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NEAR-RINGS OF POLYNOMIALS OVER GROUPS

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The set $G[x]$ of polynomials over a group $(G,+)$—as well as the polynomial functions $P(G)$ on $(G,+)$ form near-rings with respect to addition and composition (substitution). See [**1**] for polynomials and [**2**] for near-rings. A number of results on $G[x]$ can be deduced from [**2**].

Due to [**1**], the polynomials in $G[x]$ can uniquely be represented in the following "normal form":

$$g_1+z_1x+g_2+z_2x+\ldots+z_nx+g_{n+1} \tag{1}$$

with $n\in\mathbb{N}_0$, $g_1,\ldots,g_{n+1}\in G$, $z_1,\ldots,z_n\in\mathbb{Z}$, $g_2,\ldots,g_n\neq 0$ if $n>1$ and $z_i\neq 0$ if $g_{i+1}\neq 0$. In short, we write $\sum_i (g_i+z_ix)$ for (1). Another unique representation for the polynomials of $G[x]$ is given by

$$\sum_{i=1}^{n}(g_i+z_ix-g_i)+g_{n+1} \tag{2}$$

with $n\in\mathbb{N}_0$, $g_i\in G$, $z_i\in\mathbb{Z}$. Since $g_i+z_ix-g_i=z_i(g_i+x-g_i)$, another normal form is given by

$$\sum_{i=1}^{n}\sigma_i(g_i+x-g_i)+g_{n+1} \tag{3}$$

with $n\in\mathbb{N}$, $g_i\in G$ and $\sigma_i\in\{1,-1\}$. The zero-symmetric part $G_0[x]:=\{p\in G[x]\,|\,p\circ 0=0\}$ of $G[x]$ (where 0 denotes the identity in $(G,+)$) is then given by

$$G_0[x]=\left\{\sum_i \sigma_i(g_i+x-g_i)\,|\,g_i\in G,\sigma_i\in\{1,-1\}\right\}. \tag{4}$$

Note that we write groups additively, this does not imply commutativity. Moreover, $A\leqq G$ means that A is a subgroup of G. $A\subset B$ denotes strict inclusion.

The first interesting property of $G[x]$ comes directly from the normal form (4) and the fact that all g_i+x-g_i are distributive elements in $G[x]$:

Proposition 1. *$G_0[x]$ and $P_0(G)=\{f\in P(G)\,|\,f(0)=0\}$ are distributively generated (d.g.) near-rings.*

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Another interesting feature stems from the fact that all normal subgroups are left ideals. For $S, T \subseteq G$ let S^T be the set of all sums of the form $t_i + s_i - t_i$ $(t_i \in T, s_i \in S)$. S^G is then just the normal closure of S in G.

Proposition 2. *Let S be a subgroup of $(G[x], +)$. Then*

(i) *S is a left ideal in $G[x]$ iff S is a normal subgroup, which in turn is equivalent to $S^{G \cup \{x\}} = S$.*

(ii) *S is a $G_0[x]$-subgroup iff $S^G = S$.*

Proof. (a) As in 6.6 of [2] one sees that every normal $G_0[x]$-subgroup of $G_0[x]$ is a left ideal. So let us take a normal subgroup N of $G_0[x]$ or of $G[x]$ and arbitrary $p \in G_0[x]$ and $n \in N$ in order to show that $p \circ n \in N$. From (4) we see that it suffices to take $p = g + x - g$. Then $p \circ n = g + n - g \in N$ and N is a normal $G_0[x]$-subgroup, hence a left ideal. The rest of (i) and (ii) are shown similarly.

In a general near-ring N, the sum of an N_0-subgroup and a left ideal is an N_0-subgroup, but usually not a left ideal. The situation is better in $G[x]$. For that, suppose $(A:g) := \{p \in G[x] \mid p \circ g \in A\}$ for $A \subseteq G$ and $g \in G$. If $A \trianglelefteq G$ then $(A:g)$ is easily shown to be a left ideal of $G[x]$.

