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THE TRANSFER OF A COMMUTATOR LAW FROM A NIL-RING TO ITS ADJOINT GROUP

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ABSTRACT. For every field F of characteristic $p \geq 0$, we construct an example of a finite dimensional nilpotent F-algebra R whose adjoint group $A(R)$ is not centre-by-metabelian, in spite of the fact that R is Lie centre-by-metabelian and satisfies the identities $x^{2p} = 0$ when $p > 2$ and $x^8 = 0$ when $p = 2$. The existence of such algebras answers a question raised by A. E. Zalesskii, and is in contrast to positive results obtained by Krasilnikov, Sharma and Srivastava for Lie metabelian rings and by Smirnov for the class Lie centre-by-metabelian nil-algebras of exponent 4 over a field of characteristic 2 of cardinality at least 4.

1. **Introduction.** This paper is concerned with the question of how Lie identities in a nil-ring, R, influence the commutator laws in its associated adjoint (or circle) group, $A(R)$. The multiplication in $A(R)$ is given by $x \circ y = x + y + xy$. Laws which *transfer* are particularly interesting: a commutator law is said to transfer if its existence in a nil-algebra R always implies the existence of the corresponding group-commutator law in $A(R)$. These ideas extend in the obvious way to unitary rings R and their group of units, $U(R)$. We say 'extend' here because whenever a commutator law transfers in the unitary sense, it also transfers in the adjoint sense. Indeed, this follows from the fact that any nil-ring R can be embedded in a unitary ring R_1 satisfying the same Lie identities and, under this embedding, $A(R)$ is contained in $U(R_1)$.

This sort of phenomenon has been discussed by many authors. Let us now mention the results most relevant to our present purposes. First, Gupta and Levin proved in [GL] that the unit group of a Lie nilpotent ring is nilpotent of at most the same class. So, in our terminology, the nilpotent law (of a specific class) transfers. Second, Krasilnikov [K] and Sharma and Srivastava [SS] independently proved that the metabelian law transfers. In characteristic zero, Smirnov and Zalesskii ([SmZ]) proved, more generally, that solubility of any given derived length transfers. This last problem remains open in other characteristics, although Smirnov was able to show in [Sm1] that the group of units must be soluble of some bounded derived length (whenever R is necessarily without 2-torsion). Finally, Smirnov proved in [Sm2] that if R is a nil-algebra of exponent 4 over a field F of characteristic 2 with $|F| > 2$, then the centre-by-metabelian law transfers from R to $A(R)$.

In this note, we shall examine further the centre-by-metabelian law in this context. Looking at the metabelian case, it would seem tempting to ask (*cf.* [SS]) whether the

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centre-by-metabelian law transfers. The answer to this question is certainly negative in characteristic 2. Indeed, it is well-known that the full ring, $M_2(F)$, of 2×2 matrices over a field F of characteristic 2 is Lie centre-by-metabelian, and yet its unit group $\mathrm{GL}_2(F)$ contains a non-abelian free group whenever F contains a transcendental element. Consequently, the units satisfy no identity whatsoever. However, a related problem, proposed by Zalesskii (private communication, 1991), is rather more interesting:

PROBLEM 1. *Is the unit group of a Lie centre-by-metabelian, Lie nilpotent algebra always centre-by-metabelian?*

This problem, too, is now known to have negative solutions—at least in the cases of characteristic 2 and 0. Because of the special way that the even characteristics were used in the construction of these counterexamples (*cf.* [T1] and [T2]), a more complete answer is not clear. However, as we shall demonstrate below, Problem 1 fails in each characteristic $p \geq 0$. In light of this fact and Smirnov's aforementioned positive result for certain nil-algebras in characteristic 2, it makes sense to consider then the following weakened version of Problem 1:

PROBLEM 2. *Is the adjoint group of a Lie centre-by-metabelian, (associatively) nilpotent algebra always centre-by-metabelian?*

It transpires that the answer is still 'no'.

THEOREM. *For each field F of characteristic $p \geq 0$, there exists a finite dimensional nilpotent F-algebra that is Lie centre-by-metabelian but fails to have centre-by-metabelian adjoint group. Furthermore, this algebra can be taken to satisfy $x^{2p} = 0$ whenever $p > 2$ and $x^8 = 0$ when $p = 2$.*

This fully answers Problem 2, and hence also completes the answer to Problem 1. It also shows that Smirnov's positive result for nil-algebras of exponent 4 in characteristic 2 does not generalise in any obvious fashion.

2. **The tensor square of the Grassmann algebra.**

Let F be a field of characteristic $p \neq 2$. Consider the Grassmann (or exterior) algebra, E, on a countably-infinite-dimensional F-space with basis $\{v_1, v_2, \ldots\}$. A basis for E is given by the set

$$\{v_{i_1} v_{i_2} \cdots v_{i_n} \mid i_1 < \cdots < i_n\}.$$

Denote by $\overline{E}$ the central algebra extension of E by the empty monomial. Then E is a nil ideal of the unitary F-algebra $\overline{E}$.

Let E_0 be the subspace of E of those monomials of even weight in the v_is, and let E_1 be the subspace of those monomials of odd weight. Then $E = E_0 \oplus E_1$ is a superalgebra. In other words, $E_i E_j \subseteq E_{i+j}$ where the indices are added modulo 2. Since $(v_i v_j) v_k = v_k (v_i v_j)$, it follows that E_0 is central in E. Also, for every x_1, y_1 in E_1, we have $x_1 y_1 = -y_1 x_1$. Consequently, we may recover the well-known fact that E (and hence $\overline{E}$) is Lie nilpotent of class 2:

$$[E, E, E] = [E_1, E_1, E_1] \subseteq [E_0, E_1] = 0.$$

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The following lemma is a special case of Proposition 2.3 in [R]. It can also be checked by direct calculation.

