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ALGEBRAIC ELEMENTS AND SETS OF UNIQUENESS IN THE GROUP OF INTEGERS OF A p-SERIES FIELD

BY

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ABSTRACT. Let G be the group of integers of a p-series field. A class $\{E(\theta)\}$ of perfect null subsets of G is introduced and classified into M-sets and U-sets according to the arithmetical nature of the field element θ. This is analogous to the well-known classification, involving Pisot numbers, of certain Cantor sets on the circle.

1. **Introduction**. We begin by recalling one of the highlights in the classical theory of uniqueness of trigonometric series. If $\theta > 2$ then the Cantor set $C(\theta) = \{\sum_1^\infty \epsilon_n \theta^{-n} : \epsilon_n = 0 \text{ or } 1\}$ is a set of uniqueness exactly when θ is a Pisot number (see [6], [5, sec.III.3]). This means there is a nontrivial series $\sum_{-\infty}^{\infty} a_n e^{inx}$ converging to 0 outside $C(\theta)$ if and only if θ is not a Pisot number. Recall that if T is the set of algebraic integers $\theta > 1$ all of whose conjugates (other than itself) lie inside the closed unit disk $|z| \leqq 1$, then the Pisot numbers are those $\theta \in T$ whose other conjugates are all inside the open disk $|z| < 1$. The remaining numbers in T are known as Salem numbers and they are intimately related to the Pisot numbers.

Analogues of these numbers in fields of formal power series were introduced in [1] and studied further in [3], [4]. We introduce a class of Cantor sets $E(\theta)$ in the group of integers of a p-series field and obtain a characterization of the sets of uniqueness in this class in terms of the T-elements of the field (rather than the Pisot elements).

Before stating our results we give a brief review of the p-series context and refer the reader to [8] and [1] for a more detailed account.

Let p be a prime $\geqq 2$ and let k denote the field $\{0, 1, \ldots, p-1\}$ with p elements. $k[x]$ will denote the ring of polynomials in the indeterminate x with coefficients in k, and $k(x)$ its quotient field. The p-series field $k\{x^{-1}\}$ consists of formal series of the form

$$z = a_h x^h + a_{h-1} x^{h-1} + \ldots = \sum_{j=-h}^{\infty} a_{-j} x^{-j}$$

with coefficients in k and the integer h variable.

Fix a constant $c > 1$. The absolute value of z is defined by

$$|z| = c^h \quad \text{if } a_h \neq 0.$$

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This absolute value has a unique extension to the algebraic closure K of $k\{x^{-1}\}$, where it satisfies

$$|z_1 + z_2| \leqq \max\{|z_1|, |z_2|\}.$$

We denote by T the set of all elements $\theta \in k\{x^{-1}\}$, $|\theta| > 1$ such that θ is algebraic and integral over $k[x]$ with all of its remaining conjugates (with respect to $k(x)$) having absolute value $\leqq 1$. Thus T consists of the Pisot and Salem elements of $k\{x^{-1}\}$, these being defined in analogy with the real Pisot and Salem numbers.

Every $z \in k\{x^{-1}\}$ can be written uniquely as

$$z = [z] + (z), \quad [z] \in k[x], \quad |(z)| < 1$$

which splits z into integral and fractional parts.

Among the remarkable properties of the elements just defined are the following. If $\theta \in k\{x^{-1}\}$ and $|\theta| > 1$, then (i) θ is a Pisot element if and only if there exists $\lambda \in k\{x^{-1}\}$, $\lambda \neq 0$, such that $(\lambda\theta^n) \to 0$ as $n \to \infty$, in which case $\lambda \in k(x)(\theta)$; (ii) $\theta \in T$ if and only if there exists $\lambda \in k\{x^{-1}\}$, $\lambda \neq 0$, such that $|(\lambda\theta^n)| < |\theta|^{-2}$ for all sufficiently large n, in which case also $\lambda \in k(x)(\theta)$ ([1], [4]).