Proposition 3. *Let S be a $G_0[x]$-subgroup of $G[x]$, $g \in G$ and $A \trianglelefteq G$. Then $L := S + (A:g)$ is a left ideal of $G[x]$.*

Proof. Since S is a subgroup and $(A:g)$ a normal subgroup of $(G[x], +)$, L is a subgroup of $G[x]$. For $h \in G$, $s \in S$ and $p \in (A:g)$ we get $h + (s + p) - h = (h + s - h) + (h + p - h) \in S + (A:g) = L$ by Proposition 2(ii). Also,

$$x + (s + p) - x = (x - g) + (g + s - g) + (g + p - g) + (g - x)$$
$$\in (A:g) + S + (A:g) + (A:g) = L.$$

Hence L is a left ideal by Proposition 2(i).

In order to get results about the structure of $G[x]$ one needs a certain amount of knowledge about strictly maximal left ideals (i.e. left ideals which are at the same time maximal $G_0[x]$-subgroups). We start with

Theorem 1. *The collection of maximal left ideals L of $G[x]$ with $G \subseteq L$ is precisely given by*

$$L_p := \left\{ \sum_i (g_i + z_i x) \in G[x] \,\middle|\, \sum z_i \in p\mathbb{Z} \right\} \quad \textit{for } p \textit{ prime.}$$

Proof. (a) It follows readily from Proposition 2(i) that L_p is a left ideal for each prime number p. $L_p \neq G[x]$. Now suppose that U is a left ideal with $L_p \subset U$. The set U_1 of all $z \in \mathbb{Z}$ such that there is some $\sum_i (g_i + z_i x) \in U$ with $\sum z_i = z$ is a subgroup of $(\mathbb{Z}, +)$

containing $p\mathbb{Z}$. Since $L_p \neq U$ there is some $\sum_i (h_i + y_i x) \in U \backslash L_p$. This means that $\sum_i y_i \in U_1 \backslash p\mathbb{Z}$, whence $p\mathbb{Z} \subset U_1$, hence $U_1 = \mathbb{Z}$. But then $x \in U$ and (since $G \cup \{x\}$ generates $G[x]$) $U = G[x]$. Hence L_p is maximal.

(b) Now let L be a maximal left ideal and define L_1 similar to U_1 in (a). If S_1 is a proper subgroup of $(\mathbb{Z}, +)$ then $S := \{\sum_i (g_i + z_i x) \mid \sum z_i \in S_1\}$ is a proper left ideal of $G[x]$. If $L_1 \subseteq S_1$ then $L \subseteq S$. Since L is maximal and $S \neq G[x]$, we get $L = S$ and $L_1 = S_1 = p\mathbb{Z}$ for some prime p. Hence $L = L_p$.

Theorem 2. *Let L be a strictly maximal left ideal of $G[x]$ and $L_c := L \cap G$. Then $L_c = G$ or L_c is a maximal normal subgroup of G.*

Proof. By Proposition 2(ii), L_c is normal in $(G, +)$. If $L_c \subset M \trianglelefteq G$ then M is (again by Proposition 2(ii)) a $G_0[x]$-subgroup. Since L is maximal, $L + M = G[x]$. If $g \in G$ then there are $l \in L$ and $m \in M$ with $g = l + m$; $m \in M \subseteq G$ implies that $l \in L_c \subset M$. Hence $g \in M$ and $M = G$. This shows that L_c is a maximal normal subgroup of G.

For a group G let $\beta(G)$ be Baer's group radical (the intersection of all maximal normal subgroups). From Theorem 2 and Proposition 3 we get

Corollary 1. *Let L be a strictly maximal left ideal of $G[x]$. Then $\beta(G) \subseteq L$.*

From [3] we get the information that if M is a maximal normal subgroup of G and $g \in G \backslash M$ then $(M : g)$ is a strictly maximal left ideal. For groups we can generalize this result by determining all strictly maximal left ideals of $G[x]$.

