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Bernstein–Sato polynomials of arbitrary varieties

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Abstract

We introduce the notion of the Bernstein–Sato polynomial of an arbitrary variety (which is not necessarily reduced nor irreducible) using the theory of V-filtrations of M. Kashiwara and B. Malgrange. We prove that the decreasing filtration by multiplier ideals coincides essentially with the restriction of the V-filtration. This implies a relation between the roots of the Bernstein–Sato polynomial and the jumping coefficients of the multiplier ideals, and also a criterion for rational singularities in terms of the maximal root of the polynomial in the case of a reduced complete intersection. These are generalizations of the hypersurface case. We can calculate the polynomials explicitly in the case of monomial ideals.

Introduction

The notion of the Bernstein–Sato polynomial (i.e. b-function) for a function was introduced independently by Bernstein [Ber72] and Sato and Shintani [SS72]. This theory was then developed by Björk [Bjö79], Malgrange [Mal75], and by members of the Sato school (Kashiwara [Kas76/77] and Yano [Yan78] among others). Related to the theory of vanishing cycles of Deligne [Del73], it further led to the theory of V-filtrations of Kashiwara [Kas83] and Malgrange [Mal83]. It is well known and it is easy to show that the b-function depends only on the hypersurface defined by the function. Motivated by our previous work [BS05] related to multiplier ideals [Laz04], we needed to generalize the notion of b-function to the case of arbitrary subvarieties.

Let Z be a (not necessarily reduced nor irreducible) complex algebraic variety embedded in a smooth affine variety X. Let $f_1, \dots, f_r$ be nonzero generators of the ideal of Z (i.e. $f_j \neq 0$). Let $\mathcal{D}_X$ be the sheaf of linear differential operators on X. It acts naturally on $\mathcal{O}_X[\prod_i f_i^{-1}, s_1, \dots, s_r] \prod_i f_i^{s_i}$, where the s_i are independent variables. We define a $\mathcal{D}_X$-linear action of t_j on it by $t_j(s_i) = s_i + 1$ if $i = j$, and $t_j(s_i) = s_i$ otherwise. In particular, $t_j \prod_i f_i^{s_i} = f_j \prod_i f_i^{s_i}$, and the action of t_j is bijective. Let $s_{i,j} = s_i t_i^{-1} t_j$, and $s = \sum_i s_i$. The Bernstein–Sato polynomial (i.e. the b-function) $b_f(s)$ of $f := (f_1, \dots, f_r)$ is defined to be the monic polynomial of the lowest degree in s satisfying the relation

$$b_f(s) \prod_i f_i^{s_i} = \sum_{k=1}^{r} P_k t_k \prod_i f_i^{s_i},$$

where the P_k belong to the ring generated by $\mathcal{D}_X$ and the $s_{i,j}$; see §§ 2.4 and 2.10 for other formulations. Note that if we require instead $P_k \in \mathcal{D}_X[s_1, \dots, s_r]$, then there are examples in which there is no nonzero such $b_f(s)$ (see § 4.5). The above definition gives a natural generalization of the b-function in the hypersurface case (i.e. $r = 1$, see [Ber72]), and this normalization of the b-function is the same as in [BS05, Kas76/77, Sai93]. Since our definition of the b-function is closely related to the V-filtration of Kashiwara [Kas83] and Malgrange [Mal83], its existence and the rationality of its roots follow easily from their theory (see also [Gyo93]). We can show, moreover, that the

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denominators of the roots of the b-function are related to the multiplicities of the divisor obtained by an embedded resolution of (X, Z), see Proposition 3.10.

By § 2.4 below, this $b_f(s)$ coincides (up to a shift of variable) with the polynomial b_L of the minimal degree satisfying the relation in [Sab87b, I, 3.1.1] in the algebraic setting (see also [Gyo93, 2.13]), if $L(s) = s_1 + \cdots + s_r$ and if $\mathcal{M}, U$ in *loc. cit.* are chosen appropriately; see § 5.2 below. This polynomial was used to prove the main theorem in [Sab87b, I]. However, $b_f(s)$ is slightly different from a polynomial in [Sab87b, II, Proposition 1.1], because our definition requires certain additional binomial polynomials as in § 2.10 below, and Theorem 2 does not hold for that polynomial without the additional term; see § 5.4 below. In [Sab87b], Sabbah proved the existence of nonzero polynomials of *several variables* which satisfy a functional equation similar to the above one, see also [Bah05, CJG05, Gyo93]. However, its relation with $b_f(s)$ seems to be quite nontrivial, see § 5.1 below.

Let $b_Z(s) = b_f(s - r')$ with $r' = \operatorname{codim}_X Z$. This normalization is the same as in [Ber72]. Using the V-filtration of Kashiwara and Malgrange, we can show that our $b_f(s)$ is independent of the choice of a system of generators $f = (f_1, \ldots, f_r)$ if Z and $\dim X$ are fixed (but r can vary), and that $b_Z(s)$ depends only on Z; see Theorem 2.5. This was rather surprising because the assertion does not hold for the polynomial in [Sab87b, II, 1.1]. If Z is not affine, the b-function can be defined to be the least common multiple of the local b-functions (shifted appropriately if Z is not equidimensional). For $g \in \mathcal{O}_X$, we similarly define the b-function $b_{f,g}(s)$ with $\prod_i f_i^{s_i}$ replaced by $g \prod_i f_i^{s_i}$.

Our first main theorem concerns the multiplier ideals and the V-filtration. For a positive rational number α, the multiplier ideal sheaf $\mathcal{J}(X, \alpha Z)$ is defined by taking an embedded resolution of (X, Z) (see [Laz04] and also § 3.1 below) or using the local integrability of $|g|^2/(\sum_i |f_i^2|)^\alpha$ for $g \in \mathcal{O}_X$ (see [Nad90]). This gives a decreasing sequence of ideals, and there are positive rational numbers $0 < \alpha_1 < \alpha_2 < \cdots$ such that $\mathcal{J}(X, \alpha_j Z) = \mathcal{J}(X, \alpha Z) \neq \mathcal{J}(X, \alpha_{j+1} Z)$ for $\alpha_j \leqslant \alpha < \alpha_{j+1}$ $(j \geqslant 0)$ where $\alpha_0 = 0$ and $\mathcal{J}(X, \alpha_0 Z) = \mathcal{O}_X$. These α_j for $j > 0$ are called the jumping coefficients, and the minimal jumping coefficient α_1 is called the log-canonical threshold.

We will denote by V the filtration on $\mathcal{O}_X$ induced by the V-filtration of Kashiwara [Kas83] and Malgrange [Mal83] along Z.

Theorem 1. *We have $V^\alpha \mathcal{O}_X = \mathcal{J}(X, \alpha Z)$ if α is not a jumping coefficient. In general, $\mathcal{J}(X, \alpha Z) = V^{\alpha+\varepsilon}\mathcal{O}_X$ and $V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z)$ for any $\alpha \in \mathbb{Q}$, where $\varepsilon > 0$ is sufficiently small.*

This implies the following relation between the roots of the b-function and the jumping coefficients.

Theorem 2. *The log-canonical threshold of (X, Z) coincides with the smallest root α'_f of $b_f(-s)$ (in particular, $\alpha'_f > 0$), and any jumping coefficients of (X, Z) in $[\alpha'_f, \alpha'_f + 1)$ are roots of $b_f(-s)$.*

In the case when Z is a reduced complete intersection, we have an analogue of the adjoint ideal of a reduced divisor (see [EL97, MT80, Vaq94] and also [BS05, 3.7]), and the maximal root $-\alpha'_f$ of $b_f(s)$ can be used for a criterion of rational singularities as follows.

Theorem 3. *Assume Z is a reduced complete intersection of codimension r in X. Let $\pi : \widetilde{Z} \to Z$ be a resolution of singularities, and set $\widetilde{\omega}_Z = \pi_* \omega_{\widetilde{Z}}$ where $\omega_{\widetilde{Z}}$ denotes the dualizing sheaf. Then there is a coherent ideal $\mathcal{J}(X, rZ)'$ of $\mathcal{O}_X$ such that $\mathcal{J}(X, rZ) \subset \mathcal{J}(X, rZ)' \subset \mathcal{J}(X, (r - \varepsilon)Z)$ for $0 < \varepsilon \ll 1$ and*

$$\omega_X \otimes (\mathcal{O}_X / \mathcal{J}(X, rZ)') = \omega_Z / \widetilde{\omega}_Z.$$

In particular, $\omega_X \otimes (\mathcal{J}(X, (\alpha - \varepsilon)Z) / \mathcal{J}(X, \alpha Z))$ is isomorphic to a subquotient of $\omega_Z / \widetilde{\omega}_Z$ for $\alpha < r$ and $0 < \varepsilon \ll 1$.

Theorem 4. *With the assumption of Theorem 3, Z has at most rational singularities if and only if $\alpha'_f = r$ and its multiplicity is* 1.

The above theorems generalize the corresponding results in the hypersurface case (see [BS05, ELSV04, Kas76/77, Kol97, Sai93, Ste88]). In Theorem 4, r is always a root of $b_f(-s)$ by restricting to the smooth points of Z. Let α_f denote the minimal root of $b_f(-s)/(-s+r)$. Then the criterion for rational singularities in Theorem 4 is equivalent to the condition $\alpha_f > r$. In the hypersurface isolated singularity case, it is known that α_f coincides with the Arnold exponent.

In the case of rational singularities, Theorem 3 implies the following.

Corollary 1. *Under the assumption of Theorem 3, assume further that Z has at most rational singularities or, more generally, $\alpha'_f = r$. Then the jumping coefficients are the integers of at least r, and $\mathcal{J}(X, jZ) = I_Z^{j-r+1}$ for $j \geqslant r$ where I_Z is the ideal sheaf of Z.*

From Theorem 1 (together with (2.1.3) below) we also deduce the following description of multiplier ideals in terms of b-functions.