$k\{x^{-1}\}$ is a locally compact topological field whose additive group is self-dual. The group of integers of $k\{x^{-1}\}$ is the compact additive group

$$G = \{z \in k\{x^{-1}\}: |z| < 1\}$$

whose dual group is isomorphic to $k[x]$. Elements of G are of the form

$$z = a_{-1}x^{-1} + a_{-2}x^{-2} + \dots .$$

Define $\gamma_0(z) \equiv 1$ and the fundamental character γ_1 by

$$\gamma_1(z) = \exp(2\pi i a_{-1}/p), \qquad z = \sum_{j=-h}^{\infty} a_{-j}x^{-j}.$$

For $z, w \in k\{x^{-1}\}$, define γ_z by

$$\gamma_z(w) = \gamma_1(zw).$$

The mapping $z \to \gamma_z$ identifies $k\{x^{-1}\}$ with its dual group and $k[x]$ with the dual of G. Let m denote Haar measure on $k\{x^{-1}\}$, normalized so that $m(G) = 1$. Every $z \in k\{x^{-1}\}$ may be regarded as a series over the real numbers and thus evaluated there at the point $x = p$ to provide a real number $z(p)$. The mapping $z \to z(p)$ is then a continuous measure-preserving map of $k\{x^{-1}\}$ onto $[0, \infty)$ which carries G onto the unit interval $[0, 1]$, and $k[x]$ injectively onto the non-negative integers. This enumeration of $k[x]$ provides an enumeration $\{\gamma_0, \gamma_1, \dots\}$ of the dual group of G. When $p = 2$, G is the dyadic group introduced by Fine [2] and the functions $\{\gamma_n : n = 0, 1, 2, \dots\}$ are the Walsh functions with Paley's ordering. Unfortunately the overall map $z \to z(p)$ is not quite $1-1$; still, it enables a transfer of many results between $k\{x^{-1}\}$ and the reals, which is important, for example, in Walsh analysis on the real line. See [10] for a survey of the recent literature on this subject.

A subset $E \subset G$ is a *U-set* (*set of uniqueness*) if the convergence to 0 of a character series $\sum_0^\infty a_n \gamma_n$ outside E implies $a_n \equiv 0$. E is an *M-set* (*set of multiplicity*) if it is not a U-set, and is an M_o*-set* if there is a nontrivial Fourier-Stieltjes series converging to 0 outside E. The sets of greatest interest are the perfect sets of measure 0 and even here the classification problem is a delicate matter. (It is known that all countable sets are U-sets and sets of positive measure are always M-sets).

A natural analogue of the sets $C(\theta)$ on the circle is the class of sets

$$F(\theta) = \left\{ \sum_1^\infty \epsilon_n \theta^{-n} : \epsilon_n = 0 \text{ or } 1 \right\} \tag{1}$$

where $\theta \in k\{x^{-1}\}$, $|\theta| > 1$ ($|\theta| > c$ if $p = 2$). These sets are all perfect and have measure 0; however, they are not interesting in the present context, in view of the following result (proved for $p = 2$ in [11]).

THEOREM 1. *The sets $F(\theta)$ are all U-sets.*

We remark here that Theorem 1 shows a marked contrast not only with the classical case, but also with the p-adic case; there, the exact analogue of the result on the circle is known to hold for the counter-parts of the sets $C(\theta)$ (see [5, page 100]).

In what follows, it will often be convenient to work with the quantity $L(z)$ defined by $L(z) = h$ whenever $|z| = c^h$. Thus $z \in k\{x^{-1}\}$, $L(z) < -N$ means z has the form

$$z = a_1 x^{-(N+1)} + a_2 x^{-(N+2)} + \dots .$$

Our sets $E(\theta)$ are defined as follows. Let $\theta \in k\{x^{-1}\}$, $|\theta| > 1$, be arbitrary except that we require $|\theta| > c$ if $p = 3$ and $|\theta| > c^2$ if $p = 2$. Then

$$E(\theta) = \left\{ \sum_2^\infty \epsilon_n \theta^{-n} : \epsilon_n \in I_\theta \right\}, \quad \text{where } I_\theta = \{0, 1, x, \dots, x^{2L(\theta)-1}\}, \tag{2}$$

the subscript 2 on the sum ensuring that $E(\theta) \subset G$.

