ENGEL CONGRUENCES IN GROUPS OF PRIME-POWER EXPONENT

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It is a well-known result of Sanov (**5**) that groups of exponent p^k (p prime) satisfy the $(kp^k - 1)$th Engel congruence (definition below). Recently, an alternative proof of this has been given by Glauberman, Krause, and Struik (**3**). Bruck (**2**) has conjectured that such groups satisfy the $(kp^k - (k-1)p^{k-1} - 1)$th Engel congruence. In this note we go some way towards proving this.

THEOREM 1. *Groups of exponent p^k satisfy the $(kp^k - 1 - \sum_{i=0}^{k-1} p^i + k)$th Engel congruence.*

For $k = 2$, a slight modification of our argument proves Bruck's conjecture.

THEOREM 2. *Groups of exponent p^2 satisfy the $(2p^2 - p - 1)$th Engel congruence.*

This result is close to best possible for there are metabelian groups (**1**, Corollary 2) of exponent p^2 which do not satisfy the $(2p^2 - 2p - 1)$th Engel congruence.

As usual, we write $[a, b]$ for the commutator $a^{-1}b^{-1}ab$, use the left-normed convention $[a, b, c] = [[a, b], c]$ and define $[a, nb] = [a, (n-1)b, b]$ for $n \geqq 2$. The nth term $\gamma_n(G)$ of the lower central series of a group G is the normal subgroup generated by the commutators $[a_1, \ldots, a_n]$ for all $a_1, \ldots, a_n$ in G. If $[a, nb] \in \gamma_{n+2}(G)$, then G satisfies the nth *Engel congruence.*

Let p be a prime and k a positive integer; let F be free in the variety of groups of exponent p^k freely generated by $Y = \{y_0, y_1, \ldots\}$. For each commutator c with entries in Y, let $w_i(c)$ denote its weight in y_i and $w(c)$ its weight. Let Z be the subset of commutators-in-Y defined (recursively) by: $c \in Z$ if

(a) $w_0(c) \geqq 1$ and $w(c) \geqq 2$;

thus $c = [c_1, c_2]$, *and*

(b) $c_1 \in Z$, or $c_2 \in Z$, or for all i in $\{1, 2\}$, $w_0(c_i) \geqq 1$ or $w(c_i) \geqq 2$.

Clearly, Z is closed under commutation. The subgroup K generated by Z is normal because F has finite exponent. Consider $G = F/K$ and let d be the coset y_0K. Obviously, the normal closure N of d is abelian. Let Γ be the multiplicative subgroup of the endomorphism ring E of N consisting of the automorphisms induced in N by the action of G, that is, $\xi \in \Gamma$ if and only if there is an x in G such that $d^*\xi = x^{-1}d^*x$ for all $d^* \in N$. Let P be the subring of E generated by Γ, then, clearly, P is a commutative ring with identity one.

Received February 24, 1967.

https://doi.org/10.4153/CJM-1968-131-5 Published online by Cambridge University Press

Since G has exponent dividing p^k, we have (**4**, equation (3)) that

$$\text{(1)} \qquad p^h \prod_{r=h}^{k-1} \prod_{i=1}^{f(r)} (\xi_{ir}^{p^r} - 1) = 0$$

for all ξ_{ir} in Γ and all $h \in \{0, \ldots, k-1\}$, where $f(r) = p^{k-r} - p^{k-r-1}$.

We now prove by double induction on $t - h \in \{0, \ldots, k-h-1\}$ and $s \in \{0, \ldots, f(t)\}$ that

$$\text{(2)} \qquad p^h \prod_{r=h}^{t-1} \prod_{i=1}^{f(r)+\delta(r)} (\xi_{ir} - 1)^{p^r} \prod_{i=1}^{f(t)-s} (\xi_{it}^{p^t} - 1) \prod_{i=f(t)-s+1}^{f(t)+1-\delta_{h,t}} (\xi_{it} - 1)^{p^t} \times \prod_{r=t+1}^{k-1} \prod_{i=1}^{f(r)} (\xi_{ir}^{p^r} - 1) = 0$$