Corollary 2. *A function $g \in \mathcal{O}_X$ belongs to $\mathcal{J}(X, \alpha Z)$ if and only if all the roots of $b_{f,g}(-s)$ are strictly greater than α.*

In the hypersurface case this is due to [Sab87a] if $\mathcal{J}(X, \alpha Z)$ is replaced by $V^{>\alpha}\mathcal{O}_X$. If the ideal of Z is a monomial ideal, we can calculate the b-function in some cases (see § 4). Other examples can be deduced from the following Thom–Sebastiani-type theorem, which is compatible with the similar theorem for multiplier ideals (see [Mus02]) via Theorem 2. Note that it is different from a usual Thom–Sebastiani-type theorem, which applies to the sum of the pull-backs of two functions (see, for example, [Yan78]).

Theorem 5. *For $f : X \to \mathbb{A}^r$ and $g : Y \to \mathbb{A}^{r'}$, let $h = f \times g : X \times Y \to \mathbb{A}^{r+r'}$. Write*

$$b_f(s) = \prod_\alpha (s+\alpha)^{n_\alpha}, b_g(s) = \prod_\beta (s+\beta)^{m_\beta} \quad \text{and} \quad b_h(s) = \prod_\gamma (s+\gamma)^{q_\gamma}.$$

Then $q_\gamma = \max\{n_\alpha + m_\beta - 1 \mid n_\alpha > 0, m_\beta > 0, \alpha + \beta = \gamma\}$.

This paper is organized as follows. In § 1, we review the theories of V-filtrations and specializations due to Kashiwara, Malgrange and Verdier. In § 2, we define the Bernstein–Sato polynomial for an arbitrary variety (which is not necessarily reduced nor irreducible), and prove its existence and well definedness together with Theorem 5. In § 3, we show the bistrictness of the direct image, and prove Theorems 1–4. In § 4, we treat the monomial ideal case, and calculate some examples. In § 5, we explain the relation with the Bernstein–Sato polynomials in other papers.

Convention. In this paper, a variety means a (not necessarily reduced nor irreducible) separated scheme of finite type over $\mathbb{C}$, and a point of a variety means a closed point. In particular, the underlying set of a variety coincides with that of the associated analytic space.

1. Filtration of Kashiwara and Malgrange

In this section we review the theories of V-filtrations and specializations due to Kashiwara, Malgrange and Verdier.

1.1 *V*-Filtration

Let Z be a smooth closed subvariety of codimension r in a smooth complex algebraic variety X, and let I_Z be the ideal sheaf of Z in X. Let $\mathcal{D}_X$ be the sheaf of linear differential operators on X.

The filtration V on $\mathcal{D}_X$ along Z is defined by

$$V^i\mathcal{D}_X = \{P \in \mathcal{D}_X : PI_Z^j \subset I_Z^{j+i} \text{ for any } j \geqslant 0\},$$

where $I_Z^i = \mathcal{O}_X$ for $i \leqslant 0$. Let $(x_1, \dots, x_n)$ be a local coordinate system (i.e. inducing an étale morphism to $\mathbb{A}^n$) such that $Z = \{x_i = 0\,(i \leqslant r)\}$. Let $\partial_{x_i} = \partial/\partial x_i$. Then $V^k\mathcal{D}_X$ is generated over $\mathcal{O}_X$ by

$$\prod_{j\leqslant r} x_j^{\mu_j} \prod_{i\leqslant n} \partial_{x_i}^{\nu_i} \quad \text{with} \quad \sum_{j\leqslant r}\mu_j - \sum_{i\leqslant r}\nu_i = k, \tag{1.1.1}$$

where $(\mu_1, \dots, \mu_r) \in \mathbb{N}^r$ and $(\nu_1, \dots, \nu_n) \in \mathbb{N}^n$.

We say that a filtration V is discretely indexed by $\mathbb{Q}$ if there is an increasing sequence of rational numbers $\{\alpha_j\}_{j\in\mathbb{Z}}$ such that $\lim_{j\to-\infty}\alpha_j = -\infty$, $\lim_{j\to+\infty}\alpha_j = +\infty$, and V^α for $\alpha \in (\alpha_j, \alpha_{j+1})$ depends only on j. We say that a decreasing filtration V is left-continuously indexed if $V^\alpha M = \bigcap_{\beta<\alpha} V^\beta M$ for any α. Let θ be a (locally defined) vector field such that $\theta \in V^0\mathcal{D}_X$ and whose action on I_Z/I_Z^2 is the identity. For a $\mathcal{D}_X$-module M, the V-filtration of Kashiwara [Kas83] and Malgrange [Mal83] along Z is an exhaustive decreasing filtration V which is indexed discretely and left-continuously by $\mathbb{Q}$, and satisfies the following conditions (see also [Lau85, Sab87a]):

(i) the $V^\alpha M$ are coherent $V^0\mathcal{D}_X$-submodules of M;
(ii) $V^i\mathcal{D}_X V^\alpha M \subset V^{\alpha+i}M$ for any $i \in \mathbb{Z}, \alpha \in \mathbb{Q}$;
(iii) $V^i\mathcal{D}_X V^\alpha M = V^{\alpha+i}M$ for any $i > 0$, if $\alpha \gg 0$;
(iv) $\theta - \alpha + r$ is nilpotent on $\mathrm{Gr}_V^\alpha M$.

Here $\mathrm{Gr}_V^\alpha M := V^\alpha M/V^{>\alpha}M$ with $V^{>\alpha}M = \bigcup_{\beta>\alpha} V^\beta M$. Condition (iii) is equivalent to the condition $V^1\mathcal{D}_X V^\alpha M = V^{\alpha+1}M$ for $\alpha \gg 0$, assuming condition (ii). Condition (iv) is independent of the choice of θ. The shift of index by r in condition (iv) is necessary to show an assertion related to the independence of embeddings in smooth varieties.

By the theory of Kashiwara [Kas83] and Malgrange [Mal83], there exists uniquely the V-filtration on M indexed by $\mathbb{Q}$ if M is regular holonomic and quasi-unipotent, see also Remark 1.2(iv) below. (A holonomic $\mathcal{D}_X$-module is said to be regular and quasi-unipotent if its pull-back by any morphism of a curve to X has regular singularities and quasi-unipotent local monodromies.)

For example, if Z is smooth, then the V-filtration on $M = \mathcal{O}_X$ along Z is given by $V^\alpha\mathcal{O}_X = I_Z^{\lceil\alpha\rceil - r}$, where $\lceil\alpha\rceil$ is the smallest integer $\geqslant \alpha$.

Remark 1.2.

(i) The above conditions are enough to characterize the V-filtration uniquely. Indeed, if there are two filtrations V, V' satisfying the above conditions, we have, by condition (iv),

$$\mathrm{Gr}_V^\alpha \mathrm{Gr}_{V'}^\beta M = 0 \quad \text{for } \alpha \neq \beta. \tag{1.2.1}$$

So it is enough to show that

$$V^{\beta+k} \subset V'^\beta \subset V^{\beta-k} \quad \text{for } k \gg 0, \tag{1.2.2}$$

because it implies that V and V' induce finite filtrations on V^α/V^β, V'^α/V'^β for any $\alpha < \beta$ and these induced filtrations coincide by (1.2.1). The second inclusion of (1.2.2) follows from condition (i), and we need condition (iii) to show the first inclusion.

(ii) Conditions (i)–(iv) in § 1.1 imply that equality holds in condition (ii) if $i < 0$ and $\alpha \ll 0$. Indeed, $V^\beta M$ generates M over $\mathcal{D}_X$ for β sufficiently small (considering an increasing sequence of $\mathcal{D}_X$-submodules generated by $V^\beta M$), and V coincides with the filtration defined by $V'^\alpha M = V^\alpha M$ for $\alpha \geqslant \beta$, and $V'^\alpha M = V^{-i}\mathcal{D}_X V^{\alpha+i}M$ otherwise, where i is an integer such that $\beta \leqslant \alpha + i < \beta + 1$.

(iii) Let M' be a $\mathcal{D}_X$-submodule of M. Then the restriction of V to M' satisfies the conditions of the V-filtration. Indeed, $\bigoplus_{i\in\mathbb{N}} V^{\alpha+i}M'$ is finitely generated over $\bigoplus_{i\in\mathbb{N}} V^i\mathcal{D}_X$ by the noetherian property, because $\bigoplus_{i,p\in\mathbb{N}} \mathrm{Gr}_p^F V^i\mathcal{D}_X = \bigoplus_{i,p\in\mathbb{N}} I_Z^{i+p}\mathrm{Gr}_p^F\mathcal{D}_X$ by (1.1.1), where F is the filtration by the order of differential operators, see [Sab87a].

(iv) If we do not assume that M is quasi-unipotent, then, after choosing an order on $\mathbb{C}$ (for example, such that $\alpha < \beta$ if and only if $\mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$ or $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$ and $\mathrm{Im}\,\alpha < \mathrm{Im}\,\beta$), there is a V-filtration indexed by $\mathbb{C}$, see also [Sab87b]. If M is quasi-unipotent, we see that V is actually indexed by $\mathbb{Q}$ using (1.3.1) below, because the assertion in the codimension one case is well-known (and is easily proved by using a resolution of singularities).

1.3 Specialization

With the notation of § 1.1, let

$$\widetilde{X} = \mathcal{S}pec_X\Big(\bigoplus_{i\in\mathbb{Z}} I_Z^{-i}\otimes t^i\Big),$$

where $I_Z^{-i} = \mathcal{O}_X$ for $i \geqslant 0$. This is an open subvariety of the blow-up of $X\times\mathbb{C}$ along $Z\times\{0\}$. There is a natural morphism $p : \widetilde{X} \to \mathbb{A}^1 := \mathcal{S}pec\,\mathbb{C}[t]$ whose fiber over zero is the tangent cone

$$T_ZX = \mathcal{S}pec_X\Big(\bigoplus_{i\leqslant 0} I_Z^{-i}/I_Z^{-i+1}\otimes t^i\Big),$$

and $\widetilde{X}^* := p^{-1}(\mathbb{A}^1\backslash\{0\})$ is isomorphic to $X\times(\mathbb{A}^1\backslash\{0\})$. Therefore, p gives a deformation of T_ZX to X (see [Ver83]).

