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REGULAR ORDER-PRESERVING TRANSFORMATION SEMIGROUPS

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The semigroup $OT(X)$ of all order-preserving total transformations of a finite chain X is known to be regular. We extend this result to subchains of $\mathbf{Z}$; and we characterise when $OT(X)$ is regular for an interval X in $\mathbf{R}$. We also consider the corresponding idea for partial transformations of arbitrary chains and posets.

1. INTRODUCTION

If X is a set, we let $P(X)$ denote the semigroup under composition of all *partial* transformations of X (that is, mappings $\alpha : A \to B$ where $A, B \subseteq X$); and we note that $P(X)$ is *regular* (that is, for each $\alpha \in P(X)$, there exists $\beta \in P(X)$ such that $\alpha = \alpha\beta\alpha$). Following standard notation, we let $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ denote the *domain* and *range* of $\alpha \in P(X)$.

If $(X, \leqslant)$ is a poset, we say $\alpha \in P(X)$ is *order-preserving* if for all $x, y \in \operatorname{dom}\alpha$, $x \leqslant y$ implies $x\alpha \leqslant y\alpha$; and we let $OP(X)$ denote the semigroup under composition of all order-preserving partial transformations of X. Similarly, $T(X)$ denotes the semigroup of all *total* transformations of X (that is, mappings $\alpha : X \to X$) and likewise it is regular. Also, if $(X, \leqslant)$ is a poset, we let $OT(X)$ denote the subsemigroup of $T(X)$ consisting of all order-preserving total transformations of X.

It is known [**9**, p.203, Exercise 6.1.7] that $OT(X)$ is regular if $(X, \leqslant)$ is a finite chain. In Section 2, we extend this to any chain which is order-isomorphic to a subset of $\mathbf{Z}$, the set of integers with their natural order. We also prove that if X is an interval in $\mathbf{R}$, the set of real numbers, then $OT(X)$ is regular if and only if X is closed and bounded. Then we answer similar questions for $OP(X)$ and some of its subsemigroups for arbitrary chains X. And in Section 3, we suppose X is not a chain and characterise when $OP(X)$ is regular. We also list some conditions under which $OT(X)$ is (or is not) regular when X is not a chain.

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2. Order-preserving transformations of chains

If $(X, \leqslant)$ is a poset, we define the *opposite partial order* $\leqslant_{\text{opp}}$ on X via:

$$x \leqslant_{\text{opp}} y \quad \text{if and only if} \quad y \leqslant x.$$

Note that if $\alpha \in P(X)$ then α preserves $\leqslant$ if and only if it preserves $\leqslant_{\text{opp}}$. Consequently, $OP(X, \leqslant) = OP(X, \leqslant_{\text{opp}})$ and the regularity of $OP(X, \leqslant)$ holds equally for $OP(X, \leqslant_{\text{opp}})$. Similar statements are true for $OT(X, \leqslant)$ and for $OI(X, \leqslant)$, the semigroup of all order-preserving partial transformations of X which are one-to-one ("injective").

For the chain $\mathbf{Z}$ of integers, we let $\mathbf{Z}^+$ denote the set of positive integers and $\mathbf{Z}^-$ the set of negative integers. If X is a chain which is order-isomorphic to a subset of $\mathbf{Z}$ with its natural order then X has one of the following forms:

(1) $\{x_1, x_2, \dots, x_n\}$ where $n \in \mathbf{Z}^+$ and $x_1 < x_2 < \dots < x_n$,
(2) $\{x_i : i \in \mathbf{Z}^+\}$ where $x_i < x_j$ if $i < j$,
(3) $\{x_i : i \in \mathbf{Z}\}$ where $x_i < x_j$ if $i < j$, or
(4) $\{x_i : i \in \mathbf{Z}^-\}$ where $x_i < x_j$ if $i < j$.

Our first result will be needed often in what follows: its simple proof is omitted.

Lemma 2.1. *Let X be a chain. If $\alpha \in OP(X)$ and $a, b \in \operatorname{ran} \alpha$ satisfy $a < b$ then $x < y$ for all $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$.*

For any $A \subseteq X$, we let $\min(A)$ and $\max(A)$ denote the *minimum* and the *maximum* elements of A if they exist.