Theorem 3. *Let G be a group. The set of all strictly maximal left ideals L of $G[x]$ is given by the following list.*

(i) $L_A := (A : 0)$, *where A is a maximal normal subgroup of G containing the commutator subgroup $[G, G]$.*

(ii) $L_{B,g} := (B : g)$, *where B is a maximal normal subgroup of G not containing $[G, G]$ and $g \in G \backslash B$ or $g = 0$.*

(iii) $L_{\chi,p} := \{\sum_i (g_i + z_i x) \in G[x] \mid \chi(\sum_i g_i) \equiv \sum_i z_i (\mathrm{mod}\, p)\}$, *where p is a prime and $\chi \in \mathrm{Hom}(G, \mathbb{Z}_p)$.*

In cases (i) *and* (iii), *$G/L \cap G$ is cyclic of prime order, while $G/L_{B,g} \cap G \cong G/B$ holds in case* (ii).

Proof. (a) First we show that a strictly maximal left ideal L is of the form (i), (ii) or (iii). If $G \subseteq L$ then $L = L_{\zeta, p}$ (where ζ is the zero map) is of type (iii) by Theorem 1. Hence we may assume that $G \nsubseteq L$. By Theorem 2, $L_c = L \cap G$ is a maximal normal subgroup of G, and G/L_c is simple. By Proposition 2(i), $G[x]/L$ is a simple group, too. By Proposition 2(ii), L is even maximal as a subgroup of $G[x]$ normalized by G. G is an $G_0[x]$-subgroup of $G[x]$ and so is $G + L$ by 2.15 of [2]. Hence $G + L = G[x]$ since L is strictly maximal and $G \nsubseteq L$. This implies that (as groups) $G[x]/L = G + L/L \cong G/L \cap G$

$=G/L_c$ holds. This gives a natural epimorphism $\pi:G[x]\to G[x]/L\to G/L_c$. Hence there is some $g\in G$ with $\pi(x)=g+L_c$. By the well-known form of π, $\pi(g)=\pi(x)$, therefore $\pi(g-x)=0$ and $g-x\in L$. Let $K:=\langle L_c\rangle+\langle g-x\rangle$, where $\langle\ \rangle$ denotes the normal closure in $G[x]$. Now the map $\phi:G[x]\to G$, $p\to p(g)$ is clearly a group epimorphism. We claim that $\operatorname{Ker}\phi=\langle x-g\rangle$. If $p\in\langle x-g\rangle$ then $p=p_0\circ(x-g)$ for some $p_0\in G_0[x]$ by Theorem 1 of [3]. Hence $p(g)=p_0(g-g)=p_0(0)=0$ and $p\in\operatorname{Ker}\phi$. Conversely, if $k=\sum(g_i+z_ix)\in\operatorname{Ker}\phi$ then $x\equiv g(\operatorname{mod}\langle g-x\rangle)$ implies $\sum(g_i+z_ix)\equiv\sum(g_i+z_ig)=k(g)=0(\operatorname{mod}\langle g-x\rangle)$, hence $k\in\langle g-x\rangle$. This shows that the map $\psi:p+\langle x-g\rangle\to p(g)$ is an isomorphism from $G[x]/\langle g-x\rangle$ onto G. If $a\in\langle L_c\rangle$ and $s\in\langle g-x\rangle$ then $\psi(a+s+\langle g-x\rangle)=a(g)+s(g)=a(g)\in\langle L_c\rangle^G=L_c$ which shows that ψ maps $K/\langle g-x\rangle=(\langle L_c\rangle+\langle g-x\rangle)/\langle g-x\rangle$ onto L_c. By the second isomorphism theorem we get

$$G[x]/K\cong G[x]/\langle g-x\rangle\Big/K/\langle g-x\rangle\cong G/L_c\cong G[x]/L$$

which together with $K\subseteq L$ shows $K=L$ (note that $G[x]/L$ is simple).

Case I: $g\in L_c$. Then, since $g-x\in L$, $x\in L$, too, and L is the normal closure of $L_c\cup\{x\}$, i.e. $L=\{\sum(g_i+z_ix)\mid\sum g_i\in L_c\}=(L_c:0)$, and we are in (i) with $A=L_c$ or in (ii) with $B=L_c$.

