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ON THE NON-EXISTENCE OF A PROJECTION ONTO THE SPACE OF COMPACT OPERATORS

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ABSTRACT. Let X and Y be Banach spaces, $L(X, Y)$ the space of bounded linear operators from X to Y and $C(X, Y)$ its subspace of the compact operators. A sequence $\{T_i\}$ in $C(X, Y)$ is said to be an unconditional compact expansion of $T \in L(X, Y)$ if $\sum T_i x$ converges unconditionally to Tx for every $x \in X$. We prove: (1) If there exists a non-compact $T \in L(X, Y)$ admitting an unconditional compact expansion then $C(X, Y)$ is not complemented in $L(X, Y)$, and (2) Let X and Y be classical Banach spaces (i.e. spaces whose duals are some $L_p(\mu)$ spaces) then either $L(X, Y) = C(X, Y)$ or $C(X, Y)$ is not complemented in $L(X, Y)$.

1. **Introduction.** The following problem has been considered by many authors (see [1], [4], [5], [6], [7], [12], and [13]).

PROBLEM 1. Are the following two properties equivalent for every pair of Banach spaces?

(a) There is a projection from the space $L(X, Y)$ onto its subspace $C(X, Y)$.

(b) $L(X, Y) = C(X, Y)$, i.e. every operator from X to Y is compact.

For definitions and notations, see Section 2. Clearly (b) implies (a). Recently, J. Johnson [5] observed that (a) and (b) are equivalent for many pairs of classical Banach spaces. However, the following cases were left open (see [5], Table 1): $X = C(K)$ where K is non-dispersed (including $X = l_\infty$) and $Y = L_1$ or $Y = C(S)$; $X = L_1$ and $Y = C(S)$. These problems are solved in Section 4 and the L_1 preduals are also discussed. The solution is obtained using the following theorem which is a generalization of [4, Lemma 2] and of the main results of the before-mentioned papers.

THEOREM 1. *Let X and Y be Banach spaces. Suppose that there exists a non-compact $T \in L(X, Y)$ admitting an unconditional compact expansion. Then $C(X, Y)$ is uncomplemented in $L(X, Y)$.*

In particular, if X is infinite dimensional and $c_0 \subset Y$, then $C(X, Y)$ is uncomplemented in $L(X, Y)$. We go on to prove

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THEOREM 2. *Let X and Y be classical Banach spaces. Then either every operator from X to Y is compact or $C(X, Y)$ is uncomplemented in $L(X, Y)$.*

Recall that a Banach space Z is said to be "classical" if its conjugate Z^* is isometric to an $L_p(\mu)$ space. Among these spaces are: the $C(K)$ spaces, $L_p(\mu)$ spaces, l_p spaces, c_0 and the other L_1-preduals.

2. **Preliminaries.** An "operator" in this paper is always bounded and linear. X, Y, E, and F will always denote Banach spaces. $L(X, Y)$ denotes the Banach space of all operators from X to Y, with the sup norm. $C(X, Y)$ is the subspace of $L(X, Y)$ consisting of the compact operators. We will not distinguish between an operator $T:X \to Y$ and its astriction $T_a : X \to \overline{T(X)}$ (given by $T_a x = Tx$). A "projection" P is an idempotent ($P^2 = P$) operator from a Banach space X to itself. We will also regard P as an operator from X onto $P(X)$ which extends the identity $I : P(X) \to P(X)$. A subspace Y of X is complemented in X if there exists a projection from X onto Y. We denote the ith coordinate of $\xi \in l_\infty$ by ξ_i, i.e., $\xi = (\xi_i)$. ξ is supported on the set M of integers if $\xi_i \neq 0$ implies $i \in M$.

Let $T \in L(X, Y)$. A sequence $\{T_i\}$ in $C(X, Y)$ is said to be an unconditional compact expansion of T if $\sum T_i x$ converges unconditionally to Tx for every $x \in X$. In this case we shall write $\sum T_i = T$. Note that if $\xi \in l_\infty$, $\sum \xi_i T_i \in L(X, Y)$ is well defined (i.e. there is a $T_0 \in L(X, Y)$ with an unconditional compact expansion $T_0 = \sum \xi_i T_i$).

A series $\sum x_i$ in E is said to be weakly unconditionally Cauchy (w.u.C.) if $\sum |x^*(x_i)| < \infty$ for every $x^* \in E^*$. $\sum x_i$ is weakly subseries convergent if $\sum x_{n_i}$ converges weakly to some element of E for every increasing sequence of integers. This implies, by the Orlicz–Pettis theorem, that $\sum x_i$ is subseries convergent in norm, hence unconditionally convergent in norm (see Day [3, pp. 78–80]).

3. **Proof of Theorem 1.** We generalize and use ideas of Kalton [6].

Proof. Let $\{T_i\}$ be a sequence in $C(X, Y)$ such that $\sum T_i x$ converges unconditionally to Tx for every $x \in X$. We will consider two cases.

CASE 1. There is a $y^* \in Y^*$ such that $\sum T_i^* y^*$ is not weakly subseries convergent.

For every $x \in X$, $\sum (T_i^* y^*)(x) = \sum y^*(T_i x)$ is absolutely convergent. By a result of Orlicz and Mazur (see [11, p. 432, Lemma 15.1]) $\sum T_i^* y^*$ is w.u.C. By Bessaga and Pełczyński [2], X contains a complemented subspace isomorphic to l_1 since $\sum T_{\pi(i)}^* y^*$ is not weakly convergent for some permutation π of the positive integers. Since T is non-compact, Y is infinite dimensional. By Kalton [6, Lemma 3] $C(X, Y)$ is uncomplemented in $L(X, Y)$.