for all ξ_{ir} in Γ, where $\delta(r) = 1 - \delta_{h,r} - p(1 - \delta_{k-1,r})$ and $\delta_{m,n} = 0$ for $m \neq n$ and $\delta_{m,m} = 1$. For $t - h = 0$, $s = 0$ this comes from (1). Suppose that the result is true for some $t - h \in \{0, \ldots, k-h-2\}$ and $s = f(t)$, then putting $\xi_{it} = \xi_{f(t+1)+1,t+1}$ for $i \in \{f(t) + \delta(t) + 1, \ldots, f(t) + \delta(t) + p\}$ gives the result for $t - h + 1$, $s = 0$. Finally, suppose that the result is true up to some $t - h \in \{0, \ldots, k-h-1\}$ and some $s \in \{0, \ldots, f(t) - 1\}$. Let

$$\rho = p^h \prod_{r=h}^{t-1} \prod_{i=1}^{f(r)+\delta(r)} (\xi_{ir} - 1)^{p^r} \prod_{i=1}^{f(t)-s-1} (\xi_{it}^{p^t} - 1) \prod_{i=f(t)-s+1}^{f(t)+1-\delta_{h,t}} (\xi_{it} - 1)^{p^t} \times \prod_{r=t+1}^{k-1} \prod_{i=1}^{f(r)} (\xi_{ir}^{p^r} - 1),$$

then by the inductive hypothesis, $\rho(\xi_{f(t)-s,t}^{p^t} - 1) = 0$ and $p\rho = 0$ (the latter has h replaced by $h + 1$ and thus has lower "$t - h$"). The binomial theorem then gives $\rho(\xi_{f(t)-s,t} - 1)^{p^t} = 0$ which is the case $t - h$, $s + 1$. Thus (2) is proved.

Putting $h = 0$, $t = k - 1$, and $s = f(k-1)$ in (2) yields

$$\text{(3)} \qquad \prod_{r=0}^{k-1} \prod_{i=1}^{f(r)+\delta(r)} (\xi_{ir} - 1)^{p^r} = 0$$

for all ξ_{ir} in Γ. Let

$$m_r = \sum_{j=0}^{r} (f(j) + \delta(j)) \quad \text{and} \quad m = m_{k-1};$$

then (3) yields, in particular,

$$c = [y_0, y_1, \ldots, y_{m_0}, py_{m_0+1}, \ldots, py_{m_1}, \ldots, p^{k-1}y_m] \in K.$$

Hence, using a lemma of Higman's (see **6**, Lemma 5.1), c can be written as a product of elements of Z each of which has positive weight in $y_1, \ldots, y_m$. Putting $y_1 = y_2 = \ldots = y_m$ in this we have that $[y_0, kp^{k-1}(p-1)y_1]$ can be written as a product of commutators of weight at least 2 in y_0 and at least m in y_1. By a lemma of Lyndon (see **3**, Lemma 4.1) the 2 in the last sentence can

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be replaced by p. Since F is relatively free, y_0 can be replaced in the resulting expression by

$$\left[y_0, \left(kp^{k-1} - \frac{p^k - 1}{p - 1} + k - 1\right)y_1\right]$$

to yield

$$[y_0, ny_1] \in \gamma_{n+2}(F), \qquad \text{where} \quad n = kp^k - 1 - \frac{p^k - 1}{p - 1} + k.$$

Theorem 1 then follows.

The proof of Theorem 2 is similar. We have (**4**, equation (10) with $k = 2$) that

$$p \prod_{i=1}^{p(p-1)} (\xi_i - 1) = 0 \quad \text{for all } \xi_i \text{ in } \Gamma$$

and thus, taking $h = 0$ and $k = 2$ in (1) and applying the binomial theorem, we obtain

$$\prod_{i=1}^{p(p-1)} (\xi_i - 1) \prod_{i=1}^{p-1} (\eta_i - 1)^p = 0 \quad \text{for all } \xi_i, \eta_i \text{ in } \Gamma.$$

Hence, in particular,

$$[y_0, y_1, \ldots, y_{p(p-1)}, py_{p(p-1)+1}, \ldots, py_{p^2}] \in K$$

and, arguing as before, $[y_0, (2p^2 - p - 1)y_1] \in \gamma_{2p^2-p+1}(F)$, and Theorem 2 follows.

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https://doi.org/10.4153/CJM-1968-131-5 Published online by Cambridge University Press