Case II: $g\notin L_c$. Then $L=K\subseteq(L_c:g)$. Both L and $(L_c:g)$ are strictly maximal (Theorem 2 of [3]), hence we have $L=(L_c:g)$. If $L_c\not\supseteq[G:G]$ we are in case (ii). If $L_c\supseteq[G,G]$ then G/L_c is simple and abelian, hence cyclic of prime order p.

The epimorphism $\chi:G\overset{\pi}{\to}G/L_c\overset{\alpha}{\to}\mathbb{Z}_p$ with canonical π and an isomorphism α with $\alpha(g+L_c)=-1$ has kernel L_c. Hence

$$\sum(g_i+z_ix)\in L\Leftrightarrow\sum(g_i+z_ig)\in L_c\Leftrightarrow 0=\chi\left(\sum(g_i+z_ig)\right)$$

$$=\chi\left(\sum g_i\right)+\left(\sum z_i\right)\chi(g)=\chi\left(\sum g_i\right)-\left(\sum z_i\right)\Leftrightarrow\chi\left(\sum g_i\right)\equiv\sum z_i(\operatorname{mod}p)$$

and we are in case (iii).

The assertions concerning G/L are already proved or follow easily.

(b) Conversely, each L_A, $L_{B,g}$ and $L_{\chi,p}$, as in the statement of the Theorem, are strictly maximal left ideals. It is straightforward that they are left ideals. That L_A (case (i)) and $L_{B,g}$ (case (ii)) are strictly maximal follows from Theorem 2 in [3] and its proof. So consider $L_{\chi,p}$. Clearly $L_{\chi,p}\neq G[x]$. Suppose that the $G_0[x]$-subgroup U is strictly bigger than $L_{\chi,p}$ and let $u\in U\backslash L_{\chi,p}$, $u=\sum(g_i+z_ix)$. Then $\chi(\sum g_i)\not\equiv\sum z_i\ (\operatorname{mod}p)$. Let $k\in\{1,2,\dots,p-1\}$ be such that $\chi(\sum g_i)\equiv(\sum z_i)+k\ (\operatorname{mod}p)$. There exist $m,n\in\mathbb{Z}$ with $mk+np=1$. By the definition of $L_{\chi,p}$ and since $mk\equiv1\ (\operatorname{mod}p)$, $x+mu\in L_{\chi,p}$. Since also $-mu\in U$ we know that $x=(x+mu)-mu\in U$. If $g\in G$, let $r\in\{0,1,\dots,p-1\}$ be such that $\chi(g)\equiv r\ (\operatorname{mod}p)$. Then $g+rx\in L_{\chi,p}\subset U$ and $rx\in U$, hence $g\in U$. Therefore $G\cup\{x\}\subseteq U$ and so $U=G[x]$.

The proof of the preceding theorem also shows

Corollary 2. *All strictly maximal left ideals of $G[x]$ are given by either one of the following two lists*:

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(i) $(G\cup\{px\})^{G[x]}$, *p a prime, and* $(B\cup\{x-g\})^{G[x]}$, $g\in G$, *B maximal normal in G.*

(ii) $(B:g)$ *with* $g\in G\backslash B$ *or* $g=0$ *and B maximal normal in G, and* $L_{\chi,p}$, *p a prime.*

This enables us to compute the Jacobson-type radicals of $G[x]$. Recall that for a near-ring N with identity, $J_{1/2}(N)$ is defined as the intersection of all maximal left ideals of N, while $J_2(N)$ is the intersection of all strictly maximal ones. $J_0(N)=(J_{1/2}(N):N)$ and $J_1(N)=J_2(N)$ (since N has an identity). In the general case, we have $J_0(N)\subseteq J_{1/2}(N)\subseteq J_1(N)\subseteq J_2(N)$.

Theorem 4. $J_1(G[x])=J_2(G[x])=G^{G[x]}\cap(\beta(G):G)=\{\sum(g_i+z_ix)\,|\,\sum z_i=0$ *and for all* $g\in G$ $\sum(g_i+z_ig)\in\beta(G)\}$.