Let M be a regular holonomic $\mathcal{D}_X$-module. Let

$$\widetilde{M} = \bigoplus_{i\in\mathbb{Z}} M\otimes t^i.$$

This naturally has a structure of a $\mathcal{D}_X\otimes_{\mathbb{C}}\mathbb{C}[t,t^{-1}]\langle\partial_t\rangle$-module, and is identified with the pull-back of M by the projection $q : \widetilde{X}^* \to X$. Viewed as a $\mathcal{D}_X\otimes_{\mathbb{C}}\mathbb{C}[t]\langle\partial_t\rangle$-module, $\widetilde{M}$ is identified with $\rho_\bullet j_* q^*M$ where $j : \widetilde{X}^* \to \widetilde{X}$ and $\rho : \widetilde{X} \to X$ are natural morphisms. (Here $\rho_\bullet$ and j_* are direct images as Zariski sheaves. Note that the direct image of $\mathcal{D}$-modules j_* is defined by the sheaf-theoretic direct image in the case of open embeddings, see [Bor87].)

Consider the filtrations V of Kashiwara and Malgrange on M along Z, and on j_*q^*M along $T_ZX = p^{-1}(0)$. It is known that we have canonical isomorphisms

$$\rho_\bullet V^{\alpha-r+1}(j_*q^*M) = V^\alpha\widetilde{M} := \bigoplus_{i\in\mathbb{Z}} V^{\alpha-i}M\otimes t^i. \tag{1.3.1}$$

In particular,

$$\rho_\bullet \mathrm{Gr}_V^{\alpha-r+1}(j_*q^*M) = \mathrm{Gr}_V^\alpha\widetilde{M} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}_V^{\alpha-i}M\otimes t^i.$$

(These are crucial to relate Kashiwara's construction [Kas83] with the Verdier specialization [Ver83].) Here the shift of the filtrations V comes from the difference of the codimensions. The above assertion can be verified using the $\mathbb{C}^*$-action on $\widetilde{X}$ (or the action of the corresponding vector field) which comes from the natural $\mathbb{C}^*$-action on $\widetilde{X}^* = X\times(\mathbb{A}^1\backslash\{0\})$ and which corresponds to the grading by the order of t. Indeed, if $(x_1,\ldots,x_n)$ is a local coordinate system of X such that Z is locally given by $x_i = 0$ for $i \leqslant r$, then $\widetilde{X}$ has a local coordinate system $(\widetilde{x}_1,\ldots,\widetilde{x}_n,\widetilde{t})$ such that $\widetilde{x}_i = x_i/t\,(i \leqslant r)$, $\widetilde{x}_i = x_i\,(i > r)$, and $\widetilde{t} = t$ on $\widetilde{X}^*$, and hence the vector field corresponding to the

$\mathbb{C}^*$-action is given by

$$t\frac{\partial}{\partial t} = \widetilde{t}\frac{\partial}{\partial \widetilde{t}} - \sum_{i\leqslant r} \widetilde{x}_i \frac{\partial}{\partial \widetilde{x}_i}. \tag{1.3.2}$$

Note that $M \otimes t^i$ is identified with $\mathrm{Ker}(t\partial/\partial t - i) \subset \widetilde{M}$ so that $\widetilde{t}\partial/\partial\widetilde{t} - i$ is identified with $\sum_{i\leqslant r} \widetilde{x}_i\partial/\partial\widetilde{x}_i$ on $M \otimes t^i$. Actually, the existence of the filtration V can be reduced to the hypersurface case using this argument. The finite generatedness of (1.3.1) is related to condition (iii) in § 1.1 and to the property in Remark 1.2(ii).

We also have a canonical isomorphism

$$\rho_\bullet V^k\mathcal{D}_{\widetilde{X}} = \bigoplus_{i\in\mathbb{Z}} V^{k-i}\mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]t^i \quad \text{for } k \geqslant 0, \tag{1.3.3}$$

using $\partial/\partial\widetilde{x}_i = t\partial/\partial x_i$, $\widetilde{x}_i = t^{-1}x_i\,(i \leqslant r)$, etc. Then

$$\rho_\bullet \mathrm{Gr}_V^k\mathcal{D}_{\widetilde{X}} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}_V^{k-i}\mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]t^i \quad \text{for } k \geqslant 0.$$

The specialization $\mathrm{Sp}_Z M$ of M along Z is defined by

$$\mathrm{Sp}_Z M = \psi_t(j_*q^*M) := \bigoplus_{0<\alpha\leqslant 1} \mathrm{Gr}_V^\alpha(j_*q^*M),$$

and its direct image by ρ is identified with

$$\bigoplus_{r-1<\alpha\leqslant r} \mathrm{Gr}_V^\alpha \widetilde{M} = \bigoplus_{r-1<\alpha\leqslant r}\bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}_V^{\alpha-i} M \otimes t^i.$$

(Here the shift of the indices comes from the difference of the codimensions as above.) For $\lambda = \exp(-2\pi i\alpha)$, we define the λ-eigen part by

$$\mathrm{Sp}_{Z,\lambda} M = \psi_{t,\lambda}(j_*q^*M) := \mathrm{Gr}_V^\alpha(j_*q^*M),$$

which is identified with

$$\mathrm{Gr}_V^\alpha \widetilde{M} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}_V^{\alpha-i} M \otimes t^i.$$

By the Riemann–Hilbert correspondence, $\mathrm{Sp}_Z M$ corresponds via the analytic de Rham functor DR to the specialization $\mathrm{Sp}_Z K := \psi_t(\mathbf{R}j_*q^*K)$ (see [Del73, Ver83]) of $K = DR(M^{\mathrm{an}})$, where M^{an} denotes the associated analytic $\mathcal{D}$-module (see [Kas83]). Note that $\mathrm{Sp}_{Z,\lambda} M$ corresponds to $\mathrm{Sp}_{Z,\lambda} K := \psi_{t,\lambda}(\mathbf{R}j_*q^*K)$, where $\psi_{t,\lambda}$ denotes the λ-eigen part of ψ_t.

2. The Bernstein–Sato polynomial

In this section we define the Bernstein–Sato polynomial for an arbitrary variety (which is not necessarily reduced nor irreducible), and prove its existence and well definedness together with Theorem 5.

2.1 *b*-function

With the notation and the assumptions of § 1.1, let M be a quasi-unipotent regular holonomic $\mathcal{D}_X$-module. For a (local) section m of M, the *Bernstein–Sato polynomial* (i.e. the *b-function*) $b_m(s)$ along Z is defined to be the monic minimal polynomial of the action of $s := -\theta - r$ on

$$\overline{M}_m := (V^0\mathcal{D}_X)m/(V^1\mathcal{D}_X)m. \tag{2.1.1}$$

The action of θ on $\overline{M}_m$ is independent of the choice of θ. The existence of $b_m(s)$ easily follows from that of the filtration V of Kashiwara and Malgrange on M along Z. Indeed, the existence is equivalent to the finiteness of the induced filtration V on $\overline{M}_m$, and setting $M' = \mathcal{D}_X m$, it is sufficient to show for $\beta \gg 0$ that

$$V^\beta M' \subset (V^1\mathcal{D}_X)m, \quad (V^0\mathcal{D}_X)m \subset V^{-\beta}M'. \tag{2.1.2}$$

Since the induced filtration V on M' satisfies the conditions of the V-filtration (see Remark 1.2(iii)), we have for some β_0, $V^{\beta_0+i}M' = V^i\mathcal{D}_X V^{\beta_0}M'$ for any $i > 0$, and $V^{\beta_0}M' \subset (V^j\mathcal{D}_X)m$ for some j by the coherence of $V^{\beta_0}M'$. So the first inclusion follows. The second inclusion is clear.

Note that α is a root of $b_m(-s)$ if and only if $\mathrm{Gr}_V^\alpha \overline{M}_m \neq 0$. This implies

$$\max\{\alpha : m \in V^\alpha M\} = \min\{\alpha : b_m(-\alpha) = 0\}, \tag{2.1.3}$$

because the left-hand side coincides with $\min\{\alpha : \mathrm{Gr}_V^\alpha((V^0\mathcal{D}_X)m) \neq 0\}$, which is strictly smaller than $\min\{\alpha : \mathrm{Gr}_V^\alpha((V^1\mathcal{D}_X)m) \neq 0\}$; see [Sab87a] for the codimension one case.

Proposition 2.2. *Let $i : X \to Y$ be a closed embedding of smooth varieties. Let Z be a smooth closed subvariety of X. Let M be a regular holonomic $\mathcal{D}_X$-module, and m be a section of M. Let i_*M and $i_\bullet M$ denote the direct images of M as $\mathcal{D}_X$-module and as $\mathcal{O}_X$-module, respectively. Let $(y_1, \dots, y_n)$ be a local coordinate system of Y such that $X = \{y_i = 0\,(i \leqslant q)\}$. Let m' be the element of i_*M corresponding to $m \otimes 1$ by the isomorphism $i_*M \simeq i_\bullet M \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \dots, \partial_q]$ where $\partial_i = \partial/\partial y_i$ (see [Bor87]). Then the Bernstein–Sato polynomial $b_m(s)$ of m along Z coincides with $b_{m'}(s)$ of m' along Z, and*

$$V^\alpha(i_*M) = \sum_\nu i_\bullet V^{\alpha+|\nu|}M \otimes \partial^\nu, \tag{2.2.1}$$

where V is the filtration of Kashiwara and Malgrange along Z, $\partial^\nu = \prod_i \partial_i^{\nu_i}$ for $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{N}^r$, and $|\nu| = \sum_i \nu_i$.

Proof. Since the assertion is local, we may assume $\mathrm{codim}_Y X = 1$, and, furthermore, $Z = \{y_i = 0\ (1 \leqslant i \leqslant r+1)\}$ in Y. Then θ on X and Y can be given by $\theta_X := \sum_{2\leqslant i\leqslant r+1} y_i\partial_i$ and $\theta_Y := \sum_{1\leqslant i\leqslant r+1} y_i\partial_i$, respectively. Note that

$$\theta_X + r = \sum_{2\leqslant i\leqslant r+1} \partial_i y_i, \quad \theta_Y + r + 1 = \sum_{1\leqslant i\leqslant r+1} \partial_i y_i.$$

Since m' is annihilated by y_1, we see that

$$V^k\mathcal{D}_Y\, m' = \bigoplus_{i\geqslant 0}(V^{i+k}\mathcal{D}_X\, m) \otimes \partial_1^i \quad \text{for } k = 0, 1. \tag{2.2.2}$$

As $V^i\mathcal{D}_X\, m/V^{i+1}\mathcal{D}_X\, m$ is annihilated by $b_m(s+i)$, the first assertion follows. The proof of the second assertion is similar. □

Remark 2.3. With the above notation, assume M has the Hodge filtration F. Then the Hodge filtration F on i_*M is given by

$$F_p(i_*M) = \sum_\nu i_\bullet F_{p-|\nu|}M \otimes \partial^\nu. \tag{2.3.1}$$

In particular, $F_{p_0}(i_*M) = i_\bullet F_{p_0}M$ locally if $p_0 = \min\{p : \mathrm{Gr}_p^F M \neq 0\}$. Globally, we need the twist by the relative dualizing sheaf.