Theorem 2.2. *Let X be a chain. If X is order-isomorphic to a subset of $\mathbf{Z}$ then the semigroup $OT(X)$ is regular.*

Proof: We regard X as being one of the chains (1)–(4) listed above. Then, if A is any nonempty subset of X, $\max(A)$ exists if A has an upper bound in X, and $\min(A)$ exists if A has a lower bound in X. It follows from this and Lemma 2.1 that, if $\alpha \in OP(X)$ and $a \in \operatorname{ran} \alpha$, then $\max\left(a\alpha^{-1}\right)$ exists if $a < b$ for some $b \in \operatorname{ran} \alpha$, and $\min\left(a\alpha^{-1}\right)$ exists if $b < a$ for some $b \in \operatorname{ran} \alpha$.

Let $\alpha \in OT(X)$. If α is a constant map, it is clearly regular. Therefore, suppose $\operatorname{ran} \alpha$ contains at least two elements and note that, since it is a subchain of X, it takes one of the forms listed in (1)–(4) above. In cases (1)–(3), we define $\beta : X \to X$ as follows:

(1) if $\operatorname{ran} \alpha = \{a_1, a_2, \dots, a_n\}$, let

$$x\beta = \begin{cases} \max\left(a_1\alpha^{-1}\right) & \text{if } x \leqslant a_1, \\ \min\left(a_{i+1}\alpha^{-1}\right) & \text{if } a_i < x \leqslant a_{i+1} \text{ and } i \neq n, \\ \min\left(a_n\alpha^{-1}\right) & \text{if } x \geqslant a_n. \end{cases}$$

(2) if $\operatorname{ran}\alpha = \{a_i : i \in \mathbf{Z}^+\}$, let

$$x\beta = \begin{cases} \max\left(a_1\alpha^{-1}\right) & \text{if } x \leqslant a_1, \\ \min\left(a_{i+1}\alpha^{-1}\right) & \text{if } a_i < x \leqslant a_{i+1} \text{ for some } i \in \mathbf{Z}^+. \end{cases}$$

(3) if $\operatorname{ran}\alpha = \{a_i : i \in \mathbf{Z}\}$, let

$$x\beta = \max\left(a_{i+1}\alpha^{-1}\right) \quad \text{if } a_i < x \leqslant a_{i+1} \text{ for some } i \in \mathbf{Z}.$$

Now, if $x \in X$ then $x\alpha = a_k$ for some $k \in \mathbf{Z}$. By definitions (1)–(3), $a_k\beta$ equals $\max\left(a_k\alpha^{-1}\right)$ or $\min\left(a_k\alpha^{-1}\right)$: in each case, this means $(a_k\beta)\alpha = a_k$, and hence $x(\alpha\beta\alpha) = x\alpha$ for all $x \in X$; that is, $\alpha = \alpha\beta\alpha$.

To show β is order-preserving, suppose $x < y$ in X. In cases (1) and (2), if $y \leqslant a_1$ then $x\beta = y\beta$, and the same conclusion holds in case (1) if $x \geqslant a_n$. On the other hand, in each of (1)–(3), if $a_k < x < y \leqslant a_{k+1}$ for some k then $x\beta = y\beta$, by the definition of β. Suppose instead that

$$a_k < x \leqslant a_{k+1} \leqslant a_\ell < y \leqslant a_{\ell+1}$$

for some k, ℓ. Then $a_{k+1} < a_{\ell+1}$ and Lemma 2.1 imply that $u < v$ for all $u \in a_{k+1}\alpha^{-1}$ and $v \in a_{\ell+1}\alpha^{-1}$. But, by the definition of β, $x\beta \in a_{k+1}\alpha^{-1}$ and $y\beta \in a_{\ell+1}\alpha^{-1}$, hence $x\beta < y\beta$ as required. The remaining possibilities: $x \leqslant a_1 < a_n \leqslant y$ in case (1), and $x \leqslant a_1 \leqslant a_\ell < y \leqslant a_{\ell+1}$ in cases (1) and (2), lead to the same conclusion. Therefore, $\beta \in OT(X)$ is regular in cases (1)–(3).