Proof. Let $\mathcal{M}$ be the collection of all maximal normal subgroups of G and ζ the zero map. From Theorem 3 we get with $G'=[G,G]$:

$$J_2(G[x])=\bigcap_{\substack{G'\subseteq A\\ A\in\mathcal{M}}}(A:0)\cap\bigcap_{\substack{G'\nsubseteq B\\ B\in\mathcal{M}}}\ \bigcap_{\substack{g=0\text{ or}\\ g\notin B}}(B:g)\cap\bigcap_p\bigcap_{\chi\neq\zeta}L_{\chi,p}\cap\bigcap_p L_{\zeta,p}$$

$$=\Bigg(\bigcap_{\substack{G'\subseteq A\\ A\in\mathcal{M}}}(A:0)\Bigg)\cap\bigcap_{\substack{G'\subseteq B\\ B\in\mathcal{M}}}(B:(G\backslash B)\cup\{0\})\cap\bigcap_p\bigcap_{\chi\neq\zeta}L_{\chi,p}\cap\bigcap_p L_{\zeta,p}.$$

Now if f is in the second block of the intersection and $g\in B$ then $f\circ 0\in B$, hence $f\circ g=(f-f\circ 0)\circ g\in B$, since $f-f\circ 0\in G_0[x]$ and B is normal (see Proposition 2(ii)). Hence

$$\bigcap_{\substack{G'\nsubseteq B\\ B\in\mathcal{M}}}(B:(G\backslash B)\cup\{0\})=\Bigg(\Bigg(\bigcap_{\substack{G'\nsubseteq B\\ B\in\mathcal{M}}}B\Bigg):G\Bigg).$$

The first two intersections give $(\beta(G):G)$. Moreover, we get $\bigcap_p L_{\zeta,p}=\{\sum(g_i+z_ix)\,|\,\sum z_i=0\}\supseteq G^{G[x]}$. That this inclusion is in fact an equality can be seen by the same argument as for equations (3) and (4) at the beginning of this paper. Finally, take $q=\sum(g_i+z_ix)\in(\beta(G):G)\cap G^{G[x]}$. Then $\sum g_i=q\circ 0\in\beta(G)$. If χ is in $\mathrm{Hom}(G,\mathbb{Z}_p)$, $\chi\neq\zeta$, then χ is an epimorphism and $G/\ker\chi\cong\mathbb{Z}_p$. Hence $\ker\chi$ is maximal and normal in G, $pG\subseteq\ker\chi$, and $\beta(G)\subseteq\ker\chi$. Hence $\chi(\sum g_i)=0=\sum z_i$, since $q\in G^{G[x]}$. Therefore $q\in L_{\chi,p}$ and we can forget about the third part of the intersection. Hence the result (in the elegant and the explicit form.).

Examples.

(i) Since $\beta(\mathbb{Z})=\{0\}$, we get

$$J_2(\mathbb{Z}[x])=(0:\mathbb{Z})\cap\mathbb{Z}^{\mathbb{Z}[x]}=\{\textstyle\sum(g_i+z_ix)\,|\,\sum z_i=\sum g_i=0\}.$$

(ii) Let G be the direct sum of simple groups. Then similarly

$$J_2(G[x])=(0:G)\cap\{\textstyle\sum(g_i+z_ix)\,|\,\sum z_i=0\}.$$

(iii) Now let G be the group Z_p^∞. Then $\beta(G)=G$ and

$$J_2(G[x])=(G:G)\cap G^{G[x]}=\{\sum(g_i+z_ix)\big|\sum z_i=0\}.$$

(iv) The arguments in the proof of Theorem 4 showed that $(B:0)=(B:B)$ holds for all normal subgroups B of G. But in general, $(\beta(G):G)\neq(\beta(G):0)$. Let, for instance, $g\in G\backslash\beta(G)$. Then $g+x-g\in(\beta(G):0)$, but $(g+x-g)\circ g=g\notin\beta(G)$, whence $g+x-g\notin(\beta(G):G)$.