If $M = \mathcal{O}_X$, we have $\mathrm{Gr}_p^F \mathcal{O}_X = 0$ for $p \neq -n$ where $n = \dim X$. In particular, $p_0 = -n$.

2.4 The graph embedding

Let X be a smooth affine variety, and let Z be a (not necessarily reduced nor irreducible) closed subvariety with $f_1, \dots, f_r$ generators of the ideal of Z in X (where $f_j \neq 0$ for any j).

Let $i_f : X \to X' := X\times\mathbb{A}^r$ be the graph embedding of $f := (f_1,\dots,f_r) : X \to \mathbb{A}^r$, i.e. $i_f(x) = (x, f_1(x),\dots,f_r(x))$. For a quasi-unipotent regular holonomic $\mathcal{D}_X$-module M, let $M' = (i_f)_*M$ be the direct image as a $\mathcal{D}$-module. We have a natural isomorphism

$$M' = (i_f)_\bullet M \otimes_{\mathbb{C}} \mathbb{C}[\partial_1,\dots,\partial_r], \tag{2.4.1}$$

where $(i_f)_\bullet$ denotes the sheaf-theoretic direct image, and $\partial_i = \partial/\partial t_i$ with $(t_1,\dots,t_r)$ the canonical coordinates of $\mathbb{A}^r$. Furthermore, the action of $\mathcal{O}_X[\partial_1,\dots,\partial_r]$ on M' is given by the canonical one (without using f), and the action of a vector field ξ on X and that of t_i are given by

$$\xi(m\otimes\partial^\nu) = \xi m\otimes\partial^\nu - \sum_i (\xi f_i)m\otimes\partial^{\nu+1_i},$$

$$t_i(m\otimes\partial^\nu) = f_i m\otimes\partial^\nu - \nu_i m\otimes\partial^{\nu-1_i},$$

where $\partial^\nu = \prod_i \partial_i^{\nu_i}$ with $\nu = (\nu_1,\dots,\nu_r)\in\mathbb{N}^r$, and 1_i is the element of $\mathbb{Z}^r$ whose jth component is 1 if $j=i$ and 0 otherwise. In the case $M = \mathcal{O}_X$, we have a canonical injection

$$\mathcal{O}_X\otimes_{\mathbb{C}}\mathbb{C}[\partial_1,\dots,\partial_r] \hookrightarrow \mathcal{O}_X\left[\prod_i f_i^{-1}, s_1,\dots,s_r\right]\prod_i f_i^{s_i}, \tag{2.4.2}$$

such that s_i is identified with $-\partial_i t_i$ (see [Kas76/77, Mal75, Sab87b]).

Let V be the filtration of Kashiwara and Malgrange on $\mathcal{D}_{X'}$ along $X\times\{0\}$. Let $\theta = \sum_i t_i\partial_i$ and $\theta^* = -\sum_i \partial_i t_i$ $(= -\theta - r)$. (Actually * comes from the involution of the ring of differential operators, which is used in the transformation between left and right $\mathcal{D}$-modules.) For a (local) section m of M, the *Bernstein–Sato polynomial* (i.e. the *b-function*) $b_{f,m}(s)$ is defined to be that for $m\otimes 1$. This is the minimal polynomial of the action of θ^* on

$$\overline{M}_{f,m} := V^0\mathcal{D}_{X'}(m\otimes 1)/V^1\mathcal{D}_{X'}(m\otimes 1). \tag{2.4.3}$$

Note that the roots of $b_{f,m}(s)$ are rational numbers because the filtration V is indexed by rational numbers.

If $M = \mathcal{O}_X$ and $m = 1$, then $b_{f,m}(s)$ is denoted by $b_f(s)$, and $b_Z(s) = b_f(s-r')$ with $r' = \operatorname{codim}_X Z$. This definition of the Bernstein–Sato polynomial coincides with that in the introduction by (1.1.1) and (2.4.2), because θ^* belongs to the center of $\mathrm{Gr}_V^0\mathcal{D}_{X'}$. (Indeed, if $\sum_{j\leqslant r}\mu_j - \sum_{i\leqslant r}\nu_i = 1$ with $\mu_j,\nu_i\in\mathbb{N}$, then $\mu_j\geqslant 1$ for some j, and (1.1.1) for $k=0$ is contained in the $\mathbb{C}$-algebra generated by $x_i\partial_{x_j}$ $(i,j\leqslant r)$ and ∂_{x_j} $(j>r)$.)

If Z is not affine, then $b_Z(s)$ is defined to be the least common multiple of $b_{(Z,z)}(s - \dim(Z,z) + \dim Z)$ for $z\in Z$, where $b_{(Z,z)}(s)$ is the b-function of a sufficiently small affine neighborhood of z in Z.

Theorem 2.5. *The Bernstein–Sato polynomial* $b_f(s)$ *is independent of the choice of* $f = (f_1,\dots,f_r)$ *(provided that* $\dim X$ *is fixed), and* $b_Z(s)$ *depends only on* Z*.*

Proof. We first show that $b_f(s)$ is independent of the choice of the generators $f_1,\dots,f_r$ if X is fixed. Let $g_1,\dots,g_{r'}$ be other generators. Then we have $g_i = \sum_j a_{i,j}f_j$ with $a_{i,j}\in\mathcal{O}_X$. Set $f = (f_1,\dots,f_r)$ and $g = (g_1,\dots,g_{r'})$. Let $i_f : X\to X\times\mathbb{A}^r$ and $i_{f,g} : X\to X\times\mathbb{A}^{r+r'}$ be the embeddings by the graphs of f and (f,g), respectively. Consider an embedding $\phi : X\times\mathbb{A}^r \to X\times\mathbb{A}^{r+r'}$ defined by $\phi(x,s_1,\dots,s_r) = (x,s_1,\dots,s_r,s'_1,\dots,s'_{r'})$ with $s'_i = \sum_j a_{i,j}s_j$. Then $\phi\circ i_f = i_{f,g}$. So the independence of the choice of $f_1,\dots,f_r$ with X fixed follows from Proposition 2.2.

Now we have to show that $b_g(s) = b_f(s+1)$ for $g = (f_1,\dots,f_r,x)$ on $Y := X\times\mathbb{A}^1$, where x is the coordinate of $\mathbb{A}^1$ and $f_1,\dots,f_r,x$ are viewed as generators of the ideal of $Z\times\{0\}$ in $X\times\mathbb{A}^1$. Since the b-function $b_x(s)$ of x is $s+1$ as is well known, this is a special case of Theorem 5, and follows from § 2.9 below. Since the construction in § 2.4 is compatible with the pull-back by an étale morphism, the assertion in Theorem 2.5 follows from Proposition 2.2. □

Proposition 2.6. *With the notation of* § 2.4*, let* V *be the filtration on* M *induced by the* V*-filtration on* M' *along* $X\times\{0\}$ *using the isomorphism* (2.4.1) *and identifying* M *with* $(i_f)_\bullet M\otimes 1$*. Then the filtration* V *on* M *is independent of the choice of* $f_1,\ldots,f_r$*.*

Proof. This follows from (2.2.1) using the same argument as in Theorem 2.5. □

Proposition 2.7. *With the notation of* § 1.3*, let* $m\in M$*, and set* $\widetilde{m}=m\otimes 1\in\widetilde{M}$*. Let* $b_m(s)$ *and* $b_{\widetilde{m}}(s)$ *be the b-functions of* m *and* $\widetilde{m}$ *along* Z *and* $T_ZX=p^{-1}(0)$*, respectively. Then* $b_{\widetilde{m}}(s)=b_m(s+r-1)$*.*

Proof. This follows from (1.3.2) and (1.3.3). □

Corollary 2.8. *With the notation of Proposition* 2.7*, let* α_j *be the roots of* $b_m(-s)$*. Then the* $\exp(2\pi i\alpha_j)$ *are eigenvalues of the monodromy on the nearby cycle sheaf* $\psi_t(\mathbf{R}j_*q^*K)$ *in the notation of* § 1.3*. Furthermore, if* m *generates* M*, then the* $\exp(2\pi i\alpha_j)$ *coincide with the eigenvalues of the monodromy on the nearby cycle sheaf.*

Proof. This follows from Proposition 2.7 together with § 1.3 and [Kas83, Mal83]. □

2.9 Proof of Theorem 5

Let $Z=X\times Y$ and $Z'=Z\times\mathbb{A}^{r+r'}$ (similarly for X',Y' with $Z,r+r'$ replaced by X,r and Y,r', respectively). Let $M'=(i_f)_*\mathcal{O}_X$, $N'=(i_g)_*\mathcal{O}_Y$, and $R'=(i_h)_*\mathcal{O}_Z$. It is clear that R' is the external product $M'\boxtimes N'$ $(:=pr_1^*M'\otimes_{\mathcal{O}}pr_2^*N')$ of M' and N'.