For case (4), we recall from the start of this section that $OT(\mathbf{Z}^+, \leqslant) = OT(\mathbf{Z}^+, \leqslant_{\text{opp}})$, and clearly $(\mathbf{Z}^-, \leqslant)$ is order-isomorphic to $(\mathbf{Z}^+, \leqslant_{\text{opp}})$. Hence, from our conclusion in case (3), if X is order-isomorphic to $\mathbf{Z}^-$ then $OT(X)$ is regular. ∎

We now show that $OT(\mathbf{R})$ is not regular when $\mathbf{R}$ is the set of real numbers with their natural order. In fact, the following sequence of Lemmas will eventually characterise when $OT(X)$ is regular for an interval X in $\mathbf{R}$.

Lemma 2.3. *The semigroup $OT(\mathbf{R})$ is not regular.*

Proof: Fix $r \in (1, \infty)$ and let $\alpha \in OT(\mathbf{R})$ be the map: $x\alpha = r^x$ for all $x \in \mathbf{R}$. Then $\operatorname{ran}\alpha = \mathbf{R}^+$ and α is one-to-one. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in OT(\mathbf{R})$. Then, since α is one-to-one, $x = x\alpha\beta$ for each $x \in \mathbf{R}$. Thus, $\mathbf{R}^+\beta = \mathbf{R}$. Hence, since $0\beta \in \mathbf{R}$, there exists $d \in \mathbf{R}^+$ such that $0\beta = d\beta$. Choose $c \in (0, d)$. Then

$$0\beta \leqslant c\beta \leqslant d\beta = 0\beta,$$

and we deduce that $c\beta = d\beta$. Since $c, d \in \mathbf{R}^+ = \operatorname{ran}\alpha$, we can choose $x, y \in \mathbf{R}$ with $x\alpha = c$ and $y\alpha = d$. Then

$$c = x\alpha = x\alpha\beta\alpha = c\beta\alpha = d\beta\alpha = y\alpha\beta\alpha = y\alpha = d$$

which contradicts $c < d$. Hence, α is not a regular element of $OT(\mathbf{R})$, and the result follows. ∎

Lemma 2.4. *The semigroup $OT\big((a,\infty)\big)$ is not regular for any $a \in \mathbf{R}$.*

Proof: Fix $r \in [1,\infty)$ and let $\alpha \in OT\big((a,\infty)\big)$ be the map: $x\alpha = x + r$ for all $x \in (a,\infty)$. Then $\operatorname{ran}\alpha = (a+r,\infty)$ and α is one-to-one. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in OT\big((a,\infty)\big)$. Then, since α is one-to-one, $x = x\alpha\beta$ for each $x \in (a,\infty)$. Thus, $(a+r,\infty)\beta = (a,\infty)$. Hence, since $(a+r)\beta \in (a,\infty)$, there exists $d \in (a+r,\infty)$ such that $(a+r)\beta = d\beta$. Choose $c \in (a+r,d)$. Then

$$(a+r)\beta \leqslant c\beta \leqslant d\beta = (a+r)\beta,$$

and we deduce that $c\beta = d\beta$. Since $c,d \in (a+r,\infty) = \operatorname{ran}\alpha$, we can choose $x,y \in (a,\infty)$ with $x\alpha = c$ and $y\alpha = d$. Then

$$c = x\alpha = x\alpha\beta\alpha = c\beta\alpha = d\beta\alpha = y\alpha\beta\alpha = y\alpha = d$$

which contradicts $c < d$. Hence, α is not a regular element of $OT\big((a,\infty)\big)$, and the result follows. ∎

Lemma 2.5. *The semigroup $OT\big([a,\infty)\big)$ is not regular for any $a \in \mathbf{R}$.*

