BIRTH–DEATH CHAINS AND THE LOCAL TIME OF BROWNIAN MOTION

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Abstract

A connection between Brownian motion and birth–death chains is explored. Several results concerning birth–death chains are shown to be consequences of well-known results on Brownian motion.

1. Introduction

Let X_m be a Markov chain taking values on the nonnegative integers with the following transition probabilities for $n \neq 0$:

$$p_{nj} = \begin{cases} r_n & \text{if } j = n + 1, \\ l_n & \text{if } j = n - 1, \\ 0 & \text{if } |n - j| \neq 1, \end{cases}$$

with $l_n + r_n = 1$. We refer to such a Markov chain as a *birth–death chain*. (One may also find the term *state-dependent random walk* in the literature.) We set $p_{00} = 1$ and $p_{0j} = 0$ for any $j \neq 0$. Let k be the starting point of the birth–death chain, so $X_0 = k$ almost surely. Let Δ denote the extinction time of X_m. That is, $\Delta = \inf_{m>0}\{X_m = 0\}$, with $\Delta = \infty$ if X_m never reaches 0. In [5], a technique was given for realizing X_m as a Brownian motion stopped at a certain sequence $\tau(m)$ of stopping times. This idea was used to prove the following two results, the first of which is well known.

THEOREM 1.1. *Let $P_k = P(\Delta < \infty)$, the extinction probability of X_m. Then*

$$P_k = \frac{\sum_{j=k}^{\infty} t_j}{\sum_{j=0}^{\infty} t_j}$$

where this quotient is interpreted as being equal to 1 *if the sums diverge.*

THEOREM 1.2. *If $t_\infty := \lim_{n\to\infty}((l_1\cdots l_n)/(r_1\cdots r_n))$ exists, then $\lim_{m\to\infty} E[X_m]$ exists, and*

$$\lim_{m\to\infty} E[X_m] = \frac{1 + \frac{l_1}{r_1} + \cdots + \frac{l_1\cdots l_{k-1}}{r_1\cdots r_{k-1}}}{t_\infty}.$$

This result remains valid when $t_\infty = 0$ or ∞, with the quotient being interpreted as equal to 0 when $t_\infty = \infty$ and equal to ∞ when $t_\infty = 0$.

The purpose of this paper is to continue with this method by using known properties of Brownian motion to deduce several results concerning birth–death chains. The results presented can all be proved by more standard methods, but they will be seen to flow naturally from standard properties of Brownian motion, most notably the Tanaka and Itō–Tanaka formulas. In addition, the proofs given are quite short, although some time must be spent discussing the setup and relevant properties of Brownian motion. Let us begin by briefly recounting the construction given in [5]. Set

$$t_n := \frac{l_1 l_2 \cdots l_n}{r_1 r_2 \cdots r_n}$$

and $x_n = \sum_{j=0}^{n-1} t_j$, where for $n = 0$ these definitions are interpreted as $t_0 = 1$, $x_0 = 0$. Since the sequence $\{x_n\}_{n=0}^\infty$ is increasing it converges to a limit x_∞, possibly infinite, as $n \to \infty$. Let B_t be a Brownian motion stopped at the first time $\tau(\Delta)$ it hits 0 or x_∞. Unless otherwise noted, the initial starting point of B_t will be taken to be the point $x_k \in \mathcal{A}$, although we will use the standard notation E_z to denote expectation conditioned on $B_0 = z$. The recurrence properties of Brownian motion imply that $\tau(\Delta) < \infty$ almost surely. Set $\mathcal{A} = \bigcup_{n=0}^\infty \{x_n\}$. We define the stopping times $\tau(m)$ recursively by setting $\tau(0) = 0$, and having defined $\tau(m)$ we let

$$\tau(m + 1) = \inf_{t>\tau(m)} \{B_t \in \mathcal{A}, \mathcal{B}_\sqcup \neq \mathcal{B}_{\tau(\mathbb{0})}\}.$$