Define a filtration G on M',N', and R' by $G^iM'=V^i\mathcal{D}_{X'}(1\otimes 1)$, and similarly for N' and R'. Then

$$G^kR'=\sum_{i+j=k}G^iM'\boxtimes G^jN',$$

$$\mathrm{Gr}_G^0R=\bigoplus_{i+j=0}\mathrm{Gr}_G^iM'\boxtimes\mathrm{Gr}_G^jN'.$$

Put $b'_h(s)=\prod_\gamma(s+\gamma)^{q_\gamma}$ with q_γ as in Theorem 5. Since Gr_G^iM' is annihilated by $b_f(s+i)$ with $s=\sum_{1\leqslant j\leqslant r}s_j$, and similarly for Gr_G^jN', we see that $\mathrm{Gr}_G^iM'\boxtimes\mathrm{Gr}_G^jN'$ is annihilated by $b'_h(s+i+j)$ with $s=\sum_{1\leqslant j\leqslant r+r'}s_j$. Thus, $b_h(s)$ divides $b'_h(s)$. Moreover, we get the equality $b_h(s)=b'_h(s)$ by looking at the action of $\sum_{1\leqslant j\leqslant r+r'}s_j$ on $\mathrm{Gr}_G^0M'\boxtimes\mathrm{Gr}_G^0N'$. This completes the proof of Theorem 5.

2.10 Another description of the Bernstein–Sato polynomial

For $c=(c_1,\ldots,c_r)\in\mathbb{Z}^r$, let $I(c)_-=\{i:c_i<0\}$. Then the Bernstein–Sato polynomial $b_f(s)$ is the monic polynomial of the smallest degree such that $b_f(s)\prod_i f^{s_i}$ belongs to the $\mathcal{D}_X[s_1,\ldots,s_r]$-submodule generated by

$$\prod_{i\in I(c)_-}\binom{s_i}{-c_i}\cdot\prod_{i=1}^r f_i^{s_i+c_i}, \tag{2.10.1}$$

where $c=(c_1,\ldots,c_r)$ runs over the elements of $\mathbb{Z}^r$ such that $\sum_i c_i=1$. Here $s=\sum_{i=1}^r s_i$, and $\binom{s_i}{m}=s_i(s_i-1)\cdots(s_i-m+1)/m!$ as usual.

This definition of the Bernstein–Sato polynomial coincides with those in the introduction and in § 2.4. Indeed, $s_it_i^{-1}$ corresponds to $-\partial_{t_i}$ and the relation $t_i^{-1}s_i=(s_i-1)t_i^{-1}$ implies

$$(s_it_i^{-1})^{-c_i}=(-c_i)!\binom{s_i}{-c_i}t^{c_i}\quad\text{for } c_i<0.$$

Then we put $\theta_i=t_i$ if $c_i>0$, and $\theta_i=\partial_{t_i}^{-1}$ if $c_i<0$, and consider $\prod_i\theta_i^{c_i}$ for $c\in\mathbb{Z}^r$ with $\sum_i c_i=1$.

3. Proofs of Theorems 1–4

In this section, we prove Theorems 1–4 using the bistrictness of the direct image. We first recall the definition of multiplier ideals (see [Laz04, Nad90]).

3.1 Multiplier ideals

Let X be a smooth variety, and Z be a closed subvariety of X which is not necessarily reduced nor irreducible. Let $\pi : Y \to X$ be an embedded resolution of Z, i.e. Y is smooth and $D := \pi^{-1}(Z)$ is a divisor with normal crossings. Here the ideal of D is generated by the pull-back of the ideal of Z, and we assume it is locally principal. Let D_i be the irreducible components of D with multiplicity m_i. Let $\omega_{Y/X}$ be the relative dualizing sheaf $\omega_Y \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\omega_X^\vee$ (where $\omega_X^\vee$ is the dual line bundle of ω_X). Then, for a positive rational number α, the multiplier ideal $\mathcal{J}(X, \alpha Z)$ is defined by

$$\mathcal{J}(X,\alpha Z) = \pi_\bullet\biggl(\omega_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\biggl(-\sum_i [\alpha m_i] D_i\biggr)\biggr), \tag{3.1.1}$$

using the trace morphism $\pi_\bullet \omega_{Y/X} \to \mathcal{O}_X$. They define a decreasing filtration on $\mathcal{O}_X$ which is indexed discretely and right-continuously. Note that V is left-continuously indexed, see § 1.1.

The following is the key to the proof of Theorem 1.

Proposition 3.2. *Let X, Z be as in § 1.1, and let Y be a smooth projective variety. Let $pr : X \times Y \to X$ denote the first projection, and (M, F) be a filtered $\mathcal{D}$-module underlying a mixed Hodge module on $X \times Y$. Let V be the filtration of Kashiwara and Malgrange on M along $Z \times Y$. Put $p_0 = \min\{p : \mathrm{Gr}_p^F M \neq 0\}$. Then the direct image of V gives the filtration V of Kashiwara and Malgrange along Z, and the bifiltered direct image is bistrict. In particular, we have a natural isomorphism*

$$F_{p_0} V^\alpha \mathcal{H}^i pr_* M = R^i pr_\bullet(\omega_Y \otimes F_{p_0} V^\alpha M), \tag{3.2.1}$$

where pr_* denotes the direct image of bifiltered $\mathcal{D}$-modules which is defined by using the relative de Rham complex as in [BS05], and $R^j pr_\bullet$ is the sheaf-theoretic higher direct image.

Proof. This is reduced to the case of codimension one because the construction in § 1.3 is compatible with the direct image by the projection as above. Indeed, the direct image of $(M; F, V)$ by pr is defined by using the sheaf-theoretic direct image of the relative de Rham complex $DR_{X \times Y/X}(M; F, V)$ whose ith component is given by

$$\Omega_Y^{\dim Y + i} \otimes_{\mathcal{O}_Y} (M; F[-i], V),$$

where the pull-back of $\Omega_Y^{\dim Y+i}$ by the second projection is omitted to simplify the notation, and $(F[-i])_p = F_{p+i}$. Furthermore, the Hodge filtration F_{p_0-1} on $j_* q^* M$ is obtained by taking the intersection of the direct image $j_\bullet q^* F_{p_0} M$ with V^0 (see [Sai88, 3.2.3]), where the base changes of j, q are also denoted by the same symbols, and the shift of the Hodge filtration by 1 comes from the smooth pull-back q^*. Since $\rho_\bullet j_\bullet q^* F_{p_0} M$ is identified with $\bigoplus_{k \in \mathbb{Z}} F_{p_0} M \otimes t^k$ (on which the action of t is bijective), the above argument together with (1.3.1) implies

$$\rho_\bullet F_{p_0-1} j_* q^* M = \bigoplus_{k \in \mathbb{Z}} F_{p_0} V^{r-1-k} M \otimes t^k. \tag{3.2.2}$$

Note that $F_{p_0} V^{r-1-k} M = F_{p_0} M$ for $k \gg 0$.

By [Sai88, 3.3.17], we have the bistrictness of $\widetilde{pr}_*(j_* q^* M; F, V)$. By (1.3.1) and (3.2.2), the filtrations F and V are compatible with the grading by the powers of t (i.e. with the $\mathbb{C}^*$-action). So the bistrictness of $\widetilde{pr}_*(j_* q^* M; F, V)$ implies that of $pr_*(M; F, V)$ by the definition of the direct image using the relative de Rham complex, where we use the direct factor of (3.2.2) for $k \gg 0$.

Note that (3.2.1) follows from the bistrictness if V on the left-hand side is the induced filtration on the direct image $\mathcal{H}^i pr_*M$; see also (3.3.1) below. So it remains to show that the induced filtration V on $\mathcal{H}^i pr_*M$ coincides with the filtration of Kashiwara and Malgrange. However, this is proved in [Sai88, 3.3.17], for $\widetilde{pr}_*(j_*q^*M;F,V)$, and the assertion for $pr_*(M;F,V)$ follows by using (1.3.1) for M and $\mathcal{H}^i pr_*M$. This completes the proof of Proposition 3.2. □

Remark 3.3. Let $(K;F,V)$ be a bifiltered complex representing the direct image $pr_*(M;F,V)$ defined in the derived category of bifiltered $\mathcal{D}$-modules. Let G be a finite filtration on K^j defined by $G^0K^j = \operatorname{Ker} d$, $G^1K^j = \operatorname{Im} d$. Then (3.2.1) is essentially equivalent to

$$F_{p_0}V^\alpha \mathrm{Gr}^0_G K^j = \mathrm{Gr}^0_G F_{p_0}V^\alpha K^j, \tag{3.3.1}$$

and it is nontrivial because of the problem of three filtrations [Del70]. We have (3.3.1) if $(K;F,V)$ is bistrict (see [Sai88, 1.2.13]).

3.4 Proof of Theorem 1

Let $\pi : Y \to X$ be a projective morphism of smooth varieties such that $\pi^{-1}(Z)$ is a divisor with normal crossings on Y as in § 3.1, and π induces an isomorphism over $X \setminus Z$. Let $g = (g_1, \dots, g_r)$ with $g_i = f_i \circ \pi$ so that $g = \pi^*f$. Then $i_f \circ \pi = (\pi \times \mathrm{id}) \circ i_g$.

Let D be the effective divisor defined by the pull-back of the ideal of Z, and let $Y' = \mathcal{S}pec_{\mathcal{O}_Y}(\bigoplus_{m \geqslant 0} \mathcal{O}_Y(-mD))$ be the line bundle over Y corresponding to the invertible sheaf $\mathcal{O}_Y(D)$. Let $i_D : Y \to Y'$ be the closed embedding induced by $\mathcal{O}_Y(-D) \hookrightarrow \mathcal{O}_Y$. As a section of a line bundle, this corresponds to $1 \in \mathcal{O}_Y(D)$. We have an embedding $i : Y' \hookrightarrow Y \times \mathbb{A}^n$ induced by the surjective morphism

$$(g_1, \dots, g_r) : \bigoplus_j \mathcal{O}_Y \to \mathcal{O}(-D),$$

by passing to symmetric algebras over $\mathcal{O}_Y$. Then $i \circ i_D = i_g$, and we get a commutative diagram.

$$\begin{tikzcd}
Y \arrow[r, "i_D"] \arrow[d, equal] & Y' \arrow[d, "i"] \\
Y \arrow[r, "i_g"] \arrow[d, "\pi"'] & Y \times \mathbb{A}^r \arrow[d, "\pi \times \mathrm{id}"] \\
X \arrow[r, "i_f"] & X \times \mathbb{A}^r
\end{tikzcd}$$

Taking a local equation g' of D, we have a local trivialization of the line bundle $Y' \to Y$, and the embedding $i_D : Y \to Y'$ is locally identified with the graph embedding $i_{g'} : Y \to Y \times \mathbb{A}^1$. Since $g'^{-1}(0)$ is a divisor with normal crossings, the assertion of Theorem 1 for $g'^{-1}(0) \subset Y$ follows from [Sai90, Proposition 3.5]; see also [BS05, Proposition 2.3]. Here the filtration V on $\mathcal{O}_Y$ is induced by the V-filtration along the zero section of the line bundle $Y' \to Y$ using the inclusion $i_D : Y \to Y'$ (which is locally identified with $i_{g'} : Y \to Y \times \mathbb{A}^1$).