Proof: Fix $r \in [1,\infty)$ and let α be the map:

$$x\alpha = a + \frac{x-a}{x-a+r} \quad \text{for all } x \in [a,\infty).$$

Then $\operatorname{ran}\alpha = [a,a+1)$. Moreover, since the derivative of α is strictly positive on (a,∞), we know α is increasing and hence it is one-to-one. Therefore, $\alpha \in OT\big([a,\infty)\big)$. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in OT\big([a,\infty)\big)$. Then, since α is one-to-one, $x = x\alpha\beta$ for each $x \in [a,\infty)$, and so $[a,a+1)\,\beta = [a,\infty)$. Hence, since $(a+1)\beta \in [a,\infty)$, we know $(a+1)\beta = d\beta$ for some $d \in [a,a+1)$. Choose $c \in (d,a+1)$. Then

$$d\beta \leqslant c\beta \leqslant (a+1)\beta = d\beta,$$

and we deduce that $c\beta = d\beta$. Since $c,d \in [a,a+1) = \operatorname{ran}\alpha$, we can choose $x,y \in [a,\infty)$ with $x\alpha = c$ and $y\alpha = d$. The result then follows as in the proofs of the last Lemmas. ∎

Lemma 2.6. *The semigroup $OT\big((a,b)\big)$ is not regular for any $a,b \in \mathbf{R}$ with $a < b$.*

Proof: Fix $r \in (0,b-a)$ and let α be the map:

$$x\alpha = \left(1 - \frac{r}{b-a}\right)x + \frac{rb}{b-a} \quad \text{for all } x \in (a,b).$$

That is, the graph of α is a line segment with positive slope, and clearly $\operatorname{ran}\alpha = (a+r, b)$. Therefore, $\alpha \in OT\big((a,b)\big)$. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in OT\big((a,b)\big)$. Then, since α is one-to-one, $x = x\alpha\beta$ for each $x \in (a,b)$, and so $(a+r,b)\beta = (a,b)$. Hence, since $(a+r)\beta \in (a,b)$, we know $(a+r)\beta = d\beta$ for some $d \in (a+r,b)$. Choose $c \in (a+r,d)$. Then

$$d\beta = (a+r)\beta \leqslant c\beta \leqslant d\beta,$$

and we deduce that $c\beta = d\beta$. Since $c, d \in (a+r,b) = \operatorname{ran}\alpha$, we can choose $x, y \in (a,b)$ with $x\alpha = c$ and $y\alpha = d$, and the result follows as before. ∎

LEMMA 2.7. *The semigroup $OT\big([a,b)\big)$ is not regular for any $a, b \in \mathbf{R}$ with $a < b$.*

PROOF: Fix $r \in (0, b-a)$ and let α be the map:

$$x\alpha = \frac{rx}{b-a} + a - \frac{ra}{b-a} \quad \text{for all } x \in [a,b).$$

That is, the graph of α is a line segment with positive slope, and clearly $\operatorname{ran}\alpha = [a, a+r)$. Therefore, $\alpha \in OT([a,b))$. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in OT([a,b))$. Then, since α is one-to-one, $x = x\alpha\beta$ for each $x \in [a,b)$, and so $[a,a+r)\,\beta = [a,b)$. Hence, since $(a+r)\beta \in [a,b)$, we know $(a+r)\beta = d\beta$ for some $d \in [a,a+r)$. Choose $c \in (d, a+r)$. Then

$$d\beta \leqslant c\beta \leqslant (a+r)\beta = d\beta,$$

and we deduce that $c\beta = d\beta$. Since $c, d \in [a, a+r) = \operatorname{ran}\alpha$, we can choose $x, y \in [a,b)$ with $x\alpha = c$ and $y\alpha = d$, and the result follows as before. ∎

With the notation at the start of this section, if $\leqslant$ is the natural order on $\mathbf{R}$ then

(1) $\big((-\infty,a), \leqslant\big)$ is order-isomorphic to $\big((-a,\infty), \leqslant_{\text{opp}}\big)$ for each $a \in \mathbf{R}$,

(2) $\big((-\infty,a], \leqslant\big)$ is order-isomorphic to $\big([-a,\infty), \leqslant_{\text{opp}}\big)$ for each $a \in \mathbf{R}$, and

(3) $\big((a,b], \leqslant\big)$ is order-isomorphic to $\big([-b,-a), \leqslant_{\text{opp}}\big)$ for each $a, b \in \mathbf{R}$.