That is, the $\tau(m)$s are the the successive hitting times of points in $\mathcal{A}$. We see that the variables $B_{\tau(0)}, B_{\tau(1)}, B_{\tau(2)}, \ldots$ form a random process taking values in $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathbb{R}$ be defined by $\phi(x_n) = n$. It is shown in [5] that the strong Markov property of Brownian motion and the formula for the exit distribution of Brownian motion from an interval imply that $\phi(B_{\tau(0)}), \phi(B_{\tau(1)}), \phi(B_{\tau(2)}), \ldots$ is a realization of our birth–death chain. We therefore take $X_m = \phi(B_{\tau(m)})$ in what follows. An important quantity for us will be the *local time* of Brownian motion, which is the density of the occupation measure of Brownian motion with respect to Lebesgue measure. That is, the local time L_t^x satisfies

$$L_t^x \, dx = \int_0^t 1_{B_s \in dx} \, ds.$$

It is well known that L_t^x exists and that

$$L_t^x = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{|B_s - x| < \varepsilon} \, ds.$$

almost surely. The local time provides a measure of the amount of time that Brownian motion spends at a point. We will see that it can also be used to shed light upon the number of visits that a birth–death chain makes to a point, as well as the long-term behaviour of the chain.

The outline of the paper is as follows. We will begin by showing how Brownian local time can be used to give a quick derivation of the expected exit time of the birth–death chain from an interval. We will then prove an extension of Theorem 1.2 which relates to large stopping times. Some examples will be given of birth–death chains to which this result applies. We will conclude by showing that the Doob decomposition of the chain follows easily from the Itō–Tanaka formula for Brownian motion.

2. The expected exit time from an interval

For $0 \leq a < k < b$ let $\Gamma(a, b) = \inf_{m>0}\{X_m = a \text{ or } b\}$. We will calculate $E[\Gamma(a, b)]$. There is no loss of generality in assuming that the lower limit of our interval is 0. The following result has probably been noted before, although this proof is likely to be new.

THEOREM 2.1. *Suppose that* $0 < k < M$. *Then*

$$E[\Gamma(0, M)] = \frac{1}{x_M} \sum_{n=1}^{M-1} \frac{x_M \min(x_k, x_n) - x_n x_k}{r_n t_n}. \tag{2.1}$$

REMARK 2.2. If $r_n = l_n = 1/2$ for all n, X_m is simple random walk, and (2.1) is readily seen to reduce to $k(M - k)$. This agrees with the formula obtained by applying the optional stopping theorem to the martingale $X_m^2 - m$.

PROOF. Set G_m^n to be the number of times that X is equal to n on or before time $m - 1$. It is clear that

$$E[\Gamma(0, M)] = \sum_{n=1}^{M-1} E[G_{\Gamma(0,M)}^n].$$

We will use Brownian local time in order to determine $E[G_{\Gamma(0,M)}^n]$. The following lemma is the key calculation.

LEMMA 2.3. *Suppose that* $a < y, z < b$. *Let* $\gamma(a, b) = \inf_{t>0}\{B_t = a \text{ or } b\}$. *Then*

$$E_z[L_{\gamma(a,b)}^y] = \frac{2(b - a) \min(y - a, z - a) - 2(y - a)(z - a)}{b - a}. \tag{2.2}$$

PROOF. As the right-hand side of (2.2) is unchanged when $\{a, y, z, b\}$ is replaced by $\{0, y - a, z - a, b - a\}$, it is enough to assume that $a = 0$. Tanaka's formula (see [3] or [4]) states that

$$|B_t - y| = |B_0 - y| + \int_0^t \text{sgn}(B_s - y) \, dB_s + L_t^y.$$

The stochastic integral here is a martingale, and the optional stopping theorem can be applied. We find that

$$E_z[L_{\gamma(0,b)}^y] = E_z|B_{\gamma(0,b)} - y| - |y - z|.$$

Since $P(B_{\gamma(0,b)} = 0) = (b - z)/b$ and $P(B_{\gamma(0,b)} = b) = z/b$, we see that $E_z|B_{\gamma(0,b)} - y| = ((b - z)/b)y + (z/b)(b - y)$. Thus

$$\begin{aligned}
E_z[L_{\gamma(0,b)}^y] &= \frac{(b - z)y + z(b - y) - b|y - z|}{b} \\
&= \frac{b(y + z) - b|y - z| - 2yz}{b} = \frac{2b\min(y, z) - 2yz}{b}.
\end{aligned}$$