We have the factorization of $\pi : Y \to X$ by the closed embedding $i' : Y \to X \times Y$ and the projection $pr : X \times Y \to X$. Since the V-filtration is compatible with the direct image under a closed embedding by Proposition 2.2 and $F_{p_0}M$ does not essentially change by such direct images (see Remark 2.3), it is enough to consider the direct image of $M := (i_h \circ i')_* \mathcal{O}_Y$ by $pr \times \mathrm{id}$, where h is the pull-back of f by pr. Here $p_0 = -n$ with $n = \dim X$; see Remark 2.3.

By Proposition 3.2 (applied to M and $pr \times \mathrm{id}$), it is then enough to show that

$$F_{-n}V^\alpha \mathcal{H}^0(pr \times \mathrm{id})_* M = F_{-n}V^\alpha (i_f)_* \mathcal{O}_X \quad \text{for } \alpha > 0. \tag{3.4.1}$$

By the decomposition theorem for mixed Hodge modules [Sai90], $((i_f)_*\mathcal{O}_X, F)$ is a direct factor of $\mathcal{H}^0(pr\times\mathrm{id})_*(M,F)$, and the complement (N',F) is supported on $X\times\{0\}$. We have to show that $V^\alpha N'=0$ for $\alpha>0$. However, any element of N' is annihilated by a sufficiently high power of the ideal $(t_1,\dots,t_r)$, and hence it is annihilated by a polynomial $b(-s)$ in $-s$ whose roots are nonpositive integers. (Note that it is annihilated by $-s$ if it is annihilated by $(t_1,\dots,t_r)$.) So the assertion follows.

3.5 Proof of Theorem 2

With the notation of § 2.4, let $M=\mathcal{O}_X$ and $n=\dim X$ so that $F_{-n}M'=\mathcal{O}_X$. Since the filtration V on $\mathcal{O}_X$ is the induced filtration, we have the injectivity of

$$\mathrm{Gr}_V^\alpha\mathcal{O}_X\to\mathrm{Gr}_V^\alpha((V^0\mathcal{D}_{X'})(1\otimes 1)),$$

and

$$\min\{\alpha:\mathrm{Gr}_V^\alpha\mathcal{O}_X\neq 0\}=\min\{\alpha:\mathrm{Gr}_V^\alpha((V^0\mathcal{D}_{X'})(1\otimes 1))\neq 0\},$$

which will be denoted by α'_Z. (Here the inequality $\leqslant$ follows from the above injectivity, and we have the equality because $(V^0\mathcal{D}_{X'})m\subset V^\alpha(i_f)_*\mathcal{O}_X$ for $m\in V^\alpha\mathcal{O}_X\otimes 1$.) Then

$$(V^1\mathcal{D}_{X'})(1\otimes 1)\subset V^{\alpha'_Z+1}(i_f)*\mathcal{O}_X,$$

and the minimal root of $b_Z(-s)$ coincides with α'_Z. Therefore, the assertion follows from Theorem 1.

3.6 Proof of Theorem 3

Since Z is a local complete intersection and the assertion is local, we may assume $\mathrm{codim}_X Z=r$ with the notation of § 1.3. Then T_ZX is a trivial vector bundle $Z\times\mathbb{A}^r$, as

$$\bigoplus_{i\geqslant 0}I_Z^i/I_Z^{i+1}\simeq\mathcal{O}_Z\otimes_{\mathbb{C}}\mathbb{C}[u_1,\dots,u_r]$$

with u_i corresponding to f_i.

Since Z is reduced, the restriction of the specialization $\mathrm{Sp}_Z\mathbb{Q}_X$ to $T_ZX\times_Z Z_{\mathrm{reg}}\,(=Z_{\mathrm{reg}}\times\mathbb{A}^r)$ is a constant sheaf on it, where Z_{reg} is the largest smooth open subvariety of Z. This implies that the intersection complex (see [BBD82]) of T_ZX (which is the pull-back of the intersection complex of Z up to a shift of complex) is a subquotient of $\mathrm{Sp}_{Z,1}\mathbb{Q}_X$, where $\mathrm{Sp}_{Z,1}$ denotes the unipotent monodromy part of Sp_Z (see § 1.3).

To apply the theory of $\mathcal{D}$-modules, we take the embedding $i_f:X\to X'=X\times\mathbb{A}^r$ defined by the graph of f, and consider the specialization along $X\times\{0\}$. Let $M=(i_f)_*\mathcal{O}_X$ (as a direct image of a $\mathcal{D}$-module). We apply § 1.3 to these. Let $Z':=T_ZX=Z\times\mathbb{A}^r$, and $M_{Z'}$ denote the $\mathcal{D}_{X'}$-module corresponding to the intersection complex of $Z'\subset X'$, where X' is identified with the normal cone of $X\times\{0\}$ in $X'=X\times\mathbb{A}^r$. Let $\rho:X\times\mathbb{A}^r\to X$ denote the projection. Then the above argument implies that $M_{Z'}$ is a subquotient of $\mathrm{Gr}_V^1(j_*q^*M)$, i.e. $\rho_\bullet M_{Z'}$ is a subquotient of $\mathrm{Gr}_V^r\widetilde{M}$.

By [Sai88, Sai90], these $\mathcal{D}$-modules have Hodge filtrations, denoted by F. Here we use the normalization as in the case of right $\mathcal{D}$-module, see [BS05]. Let $n=\dim X$. Then $\rho_\bullet F_pM_{Z'}=F_p\mathrm{Gr}_V^r\widetilde{M}=0$ for $p<-n$, and the restrictions of $\rho_\bullet F_{-n}M_{Z'}$ and $F_{-n}\mathrm{Gr}_V^r\widetilde{M}$ to Z_{reg} are both isomorphic to the structure sheaf $\mathcal{O}$ of Z_{reg} tensored by $\mathbb{C}[u_1,\dots,u_r]$ over $\mathbb{C}$. Note that $F_{-n}M=\mathcal{O}_X$, and $F_{-n}M_{Z'}$ is isomorphic to the sheaf-theoretic direct image of the relative dualizing sheaf of a resolution of singularities of Z', see [Kol86, Sai91]. So we get

$$\rho_\bullet F_{-n}M_{Z'}=\omega_X^\vee\otimes_{\mathcal{O}}\widetilde{\omega}_Z\otimes_{\mathbb{C}}\mathbb{C}[u_1,\dots,u_r],\tag{3.6.1}$$

where $\omega_X^\vee$ is the dual of ω_X. Furthermore, the above argument implies that $\rho_\bullet F_{-n}M_{Z'}$ is a subquotient of

$$F_{-n}\mathrm{Gr}_V^r\widetilde{M} = \bigoplus_{i\in\mathbb{Z}} \mathrm{Gr}_V^{r+i}\mathcal{O}_X \otimes t^{-i} \tag{3.6.2}$$

in a compatible way with the grading induced by the natural $\mathbb{C}^*$-action.

We have $f_i \in V^{>r}\mathcal{O}_X$, and $\mathcal{O}_X/V^{>r}\mathcal{O}_X = \mathcal{O}_Z$ because $\mathcal{O}_Z$ does not contain a submodule whose support has strictly smaller dimension. Since $\omega_X^\vee \otimes_{\mathcal{O}} \widetilde{\omega}_Z$ is a subquotient of $\mathcal{O}_X/V^{>r}\mathcal{O}_X = \mathcal{O}_Z$, it is a submodule by a similar argument, and it is contained in $\mathrm{Gr}_V^r\mathcal{O}_X$. Then, defining $\mathcal{J}(X, rD)'$ to be the coherent ideal of $\mathcal{O}_X$ corresponding to $\omega_X^\vee \otimes_{\mathcal{O}} \widetilde{\omega}_Z \subset \mathrm{Gr}_V^r\mathcal{O}_X$, the assertion follows from Theorem 1.

3.7 Proof of Theorem 4

Assume first that Z has at most rational singularities. By Theorems 2 and 3, we have $\alpha'_f = r$ because $-r$ is a root of $b_f(s)$. Furthermore, $\widetilde{\omega}_Z = \omega_Z$, and hence $\rho_\bullet F_{-n}M_{Z'} = F_{-n}\mathrm{Gr}_V^r\widetilde{M}$ in the above notation. (This is closely related with Corollary 1.)

Let $\widetilde{t}$ be as in (1.3.2), and define $N = \widetilde{t}\partial/\partial\widetilde{t}$ on $\mathrm{Gr}_V^1 j_*q^*M$. Since M is pure of weight n, q^*M is pure of weight $n+1$ by the property of external products, see [Sai90, 2.17.4]. (To simplify the notation, we do not shift the filtration F as in [Sai88, Sai90] when we take q^*. Note that the filtration is shifted to the opposite direction when we take Gr_1^V in *loc. cit.*, and these two shifts cancel out. Also we do not shift the complex as in [Sai90] when we take q^*. So q^*M is a $\mathcal{D}$-module.)

By definition [Sai88, 5.1.6] the weight filtration W on $\mathrm{Gr}_V^1 j_*q^*M$ is the monodromy filtration shifted by n, i.e. it is characterized by the conditions $NW_i \subset W_{i-2}$ and

$$N^j : \mathrm{Gr}_{n+j}^W \xrightarrow{\sim} \mathrm{Gr}_{n-j}^W \quad \text{for } j > 0. \tag{3.7.1}$$

Furthermore, $\mathrm{Gr}_n^W \mathrm{Gr}_V^1 j_*q^*M$ underlies a semisimple Hodge module (see [Sai88, 5.2.13]), and $M_{Z'}$ is a direct factor of it.