Consequently, Lemmas 2.4, 2.5 and 2.7 show that the semigroups $OT\big((-\infty,a)\big)$, $OT\big((-\infty,a]\big)$ and $OT\big((a,b]\big)$ are not regular for any $a, b \in \mathbf{R}$. This covers all nonempty intervals of $\mathbf{R}$ except one: namely, $[a,b]$ with $a < b$ and we shall prove that the semigroup $OT\big([a,b]\big)$ is regular. But, to do this, we need one more Lemma.

LEMMA 2.8. *Let $\alpha \in OT\big([a,b]\big)$ where $a, b \in \mathbf{R}$ and $a < b$, and suppose $x \in (a\alpha, b\alpha)$. If $A_x = [a,x]\alpha^{-1}$ and $B_x = (x,b]\,\alpha^{-1}$ then $\{A_x, B_x\}$ is a partition of $[a,b]$ such that $c < d$ for all $c \in A_x$ and $d \in B_x$.*

PROOF: Since $x \in (a\alpha, b\alpha)$, we know $a \leqslant a\alpha < x$ and $x < b\alpha \leqslant b$, so $a \in [a,x]\alpha^{-1} = A_x$ and $b \in (x,b]\,\alpha^{-1} = B_x$. The result then follows from Lemma 2.1 and the fact that $[a,b] = [a,x] \cup (x,b]$. ∎

For the next result, we recall: if I is an *interval* in $\mathbf{R}$ and if $\{A, B\}$ is a partition of I such that $x < y$ for all $x \in A$ and $y \in B$ then either $\max(A)$ or $\min(B)$ exists (but not both).

Lemma 2.9. *The semigroup $OT([a,b])$ is regular for any $a, b \in \mathbf{R}$ with $a < b$.*

Proof: Let $\alpha \in OT([a,b])$ and note that $a\alpha \leqslant b\alpha$ and $\operatorname{ran}\alpha \subseteq [a\alpha, b\alpha]$. Define d_x for each $x \in [a,b]$ as follows:

$$d_x \begin{cases} = a & \text{if } x \in [a, a\alpha), \\ = b & \text{if } x \in (b\alpha, b], \\ \in x\alpha^{-1} & \text{if } x \in \operatorname{ran}\alpha. \end{cases}$$

To complete the definition, suppose $x \in (a\alpha, b\alpha) \setminus \operatorname{ran}\alpha$ and put $A_x = [a, x]\alpha^{-1}$ and $B_x = (x, b]\,\alpha^{-1}$. By Lemma 2.8, $\{A_x, B_x\}$ is a partition of $[a,b]$ with a special property; and, by a remark above, either $\max(A_x)$ or $\min(B_x)$ exists (but not both). Hence, we can define:

$$d_x = \begin{cases} \max(A_x) & \text{if } x \in (a\alpha, b\alpha) \setminus \operatorname{ran}\alpha \text{ and } \max(A_x) \text{ exists,} \\ \min(B_x) & \text{if } x \in (a\alpha, b\alpha) \setminus \operatorname{ran}\alpha \text{ and } \min(B_x) \text{ exists.} \end{cases}$$

Finally, we let $\beta : [a,b] \to [a,b], x \to d_x$. If $x \in [a,b]$ then $x\alpha \in \operatorname{ran}\alpha$, so the definition of d_x implies that $d_{x\alpha}\alpha = x\alpha$. Hence,

$$x\alpha\beta\alpha = \big((x\alpha)\beta\big)\alpha = d_{x\alpha}\alpha = x\alpha$$

which shows that $\alpha = \alpha\beta\alpha$ on $[a,b]$.

To show that β is order-preserving, let $x, y \in [a,b]$ and $x < y$. Then $x \in [a, y]$ and $y \in (x, b]$, and we consider six cases.

