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WEAK CONTINUITY OF THE COMPLEX k-HESSIAN OPERATORS WITH RESPECT TO LOCAL UNIFORM CONVERGENCE

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Abstract

In this paper, we study the properties of k-plurisubharmonic functions defined on domains in $\mathbb{C}^n$. By the monotonicity formula, we give an alternative proof of the weak continuity of complex k-Hessian operators with respect to local uniform convergence.

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1. Introduction

Employing an approach developed by Trudinger and Wang [16], in this paper we shall give an alternative proof of weak continuity of the complex k-Hessian measures with respect to local uniform convergence. The same technique was also used to obtain weak continuity results of Hessian measures in the Heisenberg setting [9, 11, 12, 18].

The complex Monge–Ampère operator is naturally defined for C^2 plurisubharmonic functions. Unlike the real Monge–Ampère case, it is not clear whether the complex Monge–Ampère operator can be defined for all plurisubharmonic functions. How to extend the domain of definition of the complex Monge–Ampère operator is an old topic. In [7], Chern *et al.* established an estimate for the complex Monge–Ampère operator which turned out to be very useful. Thanks to this and similar inequalities, the complex Monge–Ampère operator could be shown to admit extensions to wider classes of functions. Using Chern *et al.* estimates, Bedford and Taylor [1] defined the complex Monge–Ampère operator on continuous plurisubharmonic functions. In the same paper, they defined it for locally bounded plurisubharmonic functions by an induction argument. Later, Bedford and Taylor [2] proved the weak continuity of the complex Monge–Ampère operator on decreasing sequences of locally bounded plurisubharmonic functions. For more extensions of the definition of the complex

Monge–Ampère operator as well as the continuity property, we refer to [3, 4, 6, 8, 10, 13, 15, 19] and the references therein.

For an $n \times n$ Hermitian matrix $A = (a_{ij})$, the operator F_k on A is defined as the sum of all $k \times k$ principal minors of A, that is

$$F_k(A) = \frac{1}{k!} \sum \delta \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} a_{i_1 \bar{j}_1} a_{i_2 \bar{j}_2} \cdots a_{i_k \bar{j}_k},$$

where

$$\delta \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = \begin{cases} 1 & \text{if } j_1 j_2 \cdots j_k \text{ is an even permutation of } i_1 i_2 \cdots i_k, \\ -1 & \text{if } j_1 j_2 \cdots j_k \text{ is an odd permutation of } i_1 i_2 \cdots i_k, \\ 0 & \text{otherwise,} \end{cases}$$

is the generalised Kronecker delta symbol and all the summation indices run from 1 to n.

Let $\Omega \subset \mathbb{C}^n$ be a domain and $u \in C^2(\Omega)$ be any real-valued function. Using the standard notation

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j}\Big), \quad \frac{\partial}{\partial z_{\bar{k}}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + \sqrt{-1}\frac{\partial}{\partial y_k}\Big),$$

we write

$$u_j = \frac{\partial u}{\partial z_j}, \quad u_{\bar{k}} = \frac{\partial u}{\partial z_{\bar{k}}}, \quad u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial z_{\bar{k}}}.$$

The complex k-Hessian operator F_k on u is defined by

$$F_k[u] = F_k((u_{j\bar{k}})).$$

It is easy to see that $F_k[u]$ is real-valued. A function $u \in C^2(\Omega)$ is said to be k-plurisubharmonic if $F_j[u] \geq 0$ for $j = 1, \ldots, k$. A function $u \in C(\Omega)$ is said to be k-plurisubharmonic if there exists a sequence $\{u^m\} \subset C^2(\Omega)$ such that u^m is k-plurisubharmonic and converges to u locally uniformly. Let $\Phi^k(\Omega)$ denote the class of continuous k-plurisubharmonic functions.

Let dV be the usual Lebesgue measure in $\mathbb{C}^n$. The main result of this paper is the following theorem.