THEOREM 2. *The sets $E(\theta)$ are all perfect sets of measure 0, and $E(\theta)$ is a U-set precisely when $\theta \in T$.*

Theorem 2 shows in particular that the problem of identifying the U-sets for Walsh series is likely to be every bit as delicate as it is for trigonometric series on the circle. No such interplay with algebraic number theory has previously been uncovered in the Walsh case. Our sets $E(\theta)$ and $F(\theta)$ are not generally easy to visualize when transferred to the unit interval, but let us mention that, when $p = 2$, the sets $F(x^k)$ map onto the sets $C(2^k)$, and these sets have been known since 1949 (see [7]) to be U-sets for Walsh series on the interval. Meanwhile, the long-standing problem of deciding if the classical ternary set is a Walsh U-set or not (it is a trigonometric U-set) seems to remain as intractable as ever. It is unlikely that there is any further connection between our sets (in the Walsh case) and the remaining Cantor sets $C(\theta)$, $\theta \neq 2^k$, whose classification into Walsh U-sets and M-sets is still a complete mystery.

We note that T is a countable subset of $k\{x^{-1}\}$ so that 'most' of our sets $E(\theta)$ are M-sets (M_0-sets in fact). In contrast to the real case, however, where the Pisot numbers form a countable closed set, elements of T (indeed the Pisot elements) are everywhere dense in $\{z \in k\{x^{-1}\}: |z| > 1\}$ (see [3]). Thus the M-set $E(\theta)$ does not remain an M-set for small variations in θ, which is the case for an M-set $C(\theta)$ on the circle.

Theorems 1 and 2 will be proved in §3, after we prepared the necessary background and lemmas in §2. We note that, in broad outline, the proof of Theorem 2 parallels that used on the circle for the sets $C(\theta)$ (as in [6] or [12, Vol. II, 147–156], for example).

We close this introduction by recalling that the topology of G has a base at 0 consisting of the compact open subgroups $G_r = \{z \in G: L(z) < -r\}$, $r = 0, 1, 2, \ldots$, and that $m(G_r) = p^{-r}$.

2. **Preliminary Lemmas**. A sequence of vectors $V^{(m)} = (v_1^{(m)}, \ldots, v_n^{(m)})$ in $k[x]^n$ is called *normal* if for every nonzero vector $(a_1, \ldots, a_n)$ in $k[x]^n$ the inner product $\sum_{j=1}^n a_j v_j^{(m)} \to \infty$ in absolute value as $m \to \infty$. A subset $E \subset G$ is of *type* $H^{(n)}$ if there is a nonempty open set $\Delta \subset G^n$ and a normal sequence $V^{(m)}$ in $k[x]^n$ such that the vector $V^{(m)}(z) = (v_1^{(m)}z, \ldots, v_n^{(m)}z)$ never enters Δ, mod 1 for $z \in E$ and $m = 1, 2, \ldots$ (mod 1 means we consider only the fractional parts of the components $v_j^{(m)}z$).

LEMMA 1. ([9, Theorem 3]). *If E is of type $H^{(n)}$ for some $n \geqq 1$, then E is a U-set.*

The trigonometric counterpart of our next lemma is proved by means of the principle of localization (see [12, Vol. I, 347–348], for example). We are not aware of this lemma having previously been enunciated in the p-series case, so we provide here a proof which is simple and independent of the theory of localization.

LEMMA 2. *Suppose $E \subset G$ is closed and supports a measure $\mu \neq 0$ whose transform $\hat{\mu}(n) = \int_G \bar{\gamma}_n d\mu$ tends to 0 as $n \to \infty$. Then the Fourier–Stieltjes series $S[d\mu] = \sum \hat{\mu}(n)\gamma_n$ converges to 0 outside E, so that E is an M_0-set.*