This proves the lemma. □

We apply this lemma to our Brownian motion starting at x_k in order to calculate $E[G_{\Gamma(0,M)}^n]$. The total expected local time accumulated at x_n before hitting 0 or x_M is given by $E_{x_k}[L_{\gamma(0,x_M)}^{x_n}]$. However, if we condition on $B_{\tau(m)} = x_n$ we see that the expected local time accumulated at x_n between times $\tau(m)$ and $\tau(m + 1)$ is $E_{x_n}[L_{\gamma(x_{n-1},x_{n+1})}^{x_n}]$. We therefore obtain

$$E[G_{\Gamma(0,M)}^n] = \frac{E_{x_k}[L_{\gamma(0,x_M)}^{x_n}]}{E_{x_n}[L_{\gamma(x_{n-1},x_{n+1})}^{x_n}]}. \tag{2.3}$$

We need therefore only apply Lemma 2.3 and simplify. We have

$$\begin{aligned}
E_{x_n}[L_{\gamma(x_{n-1},x_{n+1})}^{x_n}] &= \frac{2(x_{n+1} - x_{n-1})(x_n - x_{n-1}) - 2(x_n - x_{n-1})^2}{x_{n+1} - x_{n-1}} \\
&= \frac{2(x_{n+1} - x_n)(x_n - x_{n-1})}{x_{n+1} - x_{n-1}} \\
&= \frac{2t_n t_{n-1}}{t_n + t_{n-1}} = \frac{2t_n t_{n-1}}{t_{n-1}(1 + \frac{l_n}{r_n})} = 2t_n r_n,
\end{aligned} \tag{2.4}$$

where we have used $r_n + l_n = 1$. Plugging this into (2.3) and using Lemma 2.3 to evaluate $E_{x_k}[L_{\gamma(0,x_M)}^{x_n}]$ gives

$$E[G_{\Gamma(0,M)}^n] = \frac{x_M \min(x_n, x_k) - x_n x_k}{x_M r_n t_n}.$$

Summing over all n between 1 and $M - 1$ gives (2.1), and proves Theorem 2.1. □

3. An optional stopping-like theorem for large stopping times of birth–death chains

We will prove the following result.

THEOREM 3.1. *Suppose that* $t_\infty := \lim_{n\to\infty}((l_1 \cdots l_n)/(r_1 \cdots r_n))$ *exists. Let* $\{T_m\}_{m=1}^\infty$ *be a nondecreasing sequence of stopping times satisfying* $E[T_m] < \infty$ *such that* $T_m \to \Delta$

almost surely. Then $\lim_{m\to\infty} E[X_{T_m}]$ *exists, and*

$$\lim_{m\to\infty} E[X_{T_m}] = \frac{1 + \frac{l_1}{r_1} + \cdots + \frac{l_1\cdots l_{k-1}}{r_1\cdots r_{k-1}}}{t_\infty}. \tag{3.1}$$

REMARK 3.2. This reduces to Theorem 1.2 when we take $T_m = m \wedge \Delta$. As before, the result is valid when $t_\infty = 0$ or ∞, with the same interpretation as in Theorem 1.2.

PROOF. We will assume that $t_\infty \in (0, \infty)$; easy adjustments can be made to the proof in order to handle the cases $t_\infty = 0$ or ∞. Extend ϕ to a map from $[0, x_\infty)$ onto $[0, \infty)$ by requiring it to be continuous and linear on each interval (x_n, x_{n+1}). Let ϕ'_n be the derivative of ϕ on (x_{n-1}, x_n). It was shown in [5] to be a consequence of the Itō–Tanaka formula (see [6, Theorem VI.1.5]) that

$$X_m = \phi(B_{\tau(m)}) = k + \int_0^{\tau(m)} \phi'(B_s)\, dB_s + \sum_{n=1}^{\infty} \frac{\phi'_{n+1} - \phi'_n}{2} L_{\tau(m)}^{x_n}. \tag{3.2}$$