THEOREM 1.1. *For any $u \in \Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ such that*

$$\mu_k[u](e) = \int_e F_k[u]\, dV,$$

if $u \in C^2(\Omega)$ and e is a Borel subset of Ω. Moreover, if $\{u^m\} \subset \Phi^k(\Omega)$, $u \in \Phi^k(\Omega)$, and $u^m \to u$ locally uniformly in Ω, then the corresponding measures $\mu_k[u^m] \to \mu_k[u]$ weakly, that is,

$$\int_\Omega g\, d\mu_k[u^m]\, dV \to \int_\Omega g\, d\mu_k[u]\, dV$$

for all $g \in C(\Omega)$ with compact support in Ω.

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Section 2 is devoted to the proof of Theorem 1.1. The monotonicity formula, that is, Lemma 2.2, plays a central role in our proof. In some sense, it is a substitute for the estimates of Chern *et al.* For completeness we also provide a detailed proof of Theorem 1.1, rather than referring to Trudinger and Wang [16] for similar arguments. We also remark that Theorem 1.1 does not extend to almost everywhere convergence (see Cegrell [5]) unlike the real case in Trudinger and Wang [17].

2. Weak continuity of the complex k-Hessian operators

First we state the divergence theorem in complex variables.

LEMMA 2.1. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain with C^1 boundary. If $u \in C^1(\bar{\Omega})$ is a real-valued function, then we have*

$$\int_\Omega \frac{\partial u}{\partial z_j}\, dV = \frac{1}{2}\int_{\partial\Omega} u\bar{\gamma}_j\, dS, \qquad \int_\Omega \frac{\partial u}{\partial \bar{z}_j}\, dV = \frac{1}{2}\int_{\partial\Omega} u\gamma_j\, dS,$$

where $\gamma_j = r_j + \sqrt{-1}r_{n+j}$ and $\boldsymbol{r} = (r_1, r_2, \ldots, r_{2n})$ is the unit outer normal of $\partial\Omega$.

PROOF. By the divergence theorem in Euclidean space,

$$\begin{aligned}\int_\Omega \frac{\partial u}{\partial z_j}\, dV &= \int_\Omega \Big(\frac{1}{2}\frac{\partial u}{\partial x_j} - \frac{\sqrt{-1}}{2}\frac{\partial u}{\partial y_j}\Big)\, dV \\ &= \frac{1}{2}\int_{\partial\Omega} ur_j\, dS - \frac{\sqrt{-1}}{2}\int_{\partial\Omega} ur_{n+j}\, dS \\ &= \frac{1}{2}\int_{\partial\Omega} u\bar{\gamma}_j\, dS.\end{aligned}$$

The second formula can be proved in the same way. □

Next, the monotonicity formula for complex k-Hessian operators is a straightforward modification of the real case in Trudinger and Wang [16].

LEMMA 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a C^1 bounded domain. Let $u, v \in \Phi^k(\Omega) \cap C^2(\bar{\Omega})$ satisfy $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. Define*

$$w = w(z, s) \triangleq (1 - s)u(z) + sv(z), \quad z \in \bar{\Omega},$$

for $s \in [0, 1]$. Then the function

$$s \mapsto \int_\Omega F_k[w(z, s)]\, dV$$

is nondecreasing on $[0, 1]$. In particular,

$$\int_\Omega F_k[u]\, dV \leq \int_\Omega F_k[v]\, dV.$$

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PROOF. Using the null Lagrangian property of the complex k-Hessian operator

$$\sum_{l=1}^{n} \frac{\partial}{\partial z_j}\Big(\frac{F_k[u]}{\partial u_{j\bar{l}}}\Big) = 0 \quad \text{for } j = 1, \dots, n$$