LEMMA 2.1. *The tensor product of any two Lie nilpotent algebras of class at most 2 is Lie centre-by-metabelian.*

LEMMA 2.2. *Suppose that $p > 2$ and set $R = E \otimes E$. Then R is bounded nil of index at most $2p$.*

PROOF. First observe that because E is Lie nilpotent of class 2 and generated by elements v_i with $v_i^2 = 0$, E is bounded nil of index p. Indeed, the square of any monomial is zero, and it is well-known that the identity $(x+y)^p = x^p + y^p$ holds whenever an algebra is Lie nilpotent of class at most p. Next, let $x_{i,j}$ and $y_{i,j}$ represent elements of $E_i \otimes E_j$ for $i, j \in \{0, 1\}$. Then these elements satisfy the following properties, whose verification is left to the reader.

1. $x_{0,0}$ is central in R.
2. $x_{0,1}y_{0,1} = -y_{0,1}x_{0,1}$, $x_{1,0}y_{1,0} = -y_{1,0}x_{1,0}$ and $x_{1,1}y_{1,1} = y_{1,1}x_{1,1}$.
3. $x_{0,1}x_{1,0} = x_{1,0}x_{0,1}$, $x_{0,1}x_{1,1} = -x_{1,1}x_{0,1}$ and $x_{1,0}x_{1,1} = -x_{1,1}x_{1,0}$.
4. $x_{0,1}^2 = x_{1,0}^2 = 0$ and $x_{0,0}^p = x_{1,1}^p = 0$.

Now let $x = x_{0,0} + x_{0,1} + x_{1,0} + x_{1,1}$ be an arbitrary element in R. Then there exist scalars α, β such that:

$$\begin{aligned} x^p &= x_{0,0}^p + (x_{0,1} + x_{1,0} + x_{1,1})^p \\ &= (x_{0,1} + x_{1,0})^p + x_{1,1}^p + \alpha(x_{0,1} + x_{1,0})x_{1,1}^{p-1} + \beta x_{0,1}x_{1,0}x_{1,1}^{p-2} \\ &= \alpha(x_{0,1} + x_{1,0})x_{1,1}^{p-1} + \beta x_{0,1}x_{1,0}x_{1,1}^{p-2}. \end{aligned}$$

It follows that $x^{2p} = (x^p)^2 = 0$. ∎

LEMMA 2.3. *Set $\bar{R} = \bar{E} \otimes \bar{E}$. Then $\bar{R}$ is a Lie centre-by-metabelian unitary F-algebra, but its group of units $U(\bar{R})$ is not centre-by-metabelian.*

PROOF. The first claim follows from Lemma 2.1 and the discussion preceding it. To prove the second claim, let $u_1, \ldots, u_6$ and $w_1, \ldots, w_6$ be 12 distinct elements in the generating set $\{v_1, v_2, \ldots\}$ and assign:

1. $\alpha = u_1u_2 \otimes u_3 + u_4 \otimes u_4$;
2. $\beta = u_5 \otimes u_1u_2 + u_6 \otimes u_6$;
3. $\gamma = w_1w_2 \otimes w_3 + w_4 \otimes w_4$; and
4. $\delta = w_5 \otimes w_1w_2 + w_6 \otimes w_6$.

One may verify that

$$\alpha^2 = \beta^2 = 0$$

and

$$[\alpha, \beta] = 2u_1u_2u_6 \otimes u_3u_6 + 2u_4u_5 \otimes u_1u_2u_4.$$

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Using these relations we discover that

$$\begin{aligned}(1-\alpha,1-\beta)-1 &= (1+\alpha)(1+\beta)[\alpha,\beta]\\ &= (2u_1u_2u_5u_6\otimes u_1u_2u_3u_6+2u_1u_2u_4u_5\otimes u_1u_2u_3u_4)\\ &\quad +(2u_4u_5\otimes u_1u_2u_4-2u_1u_2u_4u_6\otimes u_3u_4u_6)\\ &\quad +(2u_1u_2u_6\otimes u_3u_6-2u_4u_5u_6\otimes u_1u_2u_4u_6)\\ &\quad +(4u_1u_2u_4u_5u_6\otimes u_1u_2u_3u_4u_6).\end{aligned}$$

The four indicated partial sums are contained in $E_0\otimes E_0$, $E_0\otimes E_1$, $E_1\otimes E_0$ and $E_1\otimes E_1$, respectively.

Now put

1. $x=1-(1-\alpha,1-\beta)$; and
2. $y=1-(1-\gamma,1-\delta)$.

If U were indeed centre-by-metabelian, then

$$z=(1-x,1-y)=1+(1+x+x^2+\cdots)(1+y+y^2+\cdots)[x,y]$$

would be central in U. We intend to show that this is not the case. To see why, consider the component of z lying in $E_1\otimes E_0$:

$$\begin{aligned}&16w_4w_5u_1u_2u_4u_5u_6\otimes w_1w_2w_4u_1u_2u_3u_4u_6\\ &\qquad -16u_4u_5w_1w_2w_4w_5w_6\otimes u_1u_2u_4w_1w_2w_3w_4w_6\\ &\qquad +\text{monomials of higher degree.}\end{aligned}$$

Because this component is not central in U (it does not commute with $1+u_1\otimes u_3$, for example), z itself cannot be central. ∎

Finally, to prove the Theorem in characteristic $p\neq 2$, it suffices to observe that the construction in the proof of Lemma 2.3 required only 12 generators and that the nilpotent subalgebra of R generated by these elements is $(2^{12}-1)^2$-dimensional. The case $p=2$ follows from a closer examination of the example constructed in [T1].

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