Let T be a stopping time for X_m with $E[T] < \infty$, and set $M(t) = \int_0^t \phi'(B_s)\, dB_s$. We need to show that $E[M_{\tau(T)}] = 0$. This is not quite trivial, since even if T is bounded $\tau(T)$ will be unbounded. We must have control over $\tau(T)$ in order to apply the optional stopping theorem. The quadratic variation process of M, denoted $\langle M\rangle(t)$, is given by $\int_0^t \phi'(B_s)^2\, ds$. We then have

$$E[\langle M\rangle_{\tau(m+1)} - \langle M\rangle_{\tau(m)}|B_{\tau(m)} = x_n]$$

$$= E\left[\int_{\tau(m)}^{\tau(m+1)} \phi'(B_s)^2\, ds\,|B_{\tau(m)} = x_n\right]$$

$$= \int_{x_{n-1}}^{x_{n+1}} \phi'(x)^2 E_{x_n}[L_{\gamma(x_{n-1},x_{n+1})}^{y}]\, dy$$

$$= \frac{1}{t_{n-1}^2} \int_{x_{n-1}}^{x_n} E_{x_n}[L_{\gamma(x_{n-1},x_{n+1})}^{y}]\, dy + \frac{1}{t_n^2} \int_{x_n}^{x_{n+1}} E_{x_n}[L_{\gamma(x_{n-1},x_{n+1})}^{y}]\, dy$$

$$= \frac{1}{t_{n-1}^2} \int_{x_{n-1}}^{x_n} \frac{2(x_{n+1} - x_{n-1})(y - x_{n-1}) - 2(x_n - x_{n-1})(y - x_{n-1})}{x_{n+1} - x_{n-1}}\, dy$$

$$+ \frac{1}{t_n^2} \int_{x_n}^{x_{n+1}} \frac{2(x_{n+1} - x_{n-1})(x_n - x_{n-1}) - 2(x_n - x_{n-1})(y - x_{n-1})}{x_{n+1} - x_{n-1}}\, dy$$

$$= \frac{1}{t_{n-1}^2} \int_{x_{n-1}}^{x_n} \frac{2t_n(y - x_{n-1})}{t_n + t_{n-1}}\, dy + \frac{1}{t_n^2} \int_{x_n}^{x_{n+1}} \frac{2t_{n-1}(x_{n+1} - y)}{t_n + t_{n-1}}\, dy$$

$$= \frac{1}{t_{n-1}^2}\left(\frac{t_n t_{n-1}^2}{t_n + t_{n-1}}\right) + \frac{1}{t_n^2}\left(\frac{t_n^2 t_{n-1}}{t_n + t_{n-1}}\right) = 1.$$

The second equality is due to the occupation times formula [3, Theorem 8.10], and the fourth equality is Lemma 2.3. It follows easily from this that $E[\langle M\rangle_{\tau(T)}] = E[T]$.

This tells us that the stopped process $M(t \wedge \tau(T))$ is uniformly integrable (see, for example [3, Corollary 7.8]), and thus that $E[M_{\tau(T)}] = E[M_0] = 0$ by [3, Theorem 7.15]. We obtain

$$E[X_{T_m}] = k + \sum_{n=1}^{\infty} \frac{\phi'_{n+1} - \phi'_n}{2} E_{x_k}[L^{x_n}_{\tau(T_m)}].$$

As $m \to \infty$, $T_m \nearrow \Delta$ and $E[L^{x_n}_{\tau(T_m)}] \nearrow E[L^{x_n}_{\tau(\Delta)}]$. Letting $b \to \infty$ in Lemma 2.3 gives $E_{x_k}[L^{x_n}_{\tau(\Delta)}] = 2 \min(x_n, x_k)$. If we assume that $\sum_{n=1}^{\infty} |\phi'_{n+1} - \phi'_n| < \infty$, then we can apply the dominated convergence theorem (to the functions $((\phi'_{n+1} - \phi'_n)/2)E_{x_k}[L^{x_n}_{\tau(T_m)}]$ defined on $\mathbb{N}$) to conclude that

$$\lim_{m \to \infty} E[X_{T_m}] = k + \sum_{n=1}^{\infty} \frac{\phi'_{n+1} - \phi'_n}{2} E_{x_k}[L^{x_n}_{\tau(\Delta)}]$$

$$= k + \sum_{n=1}^{k-1} (\phi'_{n+1} - \phi'_n)x_n + x_k \sum_{n=k}^{\infty} (\phi'_{n+1} - \phi'_n)$$

$$= k + \phi'_k x_k - \phi'_1 x_1 - \sum_{n=1}^{k-1} \phi'_{n+1}(x_{n+1} - x_n) + x_k(\phi'_\infty - \phi'_k).$$