Let

$$K := \mathrm{Ker}(N : \mathrm{Gr}_V^1 j_*q^*M \to \mathrm{Gr}_V^1 j_*q^*M).$$

Then we have $M_{Z'} \subset \mathrm{Gr}_n^W K$ on $T_Z X \times Z_{\mathrm{reg}}$, and it holds everywhere by the property of an intersection complex (see [BBD82]). This implies that $F_{-n}\mathrm{Gr}_V^r M \otimes 1$ is also contained in $\mathrm{Gr}_n^W K$, and hence the action of $\sum_{i\leqslant r} \widetilde{x}_i\partial/\partial\widetilde{x}_i$ on $F_{-n}\mathrm{Gr}_V^r M \otimes 1$ vanishes; see (1.3.2). So the multiplicity of $-r$ as a root of the b-function is 1, and $\alpha_f > r$.

Assume conversely that $\alpha_f > r$, i.e. $\alpha'_f = r$ with multiplicity 1 as a root of $b_f(-s)$. Then $F_{-n}\mathrm{Gr}_V^\alpha M = 0$ for $\alpha < r$, and $F_{-n}\mathrm{Gr}_V^r M = \mathcal{O}_Z$, because $F_{-n}\mathrm{Gr}_V^r M$ is a quotient of $\mathcal{O}_Z$ and there is no nontrivial $\mathcal{O}_Z$-submodule of $\mathcal{O}_Z$ supported on $\mathrm{Sing}\, Z$. So it is enough to show that $\omega_X^\vee \otimes_{\mathcal{O}} \widetilde{\omega}_Z$ is a direct factor of $\mathcal{O}_Z$, or more generally that (3.6.1) is a direct factor of (3.6.2).

Since the multiplicity of r is 1, we see that $F_{-n}\mathrm{Gr}_V^r M \otimes 1$ is contained in K, and so is (3.6.2) because N is compatible with the action of $\bigoplus_{i\geqslant 0} I_Z^i/I_Z^{i+1} \otimes t^{-i}$. (Here (3.6.2) can be calculated as in § 3.8 below.) Then, by the semisimplicity of the Hodge module underlying $\mathrm{Gr}_n^W K$ (see [Sai88, 5.2.13]), it is sufficient to show that $F_{-n}\mathrm{Gr}_i^W K = 0$ for $i < n$. (Here $\mathrm{Gr}_i^W K = 0$ for $i > n$ by (3.7.1).) However, this follows from $F_{-n-1}\mathrm{Gr}_V^1 j_*q^*M = 0$ together with the fact that N is a morphism of type $(-1,-1)$, because the latter implies the isomorphisms (see [Sai88, 5.1.14])

$$N^j : F_p\mathrm{Gr}_{n+j}^W \xrightarrow{\sim} F_{p+j}\mathrm{Gr}_{n-j}^W \quad \text{for } j > 0.$$

This completes the proof of Theorem 4.

3.8 Proof of Corollary 1

We have $1 \in V^r\mathcal{O}_X$ by Theorem 3, and hence $\prod_i f_i^{\nu_i} \in V^{r+j}\mathcal{O}_X$ with $j = |\nu|$. Using the same argument as in § 3.6, we can show by induction on $j \geqslant 0$ that $V^{>r+j}\mathcal{O}_X$ is generated by $\prod_i f_i^{\nu_i}$ with $|\nu| = j+1$, and $\mathrm{Gr}_V^{r+j}\mathcal{O}_X$ is a free $\mathcal{O}_Z$-module generated by $\prod_i f_i^{\nu_i}$ with $|\nu| = j$ so that $\mathrm{Gr}_V^{\alpha}\mathcal{O}_X = 0$ for $\alpha \notin \mathbb{Z}$. Then the assertion follows from Theorem 1.

3.9 Proof of Corollary 2

This follows from Theorem 1 using (2.1.3).

Proposition 3.10. *With the notation and assumptions of § 3.1, the union of the subgroups of $\mathbb{Q}/\mathbb{Z}$ generated by $1/m_j$ contains the images of the roots of $b_Z(s)$ in $\mathbb{Q}/\mathbb{Z}$.*

Proof. We may assume that the ideal of Z is generated by $f_1, \ldots, f_r$. By Corollary 2.8, it is sufficient to consider the eigenvalues of the monodromy on $\psi_t(\mathbf{R}j_*q^*K)$, where $K = (i_f)_*\mathbb{C}_{X^{\mathrm{an}}}$. Since the construction of the deformation space to the tangent cone as in § 3.1 is compatible with the embedded resolution $(Y, D) \to (X, Z)$, the assertion is reduced to the case where Z is a divisor with normal crossings on a smooth variety, using the commutativity of the nearby cycle functor with the direct image under a proper morphism together with the decomposition theorem [BBD82]. In the divisor case, it is known that the eigenvalues of the monodromy for $\psi_t(\mathbf{R}j_*q^*K)$ coincide with those for $\psi_{g'}K$ where g' is a local equation of the divisor. (This also follows from (1.3.1).) Furthermore, the eigenvalues of the monodromy in the normal crossing case are $\exp(2\pi ik/m_j)$ $(k \in \mathbb{Z})$, where the m_j are the multiplicities of the irreducible components of the divisor (see also [Kas76/77, Mal83]). So the assertion follows. □

4. Calculations in the monomial ideal case

In this section we treat the case where the ideal of the subvariety Z (which is not necessarily reduced nor irreducible) is a monomial ideal, and calculate some examples.

4.1 Monomial ideal case

Assume that X is the affine space $\mathbb{A}^n$ and the f_i are monomials with respect to the coordinate system $(x_1, \ldots, x_n)$ of $\mathbb{A}^n$. Write $f_j = \prod_{i=1}^n x_i^{a_{i,j}}$. Let $\ell_i(\mathbf{s}) = \sum_{j=1}^r a_{i,j}s_j$ for $\mathbf{s} = (s_1, \ldots, s_r)$ so that

$$\prod_{j=1}^r f_j^{s_j} = \prod_{i=1}^n x_i^{\ell_i(\mathbf{s})}.$$

Let $\ell(c) = (\ell_1(c), \ldots, \ell_n(c))$, and $I'(\ell(c))_+ = \{i : \ell_i(c) > 0\}$. Let $I(c)_-$ be as in § 2.10, and define

$$g_c(s_1, \ldots, s_r) = \prod_{j \in I(c)_-} \binom{s_j}{-c_j} \cdot \prod_{i \in I'(\ell(c))_+} \binom{\ell_i(\mathbf{s}) + \ell_i(c)}{\ell_i(c)}.$$

Let $\mathfrak{a}_f$ be the ideal of $R := \mathbb{Q}[s_1, \ldots, s_r]$ generated by $g_c(s_1, \ldots, s_r)$, where $c = (c_1, \ldots, c_r)$ runs over the elements of $\mathbb{Z}^r$ such that $\sum_i c_i = 1$ (see [SMST00] for a more general case). In a forthcoming paper [BMS05], we will give a combinatorial description of the roots of the Bernstein–Sato polynomial for monomial ideals. For the proof, the following is used in an essential way.

Proposition 4.2. *With the above notation and assumption, the Bernstein–Sato polynomial $b_f(s)$ is the monic polynomial of smallest degree such that $b_f(\sum_i s_i)$ belongs to the ideal $\mathfrak{a}_f$.*

Proof. This follows from § 2.10 using the $\mathbb{Z}^n$-grading on $\mathcal{D}_X$ such that the degree of x_i is the ith unit vector e_i of $\mathbb{Z}^n$, and the degree of ∂_{x_i} is $-e_i$. □

4.3 Finite generators of $\mathfrak{a}_f$

It follows from Proposition 4.2 that, in order to compute the Bernstein–Sato polynomial in the monomial case, we need to solve the elimination problem consisting of computing $\mathbb{Q}[s_1+\cdots+s_r]\cap\mathfrak{a}_f$. This can be done using a computer algebra program (e.g. Macaulay2) if we can write down finitely many generators for the ideal $\mathfrak{a}_f$. (Here we may also consider locally the subscheme of $\mathbb{A}^n$ defined by $\mathfrak{a}_f$, and then take the direct image of its structure sheaf by the map $(s_1,\dots,s_r)\mapsto\sum_i s_i$, because it is enough to determine the *annihilator* of the direct image sheaf. Using this, we can calculate the examples below by hand.)

We now explain how to find finite generators of $\mathfrak{a}_f$. With the notation and the assumption of § 4.1, let $A_\delta=\{\sum_i c_i=\delta\}\subset\mathbb{Z}^r$ for $\delta=0,1$. Let $\ell_{n+i}(c)=-c_i$ for $i=1,\dots,r$. For $\varepsilon=(\varepsilon_1,\dots,\varepsilon_{n+r})\in\{1,-1\}^{n+r}$, define

$$A_\delta^\varepsilon=\{c\in A_\delta:\varepsilon_i\ell_i(c)\geqslant 0\quad\text{for } i=1,\dots,n+r\}.$$

If there is a subset $I(\varepsilon)$ of A_1^ε for any ε such that

$$A_1^\varepsilon=\bigcup\nolimits_{c\in I(\varepsilon)}(c+A_0^\varepsilon),\tag{4.3.1}$$

then we see that $\mathfrak{a}_f$ is generated by the g_c for $c\in I(\varepsilon)$ and $\varepsilon\in\{1,-1\}^{n+r}$. For each ε it is easy to show the existence of a finite subset $I(\varepsilon)$ satisfying (4.3.1). In fact, it is probable that some algorithms to find such finite subsets are already known. (For example, if we find some $v\in A_1^\varepsilon$, we can proceed by induction on r, considering the complement of $v+A_0^\varepsilon$ and taking the restrictions of A_1^ε to the hyperplanes $\{\varepsilon_i\ell_i(c)=m_i\}$ for $0\leqslant m_i<\varepsilon_i\ell_i(v)$ and for every i.)