(see Reilly [14] for a similar argument in the real Hessian case), we then obtain by Lemma 2.1,

$$\begin{aligned}\frac{d}{ds}\int_\Omega F_k[w]\,dV &= \int_\Omega \frac{\partial F_k[w]}{\partial w_{j\bar{l}}}(w_s)_{j\bar{l}}\,dV\\ &= \frac{1}{2}\int_{\partial\Omega} \frac{\partial F_k[w]}{\partial w_{j\bar{l}}}(w_s)_{\bar{l}}\bar{\gamma}_j\,dS,\end{aligned}$$

where w_s is the partial derivative $\partial w/\partial s = v - u$. If we denote $\nabla u = (u_1, \dots, u_n)$, then $(w_s)_{\bar{l}} = \gamma_l \cdot |\nabla w_s|$. Hence by ellipticity,

$$\frac{d}{ds}\int_\Omega F_k[w]\,dV = \frac{1}{2}\int_{\partial\Omega} \frac{\partial F_k[w]}{\partial w_{j\bar{l}}}\gamma_l\bar{\gamma}_j|\nabla w_s|\,dS \geq 0.$$

This concludes the proof. □

LEMMA 2.3. *Let $u^1, \dots, u^m \in \Phi^k(\Omega)$ and f be a convex function on $\mathbb{R}^m$. Assume further f is nondecreasing in each variable. If we define $w = f(u^1, \dots, u^m)$, then $w \in \Phi^k(\Omega)$.*

PROOF. Define

$$\Gamma_k = \{A \in \mathscr{S}_n : F_j(A) \geq 0,\ j = 1, \dots, k\},$$

where $\mathscr{S}_n$ is the set of all $n \times n$ Hermitian matrices. It is easy to see that Γ_k is a convex cone. Therefore, the linear combination of k-plurisubharmonic functions with nonnegative coefficients is also k-plurisubharmonic. For the general case, we may assume $u^1, \dots, u^m \in \Phi^k(\Omega) \cap C^2(\Omega)$ and $f \in C^2(\mathbb{R}^m)$. By straightforward calculation,

$$w_{j\bar{k}} = \frac{\partial f}{\partial u^p}u^p_{j\bar{k}} + \frac{\partial^2 f}{\partial u^p \partial u^q}u^p_j u^q_{\bar{k}}.$$

Recalling the assumptions we imposed on f, we have $(w_{j\bar{k}}) \in \Gamma_k$, that is, $w \in \Phi^k(\Omega)$. □

LEMMA 2.4. *If $u \in \Phi^k(\Omega) \cap C^2(\Omega)$, then for any subdomain $\Omega' \Subset \Omega$,*

$$\int_{\Omega'} F_k[u]\,dV \leq C(\operatorname{osc}_\Omega u)^k,$$

where C is a constant depending on n, k, Ω' and Ω.

PROOF. Let $B_R = B_R(z_0) \Subset \Omega$ be a ball with centre z_0 and radius R. For $0 < \sigma < 1$, $B_{\sigma R}$ denotes the concentric ball of radius σR. Without loss of generality, we assume $z_0 = 0$ and $u < -\varepsilon$ in B_R for some given positive constant ε. For $z \in B_R$, define

$$\begin{aligned}\psi(z) &= \frac{\inf_{B_R} u}{1 - \sigma^2}\Big(1 - \frac{|z|^2}{R^2}\Big),\\ w(z) &= \max\{u, \psi\}.\end{aligned}$$

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It is easy to see that $w \geq \psi$ in $B_{\sigma R}$ and $w = \psi$ on ∂B_R. In order to apply Lemma 2.2 to the functions w and ψ, we replace w by its mollification $w_h = f_h(u, \psi)$ for $h > 0$ small, where

$$f_h(x) = h^{-2} \int_{\mathbb{R}^2} \zeta\Big(\frac{x-y}{h}\Big) \max\{y_1, y_2\}\, dy,$$

and

$$\zeta(x) = \begin{cases} \dfrac{1}{C} e^{-1/(1-|x|^2)} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

is the usual mollifier with $\int_{\mathbb{R}^2} \zeta(x)\, dx = 1$.