Using $x_{n+1} - x_n = t_n = 1/\phi_{n+1}$ and $\phi'_1 x_1 = 1$, this reduces to

$$\lim_{m \to \infty} E[X_{T_m}] = x_k \phi'_\infty.$$

Recalling the values for x_k and ϕ'_∞, this is seen to be equivalent to (3.1). It remains to remove the restriction $\sum_{n=1}^{\infty} |\phi'_{n+1} - \phi'_n| < \infty$. If there is a sequence of numbers N_m such that $\sup_{0 \le m' \le T_m} X_{m'} \le N_m$ almost surely, then we can use [5, Lemma 3] and follow steps (22) through (26) of [5] in order to obtain the desired convergence. Suppose now that T_m is an arbitrary sequence of stopping times as in the statement of the theorem. As $E[T_m] < \infty$, there is a number H_m such that if $S_m = T_m \wedge H_m$, then $E[T_m - S_m] < 1/m$. Since $|X_{m+1} - X_m| = 1$ almost surely, we see that

$$|E[X_{T_m}] - E[X_{S_m}]| \le E[T_m - S_m] < \frac{1}{m}.$$

Since $\sup_{0 \le m' \le S_m} X_{m'} \le k + H_m$, we see by our previous work that $\lim_{m \to \infty} E[X_{S_m}] = x_k/t_\infty$, and it follows that the same is true for $\lim_{m \to \infty} E[X_{T_m}]$. This completes the proof of the theorem. □

4. Examples to which Theorem 3.1 can be applied

EXAMPLE 4.1. This example appeared in [5]. Let $l_n = n/(2n + 1)$, $r_n = (n + 1)/(2n + 1)$ for $n \ge 1$. Then $t_n = 1/(n + 1)$, so that $t_\infty = 0$. On the other hand, $x_\infty = 1 + \sum_{n=1}^{\infty} t_n = \infty$. We see that the birth–death chain X_m built upon these transition probabilities has an extinction probability of 1, but $E[X_m] \to \infty$ as $m \to \infty$.

EXAMPLE 4.2. Suppose that there is an $M \ge 1$ such that $t_n = r_n = 1/2$ for all $n > M$. Then $t_\infty = (l_1 \cdots l_M)/(r_1 \cdots r_M)$, and so, for a sequence $\{T_m\}_{m=1}^{\infty}$ satisfying the

requirements for our theorem,

$$E[X_{T_m}] \to \frac{1 + \frac{l_1}{r_1} + \cdots + \frac{l_1 \cdots l_{k-1}}{r_1 \cdots r_{k-1}}}{\frac{l_1 \cdots l_M}{r_1 \cdots r_M}}.$$

EXAMPLE 4.3. In order for t_∞ to exist and lie in $(0, \infty)$, it is clearly necessary that $r_n, l_n \to 1/2$ as $n \to \infty$, but it is not sufficient. Example 4.1 shows this, as $t_\infty = 0$. Exchanging r_n and l_n in Example 4.1 will give the case where $t_\infty = \infty$. It is not hard to interlace these series in a way so that t_∞ does not exist. The theory of infinite products (most complex analysis texts contain this; see, for example, [1]) gives us the standard necessary condition for t_∞ to exist and lie in $(0, \infty)$. The following proposition is obtained.