To prove Lemma 2, it is enough to show that the partial sums $S_n = \sum_{j=0}^{n-1} \hat{\mu}(j)\gamma_j$ converge to 0 on any coset $z + G_r (z \in G, r > 0)$ which does not intersect E (see the final paragraph in §1). Let H be such a coset of G_r, assume $z_0 \in H$, and let the non-negative integer k be arbitrary. Then (with $S_0 \equiv 0$)

$$
\begin{aligned}
S_{(k+1)p^r}(z_0) - S_{kp^r}(z_0) &= \sum_{j=0}^{p^r-1} \hat{\mu}(kp^r + j)\gamma_{kp^r+j}(z_0) \\
&= \sum_{j=0}^{p^r-1} \gamma_{kp^r+j}(z_0) \int_G \bar{\gamma}_{kp^r+j}(z)\mu(dz) \\
&= \gamma_{kp^r}(z_0) \int_G \bar{\gamma}_{kp^r}(z)\, D_{p^r}(z_0\text{-}z)\mu(dz),
\end{aligned}
$$

where $D \equiv \sum_{n=0}^{\infty} \gamma_n$ is the Dirichlet kernel on G. Since $D_{p^r} = p^r\, 1_{G_r}$, and $z_0 - z \in G_r$ only when $z \in H$, and since $\mu(H) = 0$, this last integral is 0. Using now the fact that $\hat{\mu}(n) \to 0$, we see immediately that $S_n \to 0$ uniformly on H. This completes our proof.

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LEMMA 3. ([4, Théorème 2.2]). *Suppose* $\theta \in k\{x^{-1}\}$, $|\theta| > 1$, *and there exists* $\lambda \in k\{x^{-1}\}$, $\lambda \neq 0$, *such that* $|(\lambda\theta^n)| < |\theta|^{-2}$ *for all sufficiently large* n. *Then* θ *belongs to* T.

Our final lemma comes from the proof of Theorem 1.2 in [1] and is a special analogue of Minkowski's theorem on complex linear forms.

LEMMA 4. *If* $\theta \in k\{x^{-1}\}$ *is algebraic of degree* s *over* $k(x)$ *and* m, N *are given positive integers, then there exists a Pisot element* $\lambda \in k(x)(\theta)$, $|\lambda| \leq c^m$, *with conjugates* $\lambda_1, \ldots, \lambda_{s-1}$ *satisfying* $|\lambda_j| < c^{-N}$ $(j = 1, \ldots, s-1)$, *provided that* $m \geq N(s-1) + B$, *where the integer* B *depends only on* θ.

3. **Proofs of Theorems**. We first prove Theorem 1 by showing that the sets $F(\theta)$ are all of type $H^{(1)}$ and then applying Lemma 1. (We could proceed here by considering an appropriate map of $k\{x^{-1}\}$ to itself. For example, when $p > 2$, the set $F = \{\sum_1^\infty \epsilon_n x^{-n}: \epsilon_n = 0 \text{ or } 1\}$ is clearly of type $H^{(1)}$ with respect to the sequence $1, x, x^2, \ldots$, and an obvious map of $k\{x^{-1}\}$ into itself sends F onto $F(\theta)$. However, we prefer to give the direct argument below.) For any θ used in the definition of a set $F(\theta)$ in (1), the sequence $\{\theta^m\}$ is sufficiently lacunary that the equations

$$\gamma_{\theta^m}(\lambda) = 1, \quad m \geq 0 \ (m \geqq 1 \text{ if } p > 2) \tag{3}$$

can always be solved for a nonzero $\lambda \in G$. For $z = \sum_1^\infty \epsilon_n \theta^{-n}$ and such a λ satisfying (3), we have, for $m > 0$,

$$\lambda\theta^m z = \sum_{j=0}^{m-1} \epsilon_{m-j}\lambda\theta^j + \sum_{j=1}^{\infty} \epsilon_{m+j}\lambda\theta^{-j},$$

and hence, for the sequence of integral parts $[\lambda\theta^m]$,

$$\gamma_1([\lambda\theta^m]z) = \gamma_1(\lambda\theta^m z) = \prod_{j=0}^{m-1} \gamma_1(\epsilon_{m-j}\lambda\theta^j) = \gamma_1(\epsilon_m\lambda),$$

which is always either 1 or $\gamma_1(\lambda)$ (always 1 if $p = 2$), showing that $F(\theta)$ is of type $H^{(1)}$ (take $\Delta = \{z \in G: \gamma_1(z) = \xi\}$, where $\xi^p = 1$ and $\xi \neq 1$ or $\gamma_1(\lambda)$).