Concerning the other two Jacobson-type radicals J_0 and $J_{1/2}$ of [**2**] we get from 5.2 and 5.35 of [**2**]

Corollary 4. *$J_0(G)=(\beta(G[x]):G[x])$ and $J_{1/2}(G)=\beta(G[x])$.*

We close this topic with some remarks on $G[x]$.

(i) All $G[x]$-groups of type 2 arise as $G[x]/L$ for L a strictly maximal left ideal. If $G[x]/L$ is cyclic of prime order then x acts as the identity and G induces the constant maps. Hence $G[x]/(0:G[x]/L)\cong\mathbb{Z}_p\times\mathbb{Z}_p$, where the first $\mathbb{Z}_p$ is generated by the image of x and the second $\mathbb{Z}_p$ is given by the constant maps. If $G[x]/L$ is not of this kind, it is isomorphic to the non-abelian simple group $G/L\cap G$. Then $G_0[x]$ induces the near-ring $I(G/L\cap G)$ generated by all inner automorphisms of $G/L\cap G$. Adding the constants we get $G[x]/(0:G/L\cap G)\cong I(G/L\cap G)+G/L\cap G$. Observe that by 7.46(c) of [**2**], $I(G/L\cap G)=M_0(G/L\cap G)$ if $G/L\cap G$ is finite.

(ii) The $G[x]$-groups of type 0 which are not of type 2 are induced by maximal normal subgroups L of $G[x]$ where $G+L/L\subset G[x]/L$. The latter creature is simple. In this case, $G[x]/(0:G[x]/L)\cong(R,S)+G/L\cap G$, where (R,S) is the *d.g.* near-ring generated by the inner automorphisms of $G[x]/L$ induced by $G/L\cap G$. Observe that $G[x]/L$ need not be finite, nor need $G/L\cap G$ be simple.

Life becomes very simple if we change from the variety of all groups to $\mathscr{A}$, the one of all abelian groups. In this case, for all $G\in\mathscr{A}$ we have other polynomial algebras, namely $G^{\mathscr{A}}[x]=\{g+zx\big|g\in G,z\in\mathbb{Z}\}=:G]x[$.

Proposition 4. *For $G\in\mathscr{A}$, $(G]x[,+,\circ)$ is an abstract affine near-ring.*

The proof is easy and hence omitted.
Theorem 9.77 of [**2**] gives us the following

Corollary 5. *For $G\in\mathscr{A}$, all radicals of $G]x[$ are equal to $\beta(G)$, which is the Frattini subgroup in this case.*

One knows from universal algebra (see e.g. [**1**]) that $G]x[$ must be a factor near-ring of $G[x]$ if $G\in\mathscr{A}$. In fact:

Theorem 5. *Let G be abelian.*

(i) $\vartheta: G[x] \to G]x[: \sum(g_i + z_i x) \to (\sum g_i) + (\sum z_i)x$ *is a near-ring epimorphism.*

(ii) $G[x]/\{\sum g_i + z_i x \mid \sum g_i = 0 \wedge \sum z_i = 0\} \cong G]x[$.

Proof. (i) follows from [**1**] (it is not trivial that ϑ is well-defined!) and from this we get (ii) by the homomorphism theorem.

Example. $\mathbb{Z}[x]/(0:\mathbb{Z}) \cong \mathbb{Z}]x[$.

Remarks.

(i) There is a striking similarity between Theorem 4 and (ii) in Theorem 5. It is, however, unknown how far these results are related.

(ii) One may switch to the variety of R-modules (see [**4**]). One then gets, for an R-module M, a polynomial algebra (near-ring and R-module at the same time) $M_R[x] = \{m + rx \mid m \in M, r \in R\}$. $M_R[x]$ is again an abstract affine near-ring (in the paper [**4**] we show that all abstract near-rings are isomorphic (!) to some $M_R[x]$). Hence all radicals are equal to $J(M) + J(R)x$, where $J(M)$ is the intersection of all maximal R-submodules of M and $J(R)$ is the Jacobson-radical of R.

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