 345–355.
[21] ———, *A geometric property of the boundary of symmetric convex bodies and convexity of flotation surfaces.* Geom. Dedicata **37**(1991), 327–337.
[22] C. M. Petty, *Affine isoperimetric problems.* Ann. New York Acad. Sci. **440**(1985), 113–127.
[23] J. Saint Raymond, *Sur le volume des corps convexes symétriques.* Initiation à l'analyse, Exp. No. 11, Publ. Math. Univ. Pierre et Marie Curie 46, Univ. Paris VI, Paris, 1981.
[24] L. A. Santaló, *Un invariante affine para los cuerpos convexos del espacio de n dimensiones.* Portugaliae Math. **8**(1949), 155–161.
[25] M. Schmuckenschläger, *The distribution function of the convolution square of a convex symmetric body in* $\boldsymbol{R}^n$. Israel J. Math. **78**(1992), 309–334.
[26] R. Schneider, *Convex bodies: The Brunn-Minkowski theory.* Encyclopedia of Mathematics and Its Applications 44, Cambridge University Press, Cambridge, 1993.
[27] C. Schütt, *The convex floating body and polyhedral approximation.* Israel J. Math. **73**(1991), 65–77.
[28] ———, *On the affine surface area.* Proc. Amer. Math. Soc. **118**(1993), 1213–1218.
[29] ———, *Floating body, illumination body, and polytopal approximation.* In: Convex Geometric Analysis, Math. Sci. Res. Inst. Publ. 34. Cambridge University Press, Cambridge, 1999, pp. 203–229.
[30] C. Schütt and E. Werner, *The convex floating body.* Math. Scand. **66**(1990), 275–290.
[31] A. Stancu, *The floating body problem.* Bull. London Math. Soc. **38**(2006), 839–846.
[32] A. Stancu and E. Werner, *New higher-order equiaffine invariants.* To appear in Israel J. Math.
[33] E. Werner, *Illumination bodies and affine surface area.* Studia Math. **110**(1994), 257–269.
[34] ———, *The p-affine surface area and geometric interpretations.* Rend. Circ. Mat. Palermo **70**(2002), part II, 367–382.

Department of Mathematics and Statistics, Concordia University, Montréal, QC H3G 1M8
e-mail: stancu@mathstat.concordia.ca