PROOF OF THEOREM 2. First we show that $E(\theta)$ as defined in (2) is a perfect set of measure 0 supporting a natural probability measure μ whose transform $\hat{\mu}$ vanishes at ∞ whenever $\theta \notin T$. This μ is the 'Lebesgue measure' on $E(\theta)$ and is simply the distribution measure of the random variable $\sum_2^\infty \epsilon_n \theta^{-n}$ when the sequence $\{\epsilon_n\}$ is regarded as a sequence of independent random variables each uniformly distributed in I_θ. More formally, give I_θ the discrete topology and assign mass $(2L(\theta) + 1)^{-1}$ to each of its points. The obvious map from the compact measure space $\Pi_1^\infty I_\theta$ onto $E(\theta)$ is continuous (showing that $E(\theta)$ is perfect) and gives rise to the distribution μ on $E(\theta)$. To see that $E(\theta)$ has Haar measure 0, write

$$E(\theta) = \left\{\sum_2^{N+1} \epsilon_n\theta^{-n}: \epsilon_n \in I_\theta\right\} + \left\{\sum_{N+2}^{\infty} \epsilon_n\theta^{-n}: \epsilon_n \in I_\theta\right\}$$
$$= Q_N + H_N, \text{ say.}$$

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Recalling that $G_r = \{z \in G: L(z) < -r\}$ and noting that

$$L(\epsilon_n\theta^{-n}) = -nL(\theta) + L(\epsilon_n) \leqq -(n-2)L(\theta) - 1,$$

we have $H_N \subset G_{NL(\theta)}$ and hence $m(H_N) \leqq p^{-NL(\theta)}$.

Now Q_N has not more than $(2L(\theta)+1)^N$ points, so

$$m(E(\theta)) \leqq (2L(\theta+1)^N p^{-NL(\theta)},$$

and since the restrictions on θ in (2) ensure that $2L(\theta)+1 < p^{L(\theta)}$, we see that $m(E(\theta)) = 0$. Noting that $E(\theta) = \bigcap_{N>1}(Q_N + G_{NL(\theta)})$ shows also that $E(\theta)$ is perfect.

We now break the proof up into two parts. Part 1 will show that $\hat{\mu}$ vanishes at ∞ when $\theta \notin T$, Lemma 2 then showing that $E(\theta)$ is an M_0-set. That $\hat{\mu}(n)$ cannot tend to 0 if $\theta \in T$ will be a consequence of Lemma 2 and Part 2, where we show that if $\theta \in T$ and θ has degree s over $k(x)$, then $E(\theta)$ is of type $H^{(s)}$ and hence a U-set by Lemma 1.

PART 1. Define

$$\phi(z) = (2L(\theta)+1)^{-1}\left(1 + \sum_{j=0}^{2L(\theta)-1} \gamma_{p^j}(z)\right), \quad z \in k\{x^{-1}\},$$

and notice that there is a constant $\rho < 1$ such that

$$\phi(z) = 1 \text{ if } |(z)| < |\theta|^{-2}, \quad |\phi(z)| \leqq \rho \text{ otherwise.} \tag{4}$$

Denoting $\sigma(u) = \overline{\hat{\mu}(u)} = \int_G \gamma_u(z)\mu(dz), \quad u \in k\{x^{-1}\}$, we compute

$$\sigma(u) = \prod_{n=2}^{\infty} (2L(\theta)+1)^{-1} \textstyle\sum_{\epsilon_n \in I_\theta} \gamma_u(\epsilon_n\theta^{-n}) = \displaystyle\prod_{n=2}^{\infty} \phi(u\theta^{-n}).$$

From this we obtain, for any $\lambda \in k\{x^{-1}\}$,

$$\sigma(\lambda\theta^m) = \sigma(\lambda) \prod_{j=-1}^{m-2} \phi(\lambda\theta^j), \quad m \geqq 1. \tag{5}$$

Suppose now that $\theta \notin T$. Lemma 3 and (4) show that, for every $\lambda \neq 0$, $|\phi(\lambda\theta^j)| \leq \rho$ for infinitely many $j \geq 0$. It follows from (5) then that $|\sigma(\lambda\theta^m)|$ decreases to 0 as $m \to \infty$ for all $\lambda \neq 0$, and since σ is continuous, this convergence is uniform on the compact set $\{\lambda: 1 \leqq |\lambda| \leqq |\theta|\}$. Since an arbitrary $u \neq 0$ can be written in the form $u = \lambda\theta^m$, where $1 \leqq |\lambda| < |\theta|$, we conclude that $\sigma(u) \to 0$ as $|u| \to \infty$.