4.4 Multiplier ideals of monomial ideals

With the above notation and assumption, let $\mathbf{a}_j=(a_{1,j},\dots,a_{n,j})$, and let P be the convex hull of $\bigcup_j(\mathbf{a}_j+\mathbb{R}^n_{\geqslant 0})$ in $\mathbb{R}^n$. Set $\mathbf{1}=(1,\dots,1)\in\mathbb{Z}^n$. By [Laz04] (see also [How01]), $\mathcal{J}(X,\alpha Z)$ is generated by the monomials $x^\nu:=\prod_i x_i^{\nu_i}$ with $\nu+\mathbf{1}\in(\alpha+\varepsilon)P$ where $\varepsilon>0$ is sufficiently small.

As a corollary, the jumping coefficients are the numbers α such that $(\alpha\cdot\partial P)\cap\mathbb{Z}^n_{>0}\neq\emptyset$, where ∂P is the boundary of P. Let ϕ_j be linear functions on $\mathbb{R}^n$ such that the maximal dimensional faces of P are contained in $\phi_j=1$. Then the jumping coefficient corresponding to $x\in\mathbb{Z}^n_{>0}$ (i.e. the number α such that $x\in\alpha\cdot\partial P$) is given by $\min\{\phi_j(x)\}$.

4.5 Examples

(i) With the notation of § 4.1, let $f_i=\prod_{j\neq i}x_j$ for $i=1,\dots,n$. If $n=3$, then $b_f(s)=(s+3/2)(s+2)^2$. Indeed, we have $\ell_i(\mathbf{s})=\sum_{j\neq i}s_j$, and $\mathfrak{a}_f$ is generated by $g_{(1,0,0)}=(\ell_2(\mathbf{s})+1)(\ell_3(\mathbf{s})+1)$ and $g_{(1,1,-1)}=s_3(\ell_3(\mathbf{s})+1)(\ell_3(\mathbf{s})+2)$ up to a permutation, see § 4.3. So the assertion follows by using $u_i:=\ell_i(\mathbf{s})+1$. Note that if we consider only g_c with all $c_i\geqslant 0$, then we get only $(\ell_2(\mathbf{s})+1)(\ell_3(\mathbf{s})+1)$ up to a permutation, and there is no nonzero polynomial $b(s)$ such that $b(\sum s_i)$ belongs to the ideal generated by these g_c.
In this case, the polyhedron P in § 4.4 is defined by the inequalities $\sum_i x_i\geqslant 2$ and $\sum_{i\neq j}x_i\geqslant 1$ for all j. Thus the jumping coefficients of this ideal are $k/2$ for $k\geqslant 3$ by § 4.4, and this is compatible with Theorem 2.

(ii) The above calculation can be extended to the case of a monomial ideal generated by x_ix_j for $1\leqslant i<j\leqslant n$ with $n>3$ (i.e. Z is the union of coordinate axes). In this case we have $b_f(s)=(s+n/2)(s+(n+1)/2)(s+n-1)$. Note that $n/2,(n+1)/2$ are roots of $b_f(-s)$ by § 4.4 together with Theorem 2, and $n-1$ is a root of $b_f(-s)$ by restricting to a smooth point of Z. (So it is enough to show that the above polynomial belongs to $\mathfrak{a}_f$ to determine the b-function in this case, although it is not difficult to generalize the above calculation.)

(iii) With the notation of (i), assume $n = 4$. In this case we can show that $b_f(s) = (s + 4/3)(s + 5/3)(s + 3/2)(s + 2)^3$. Note that $4/3$ and $3/2$ are jumping coefficients, but $5/3$ is not. (Indeed, the linear functions in § 4.4 are given by $\sum_k x_k/3$, $\sum_{k \neq i} x_k/2$, and $\sum_{k \neq i,j} x_k$ for $i \neq j$.)

(iv) Assume $n = 3$ and $f_i = x_i \prod_{j=1}^{3} x_j$ for $i = 1, 2, 3$. Then we have $b_f(s) = (s + 3/4)(s + 5/4)(s + 6/4)(s + 1)^3$. Here $3/4$ and $6/4$ are jumping coefficients, but $5/4$ is not. Note also that the support of $R/\mathfrak{a}_f$ is not discrete in this case.

4.6 Integral closure of the ideal and *b*-function

It is known that the jumping coefficients depend only on the integral closure of the ideal, see [Laz04, 11.1]. For monomial ideals, this also follows from the description in § 4.4. However, this does not hold for the b-function. For example, consider the ideal generated by $x^2, y^2 \in \mathbb{C}[x, y]$. Its b-function is $(s + 1)(s + 3/2)(s + 2)$ by Theorem 4. However, the integral closure of this ideal is generated by $x^2, xy, y^2 \in \mathbb{C}[x, y]$, and one can check that its b-function is $(s + 1)(s + 3/2)$.

5. Relation with other Bernstein–Sato polynomials

5.1 Bernstein–Sato polynomials of several variables

With the notation of § 2.4, let $w = (w_i)_i \in \mathbb{N}^r$, and consider the ideal $B_w(f) \subseteq \mathbb{C}[s_1, \ldots, s_r]$ consisting of those polynomials b satisfying

$$b(s_1, \ldots, s_r) \prod_i f_i^{s_i} \in \mathcal{D}_X[s_i, \ldots, s_r] \cdot \prod_i f_i^{s_i + w_i}.$$

It was shown by Sabbah [Sab87b] that $B_w(f) \neq (0)$.

Suppose now that the f_i are monomials as in § 4.1. Then with the notation of § 4.1, we can show that the ideal $B_w(f)$ is generated by

$$b_{f,w}(s_1, \ldots, s_r) := \prod_{i=1}^{n} (\ell_i(\mathbf{s}) + 1) \cdots (\ell_i(\mathbf{s}) + \ell_i(w)),$$

using the $\mathbb{Z}^n$-grading on $\mathcal{D}_X$ as in the proof of Proposition 4.2. So this can be called the Bernstein–Sato polynomial of several variables in this monomial case. However, it does not seem easy to relate this with our Bernstein–Sato polynomial of one variable. For example, if $f_1 = x^\alpha y$, $f_2 = xy^\beta$ with $\alpha, \beta \geqslant 2$ and $n = r = 2$, then

$$\ell_1 = \alpha s_1 + s_2, \quad \ell_2 = s_1 + \beta s_2,$$

and the roots of the Bernstein–Sato polynomial of one variable $b_f(s)$ are

$$-\frac{(\beta - 1)i + (\alpha - 1)j}{\alpha\beta - 1} \quad \text{for } 1 \leqslant i \leqslant \alpha, 1 \leqslant j \leqslant \beta,$$

and -1 is also a root if it is not included there.

5.2 Bernstein–Sato polynomials of one variable in [Sab87b, I]

With the notation of § 2.4, $b_f(s)$ coincides (up to a shift of variable) with the polynomial b_L of the minimal degree satisfying the relation in [Sab87b, I, 3.1.1], in the algebraic setting, if $\mathcal{M}$ in *loc. cit.* is the direct image of $\mathcal{O}$ by the graph embedding, the multi-filtration U is induced by that on $\mathcal{D}_X$ using the action of $\mathcal{D}_X$ on $1 \otimes 1 \in \mathcal{M}$, and $L(s) = s_1 + \cdots + s_r$ (see also [Gyo93, 2.13]).

If X is affine space $\mathbb{A}^n$, then $b_f(s)$ coincides with the b-function in [SMST00, p. 194], if the weight vector is chosen appropriately. Algorithms to compute this b-function are given in [SMST00, p. 196].

Remark 5.3. The existence and the rationality of the roots of the polynomial b_α in [Gyo93, 2.13] can be reduced to the case $\alpha = (1, \dots, 1)$ by considering the pull-back by the base change of the finite covering $\pi : (\tilde{\tau}_j) \in \mathbb{C}^l \mapsto (\tau_j) = (\tilde{\tau}_j^{\alpha_j}) \in \mathbb{C}^l$ and taking the invariant by the covering transformation group $\prod_j (\mathbb{Z}/\alpha_j\mathbb{Z})$. (Here we may assume that $\alpha \in \mathbb{Z}^r_{>0}$ by replacing r and changing the decomposition $X \times E$ in *loc. cit.* if necessary.) Then the assertions easily follow from the theory of Kashiwara and Malgrange using (1.3.1).

5.4 Bernstein–Sato polynomials of one variable in [Sab87b, II]

With the notation of § 2.4, we can define a polynomial b'_L associated to a linear form $L : \mathbb{Z}^r \to \mathbb{Z}$ to be the monic polynomial of the smallest degree satisfying the relation

$$b'_L(L(\mathbf{s})) \prod_i f_i^{s_i} \in \sum_\nu \mathcal{D}_X[s_1, \dots, s_r] \prod_i f_i^{s_i+\nu_i}, \tag{5.4.1}$$

where the summation is taken over $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{Z}^r$ such that $L(\nu) = 1$, see [Sab87b, II, Proposition 1.1]. By § 2.10, this polynomial b'_L for $L(\mathbf{s}) = \sum_i s_i$ divides our Bernstein–Sato polynomial $b_f(s)$, but they do not coincide, and, furthermore, Theorem 2 does not hold for b'_L in general. For example, consider the case $f_1 = x^3, f_2 = x^2y$ with $n = r = 2$. Then

$$f_1^{s_1+1-k} f_2^{s_2+k} = x^{\ell_1+3-k} y^{\ell_2+k} \quad \text{for } 0 \leqslant k \leqslant 3,$$

where $\ell_1 = 3s_1 + 2s_2, \ell_2 = s_2$. Applying $\partial_x^{3-k}\partial_y^k$ to this, we see that (5.4.1) holds with $b'_L(L(\mathbf{s}))$ replaced by

$$\prod_{1 \leqslant i \leqslant 3-k} (\ell_1 + i) \cdot \prod_{1 \leqslant j \leqslant k} (\ell_2 + j).$$

This implies that $b'_L(s) = (s + 2/3)(s + 1)(s + 4/3)$, and $-1/2$ is not a root of it. However, this is not compatible with the jumping coefficients which include $1/2$ (see § 4.4), and Theorem 2 does not hold for this polynomial.

Note also that the polynomial $b'_L(s)$ depends on the choice of generators of the ideal of Z. Indeed, if we put $g_i = f_i$ for $i \leqslant r$ and $g_{r+1} = f_1^2$, then $g_1^2 g_{r+1}^{-1} = 1$ and the polynomial associated to $g_1, \dots, g_{r+1}$ is 1.

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