CASE 2. For every $y^* \in Y^*$, $\sum T_i^* y^*$ is weakly subseries convergent.

There is a separable subspace E of X such that the restriction $T \mid E$ is non-compact. If $J: E \to X$ denotes the inclusion map, then $T \mid E = TJ$. Since TJ is non-compact, the series $\sum T_i J$ diverges in the norm topology of $C(E, Y)$. Thus we may assume, without loss of generality, that $\inf_i \|T_i J\| > 0$ (otherwise, replace $\{T_i\}$ by a suitable sequence of blocks $\{B_n = \sum_{i=p_n+1}^{p_{n+1}} T_i\}$ with $\inf_n \|B_n J\| > 0$; $T = \sum B_n$ is also an unconditional compact expansion of T). Put $F = \overline{\text{span}} \cup T_i(E)$. F is a separable subspace of Y. Let $S_0 : F \to l_\infty$ be an isometrical embedding and let $S: Y \to l_\infty$ be an extension of S_0. Assume now to the contrary, that there exists a projection $p : L(X, Y) \to L(X, Y)$ onto $C(X, Y)$. Consider the following string of maps

$$l_\infty \xrightarrow{w} L(X, Y) \xrightarrow{q} L(X, Y) \xrightarrow{v} L(E, l_\infty)$$

where $w(\xi) = \sum \xi_i T_i$, $q = id_{L(x,y)} - p$ and $v(R) = SRJ$. Let $\phi = v \circ q \circ w : l_\infty \to L(E, l_\infty)$. Clearly, $\phi(c_0) = 0$. By Kalton [6, Prop. 5] there is an infinite set M of integers such that $\phi(\xi) = 0$ whenever ξ is supported on M. Let $\xi \in l_\infty$ be supported on M; then $S \sum \xi_i T_i J = S[p(\sum \xi_i T_i)]J$ is compact. Hence $K_\xi = \sum \xi_i T_i J$ is compact. Let $y^* \in Y^*$ and $\xi \in l_\infty$ be given. By our assumption, $\sum T_i^* y^*$ is weakly subseries convergent. (By the Orlicz–Pettis Theorem, it is also unconditionally convergent.) For every $\xi \in l_\infty$ which is supported on M, $\sum \xi_i (T_i J)^* y^* = \sum \xi_i J^* T_i^* y^*$ converges unconditionally to $K_\xi^* y^*$. By Kalton [6, Cor. 3], $\sum_{i \in m} T_i J$ is weakly subseries convergent in $C(E, Y)$. By the Orlicz–Pettis Theorem, $\sum_{i \in M} T_i J$ converges in norm, contrary to the assumption that $\inf \|T_i J\| > 0$. Hence no projection from $L(X, Y)$ to $C(X, Y)$ exists.

The following corollary generalizes Theorem 4 of J. Johnson [5].

COROLLARY 1. *Let X be infinite dimensional and suppose that Y contains an isomorphic copy of c_0. Then $C(X, Y)$ is not complemented in $L(X, Y)$.*

Proof. A result of Josefson and Nissenzweig (see [4] or [5]) says that if $\dim X = \infty$ then there is a non-compact operator $T : X \to c_0$. Let $V : c_0 \to Y$ be an isomorphism (into). VT is non-compact and has an unconditional compact expansion. Now use Theorem 1.

4. **Proof of Theorem 2.** A result of Zippin [14] states that every infinite dimensional L_1-predual contains a copy of c_0. Combining this with Corollary 1 we get

COROLLARY 2. Let X and Y be infinite dimensional and let Y be an L_1-predual. Then $C(X, Y)$ is uncomplemented in $L(X, Y)$.

Proof of Theorem 2. Assume that $L(X, Y) \neq C(X, Y)$. We have to prove that $C(X, Y)$ is uncomplemented in $L(X, Y)$.

(1) If Y is an L_1-predual, we are done by Corollary 2.

(2) If $X = L_1(\mu)$, Kalton [6, Lemma 3] gives the result.

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(3) If $X = L_p(\mu)$, $1 < p < \infty$ and $Y = L_r(\nu)$, $1 \leq r < \infty$ then by the proof of Theorem A2 of Rosenthal [10], there is a non-compact $T : X \to Y$ factoring through l_p or l_2 (since we assumed that $L(X, Y) \neq C(X, Y)$). Now use Theorem 1.

(4) If $X^* = L_1(\mu)$ and $Y = L_r(\nu)$ $(1 \leq r < \infty)$ then μ cannot be purely atomic (otherwise, $L(Y^*, l_1(\Gamma)) \neq C(Y^*, l_1(\Gamma))$ for some Γ, since $L(X, Y) \neq C(X, Y)$) Also $r \geq 2$ or ν is not purely atomic. This implies that there is a non-compact $T : X \to Y$ factoring through l_2. Again, use Theorem 1.

We conclude with

PROBLEM 2. Is there a pair of Banach spaces X and Y such that $L(X, Y) \neq C(X, Y)$ and yet no non-compact operator from X to Y admits an unconditional compact expansion?

Clearly a negative answer to Problem 1 would answer Problem 2 positively.

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