PART 2. We suppose now that $\theta \in T$ and the degree of θ over $k(x)$ is $s \geqq 1$. Then it is readily seen that the vector sequence given by

$$V^{(m)} = ([\lambda\theta^m], \ldots, [\lambda\theta^{m+s-1}]) \tag{6}$$

is normal for any $\lambda \neq 0$. We will find a $\lambda \neq 0$ so that $E(\theta)$ becomes an $H^{(s)}$-set with respect to (6).

For a positive integer N to be specified later, we use Lemma 4 to choose a Pisot element $\lambda \in k(x)(\theta)$ such that

(7) $$|\lambda| \leqq c^{N(s-1)+B_1}, \quad \text{where } B_1 = 2L(\theta)(s-1) + B,$$

and such that $|\lambda_i| < c^{-(N+2L(\theta))}$ for each remaining conjugate λ_i of λ.

For each $j \geqq 0$, the trace of $\lambda\theta^j$ over $k(x)$ belongs to $k[x]$. This, together with the bound on the λ_i and the fact that $\theta \in T$, leads to

(8) $$|(\lambda\theta^j)| < c^{-(N+2L(\theta))} \ (j \geqq 0).$$

If $z = \sum_2^\infty \epsilon_n \theta^{-n} \in E(\theta)$, then

(9) $$\lambda\theta^m z = \sum_{j=0}^{m-2} \epsilon_{m-j}\,\lambda\theta^j + \sum_{j=1}^{\infty} \epsilon_{m+j}\lambda\theta^{-j} \ (m \geqq 2).$$

For ϵ in I_θ, (7) implies that

$$L(\epsilon\lambda\theta^{-j}) \leqq 2L(\theta) - 1 + N(s-1) + B_1 - jL(\theta),$$

and hence

(10) $$L(\epsilon\lambda\theta^{-j}) < -N \text{ if } j > J,$$

where $J = [NsL(\theta)^{-1} + B_2]$, and where B_2 depends only on θ. It follows from (8), (9) and (10) that we can write $d_m = ([\lambda\theta^m]z)$ as

$$d_m = \left(\sum_{j=1}^{J} \epsilon_{m+j}\lambda\theta^{-j}\right) + u_m(z),$$

where

$$u_m(z) \in G_N = \{z \in G\colon |z| < c^{-N}\}.$$

Now the number of open rectangles $\triangle \subset G^s$ which are products of cosets in G of G_N is p^{Ns}, and to just which one of these the vector

$$V^{(m)}(z) = (d_m, \ldots, d_{m+s-1})$$

belongs is completely determined by the sequence $\epsilon_{m+1}, \ldots, \epsilon_{m+J+s}$. Hence, as z ranges over $E(\theta)$ and m over the positive integers, the vector $V^{(m)}(z)$ cannot possibly meet more than $(2L(\theta)+1)^{J+s}$ of these rectangles $\triangle$. Since $2L(\theta) + 1 < p^{L(\theta)}$ (recall that we needed this to show $m(E(\theta)) = 0$), we know that $2L(\theta) + 1 = rp^{L(\theta)}$ for a positive constant $r < 1$. Thus we have

$$(2L(\theta) + 1)^{J+s} = r^{J+s}p^{L(\theta)(J+s)},$$

and a glance at the expression for J following (10) shows that

$$r^{J+s}p^{L(\theta)(J+s)} < Ar^{Ns/L(\theta)}p^{Ns}$$

for a constant $A > 0$ independent of N. Hence a large enough N will ensure that $(2L(\theta) + 1)^{J+s} < p^{Ns}$, providing us with a $\triangle$ free of any vector $V^{(m)}(z)$. This finishes the proof.

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