For sufficiently small h, it follows by Lemma 2.2, that

$$\begin{aligned} \int_{B_{\sigma R}} F_k[u] dV = \int_{B_{\sigma R}} F_k[w_h]\, dV \leq \int_{B_R} F_k[w_h] dV \leq \int_{B_R} F_k[\psi]\, dV \\ = \binom{n}{k} \Big(\frac{-2 \inf_{B_R} u}{1-\sigma^2}\Big)^k \omega_n R^{n-2k}. \end{aligned}$$

By letting $\varepsilon \to 0$ and covering Ω' with balls, we obtain the desired estimates. □

We are now at a stage to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose $u \in \Phi^k(\Omega)$, $\{u^m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ and $u^m \to u$ locally uniformly in Ω. By Lemma 2.4, the integrals

$$\int_{\Omega'} F_k[u^m]\, dV$$

are uniformly bounded for any subdomain $\Omega' \Subset \Omega$. Hence there is a subsequence $\{F_k[u^{m_p}]\}$ that converges weakly to a Borel measure $\mu_k[u]$. The main task is to prove that the measure $\mu_k[u]$ is uniquely determined by the function u. Assume there exist two sequences $\{u^m\}, \{v^m\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ which both converge to u locally uniformly, but the corresponding sequences $\{F_k[u^m]\}$ and $\{F_k[v^m]\}$ weakly converge to Borel measures ν_1 and ν_2, respectively. Let $B_R = B_R(z_0) \Subset \Omega$ be a ball with centre z_0 and radius R. Fix some $\sigma \in (0, 1)$. Let $\eta \in C^2(\bar{B}_R)$ be a convex function satisfying $\eta = 0$ in $B_{\sigma R}$, $\eta = 1$ on ∂B_R. For fixed $\varepsilon > 0$, it then follows from the uniform convergence of $\{u^m\}$ and $\{v^m\}$ on $\bar{B}_R$, that

$$-\frac{\varepsilon}{2} \leq u^m - v^m \leq \frac{\varepsilon}{2} \quad \text{on } \bar{B}_R,$$

for sufficiently large m. Hence,

$$u^m + \frac{\varepsilon}{2} \leq v^m + \varepsilon\eta \quad \text{on } \partial\bar{B}_R.$$

Define

$$\Omega_m = \Big\{g \in B_R \mid u^m + \frac{\varepsilon}{2} > v^m + \varepsilon\eta\Big\}.$$

Without loss of generality, we may assume that $\partial\Omega_m$ is sufficiently smooth so that from Lemma 2.2,

$$\int_{\Omega_m} F_k[u^m]\, dV \leq \int_{\Omega_m} F_k[v^m + \varepsilon\eta]\, dV.$$

Recalling Lemma 2.4 and expanding $F_k[v^m + \varepsilon\eta]$ as the sum of mixed k-Hessian operators,

$$\int_{\Omega_m} F_k[v^m + \varepsilon\eta]\, dV \leq \int_{\Omega_m} F_k[v^m]\, dV + \varepsilon C,$$

where the constant C depends on n, k, η, u and B_R.

Since $\eta = 0$ in $B_{\sigma R}$, by the definition of Ω_m, we have $B_{\sigma R} \subset \Omega_m$, and hence

$$\int_{B_{\sigma R}} F_k[u^m]\, dV \leq \int_{B_R} F_k[v^m]\, dV + \varepsilon C.$$

Letting $\varepsilon \to 0$, $\sigma \to 1$ and $m \to \infty$, we then obtain

$$\nu_1(B_R) \leq \nu_2(B_R).$$

By interchanging $\{u^m\}$ and $\{v^m\}$, we have $\nu_1(B_R) = \nu_2(B_R)$, whence $\nu_1 = \nu_2$. This completes the proof of the theorem, as the above argument shows that $\mu_k[u]$ is well defined and the mapping μ_k is weakly continuous from $\Phi^k(\Omega)$ to the space of locally finite Borel measures in Ω. □

Let us conclude with two corollaries.