PROPOSITION 4.4. *Suppose that*

$$\sum_{n=1}^{\infty} \left| 1 - \frac{l_n}{r_n} \right| < \infty.$$

Then $\lim_{m \to \infty} E[X_{T_m}]$ *exists and lies in* $(0, \infty)$, *where* $\{T_m\}_{m=1}^{\infty}$ *is any nondecreasing sequence of stopping times satisfying* $E[T_m] < \infty$ *such that* $T_m \to \Delta$ *almost surely.*

5. The Doob decomposition of X_m

Let $\mathcal{F}_t$ be the filtration for B_t. Equation (3.2) shows that the σ-fields $\mathcal{F}_{\tau_m}$ can be taken as the filtration for the birth–death chain X_m, although of course this is far from the smallest possible filtration as it contains a great deal of extra information concerning the behaviour of B_t. Equation (3.2) comes from the Doob–Meyer decomposition of $\phi(B_t)$, but is not the discrete-time Doob decomposition of X_m (see [2]), since $L_{\tau(m)}^{x_n}$ is not $\mathcal{F}_{\tau(m-1)}$ measurable. On the other hand, it leads very easily to the Doob decomposition.

THEOREM 5.1. *The Doob decomposition of X_m is given by*

$$X_m = \mathcal{M}_m + \sum_{n=1}^{\infty} G_m^n (r_n - l_n),$$

where $\mathcal{M}_m$ is the martingale given by

$$\mathcal{M}_m = k + \int_0^{\tau(m)} \phi'(B_s) \, dB_s + \sum_{n=1}^{\infty} \frac{\phi'_{n+1} - \phi'_n}{2} (L_{\tau(m)}^{x_n} - 2 G_m^n r_n t_n).$$

PROOF. By the standard arguments [2, Lemma 6.10] we have $X_m = M_m + A_m$, where M_m is a martingale and $A_m = \sum_{m'=1}^{m} E[X_m - X_{m'-1} \mid \mathcal{F}_{\tau(m'-1)}]$. Since $\int_0^{\tau(m)} \phi'(B_s) \, dB_s$ is a martingale, (3.2) shows that

$$E[X_m - X_{m'-1} \mid \mathcal{F}_{\tau(m'-1)}] = E\left[\sum_{n=1}^{\infty} \frac{\phi'_{n+1} - \phi'_n}{2} (L_{\tau(m')}^{x_n} - L_{\tau(m'-1)}^{x_n}) \,\Big|\, \mathcal{F}_{\tau(m'-1)} \right].$$

The result then follows from the following calculation:

$$
\frac{\phi'_{n+1} - \phi'_n}{2} \sum_{m'=1}^{m} E[L^{x_n}_{\tau(m')} - L^{x_n}_{\tau(m'-1)} \mid \mathcal{F}_{\tau(m'-1)}]
$$

$$
= \frac{\phi'_{n+1} - \phi'_n}{2} \sum_{m'=1}^{m} E_{x_n}[L^{x_n}_{\gamma(x_{n-1}, x_{n+1})}] 1_{\{B_{\tau(m'-1)} = x_n\}}
$$

$$
= \frac{\phi'_{n+1} - \phi'_n}{2} \sum_{m'=1}^{m} 2 r_n t_n 1_{\{B_{\tau(m'-1)} = x_n\}}
$$

$$
= \frac{\phi'_{n+1} - \phi'_n}{2} 2 r_n t_n G^n_m
$$

$$
= \left(\frac{1}{t_n} - \frac{1}{t_{n-1}} \right) r_n t_n G^n_m = (r_n - l_n) G^n_m.
$$

Note that the first equality above is due to the independent increments and strong Markov property of Brownian motion, while the second follows from Lemma 2.3 (see (2.4)). □

Acknowledgement

References

[1] L. V. Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable* (McGraw-Hill, New York, 1966).

[2] O. Kallenberg, *Foundations of Modern Probability* (Springer, New York, 2002).

[3] F. C. Klebaner, *Introduction to Stochastic Calculus with Applications* (Imperial College Press, London, 2005).

[4] M. B. Marcus and J. Rosen, *Markov Processes, Gaussian Processes and Local Times* (Cambridge University Press, Cambridge, 2006).

[5] G. Markowsky, 'Applying Brownian motion to the study of birth–death chains', *Statist. Probab. Lett.* **81**(8) (2011), 1173–1178.

[6] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion* (Springer, Berlin, 1999).

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