Corollary 2.5. *Let $u \in \Phi^k(\Omega)$. Then for any subdomain $\Omega' \Subset \Omega$,*

$$\mu_k[u](\Omega') \leq C(\mathrm{osc}_\Omega u)^k,$$

where C is a constant depending on n, k, Ω and Ω'.

Corollary 2.6. *Let $u, v \in \Phi^k(\Omega) \cap C(\bar{\Omega})$ satisfy $u = v$ on $\partial\Omega$. Then $u \geq v$ in Ω if and only if the corresponding measures μ_k satisfy $\mu_k[u] \leq \mu_k[v]$ in Ω.*

References

[1] E. Bedford and B. A. Taylor, 'The Dirichlet problem for a complex Monge–Ampère equation', *Invent. Math.* **37**(1) (1976), 1–44.

[2] E. Bedford and B. A. Taylor, 'A new capacity for plurisubharmonic functions', *Acta Math.* **149**(1–2) (1982), 1–40.

[3] E. Bedford and B. A. Taylor, 'Fine topology, Šilov boundary, and $(dd^c)^n$', *J. Funct. Anal.* **72**(2) (1987), 225–251.

[4] Z. Blocki, 'The domain of definition of the complex Monge–Ampère operator', *Amer. J. Math.* **128**(2) (2006), 519–530.

[5] U. Cegrell, 'Discontinuité de l'opérateur de Monge–Ampère complexe', *C. R. Acad. Sci. Paris Sér. I Math.* **296**(21) (1983), 869–871.

[6] U. Cegrell, 'Sums of continuous plurisubharmonic functions and the complex Monge–Ampère operator in $\mathbb{C}^n$', *Math. Z.* **193**(3) (1986), 373–380.

[7] S. S. Chern, H. I. Levine and L. Nirenberg, 'Intrinsic norms on a complex manifold', in: *Global Analysis (Papers in Honor of K. Kodaira)* (University Tokyo Press, Tokyo, 1969), 119–139.

[8] J.-P. Demailly, 'Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines', *Mém. Soc. Math. France (N.S.)*(19) (1985).

[9] N. Garofalo and F. Tournier, 'New properties of convex functions in the Heisenberg group', *Trans. Amer. Math. Soc.* **358**(5) (2006), 2011–2055.

[10] V. Guedj and A. Zeriahi, 'The weighted Monge–Ampère energy of quasiplurisubharmonic functions', *J. Funct. Anal.* **250**(2) (2007), 442–482.

[11] C. E. Gutiérrez and A. Montanari, 'Maximum and comparison principles for convex functions on the Heisenberg group', *Comm. Partial Differential Equations* **29**(9–10) (2004), 1305–1334.

[12] C. E. Gutiérrez and A. Montanari, 'On the second order derivatives of convex functions on the Heisenberg group', *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**(2) (2004), 349–366.

[13] C. O. Kiselman, 'Sur la définition de l'opérateur de Monge–Ampère complexe', in: *Complex Analysis (Toulouse, 1983)*, Lecture Notes in Mathematics, 1094 (Springer, Berlin, 1984), 139–150.

[14] R. C. Reilly, 'On the Hessian of a function and the curvatures of its graph', *Michigan Math. J.* **20** (1973), 373–383.

[15] N. Sibony, 'Quelques problèmes de prolongement de courants en analyse complexe', *Duke Math. J.* **52**(1) (1985), 157–197.

[16] N. S. Trudinger and X.-J. Wang, 'Hessian measures. I', *Topol. Methods Nonlinear Anal.* **10**(2) (1997), 225–239.

[17] N. S. Trudinger and X.-J. Wang, 'Hessian measures. II', *Ann. of Math.* (2) **150**(2) (1999), 579–604.

[18] N. S. Trudinger and W. Zhang, 'Hessian measures on the Heisenberg group', *J. Funct. Anal.* **264**(10) (2013), 2335–2355.

[19] Y. Xing, 'Continuity of the complex Monge–Ampère operator', *Proc. Amer. Math. Soc.* **124**(2) (1996), 457–467.

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