Case 1. $x < a\alpha$. Then $x\beta = d_x = a$, so $x\beta \leqslant y\beta$.

Case 2. $y > b\alpha$. Then $y\beta = d_y = b$, so $x\beta \leqslant y\beta$.

Case 3. $x, y \in \operatorname{ran}\alpha$. By Lemma 2.1, $u < v$ for all $u \in x\alpha^{-1}$ and $v \in y\alpha^{-1}$. In particular, by definition, $x\beta = d_x < d_y = y\beta$.

Case 4. $x \in \operatorname{ran}\alpha$ and $y \in (a\alpha, b\alpha) \setminus \operatorname{ran}\alpha$. Then $d_x \in x\alpha^{-1} \subseteq [a, y]\alpha^{-1} = A_y$. Hence, if $\max(A_y)$ exists then

$$x\beta = d_x \leqslant \max(A_y) = d_y = y\beta.$$

On the other hand, if $\min(B_y)$ exists then, by Lemma 2.8, we have:

$$x\beta = d_x < d_y = \min(B_y) = y\beta.$$

Case 5. $x \in (a\alpha, b\alpha) \setminus \operatorname{ran}\alpha$ and $y \in \operatorname{ran}\alpha$. An argument similar to that in case (4) shows $x\beta \leqslant y\beta$ in this case also.

CASE 6. $x, y \in (a\alpha, b\alpha) \setminus \operatorname{ran}\alpha$. If $[x,y] \cap \operatorname{ran}\alpha = \emptyset$ then

$$A_x = [a,x]\alpha^{-1} = [a,y]\alpha^{-1} = A_y \quad \text{and} \quad B_x = (x,b]\,\alpha^{-1} = (y,b]\,\alpha^{-1} = B_y$$

and hence, by definition, $x\beta = d_x = d_y = y\beta$. However, if $[x,y] \cap \operatorname{ran}\alpha \neq \emptyset$ then there exists $c \in \operatorname{ran}\alpha$ such that $x < c < y$. In this event, we can choose $p \in [a,b]$ with $p\alpha = c$. Then

$$p \in [a,y]\alpha^{-1} \cap (x,b]\,\alpha^{-1} = A_y \cap B_x$$

and so, using a standard property of $\mathbf{R}$, we have:

$$\sup(A_x) \leqslant \inf(B_x) \leqslant p \leqslant \sup(A_y) \leqslant \inf(B_y).$$

Note that $\sup(A_x)$ equals $\max(A_x)$ if this maximum exists, and it equals $\min(B_x)$ if this minimum exists; and a similar comment can be made for $\inf(B_y)$. Hence, we conclude from the above that $d_x \leqslant d_y$ and so $x\beta \leqslant y\beta$. □

The combination of Lemmas 2.3-2.9, and the remarks between them, give us the following result.

THEOREM 2.10. *For any interval X of $\mathbf{R}$, the semigroup $OT(X)$ is regular if and only if X is closed and bounded.*

In passing, we note that in [**11**] Howie showed that $OT(X)$ is also idempotent-generated if X is a finite chain, and in [**8**] the authors extended this to $OP(X)$, while in [**7**] Garba investigated the same idea for various subsemigroups of $OP(X)$: see [**12**] for a brief survey of this and related work; and see [**10**] for an alternative approach to the same idea for $OT(X)$ and its subsemigroup consisting of all *decreasing* transformations (that is, $x\alpha \leqslant x$ for all $x \in X$). For an arbitrary chain X, the elements of $OT(X)$ which are products of idempotents were described in [**14**]; and the corresponding notion for products of "nilpotents" in $OP(X)$ and $OI(X)$ has been examined in [**6**] and [**5**] for finite chains.

In [**3**] the authors considered the semigroup $OP'(X)$ consisting of all order-preserving transformations α whose domains are *final segments* in a chain X (that is, $x \in \operatorname{dom}\alpha$ and $x \leqslant y$ imply $y \in \operatorname{dom}\alpha$); and they observed that this semigroup need not be regular. By contrast with this fact and the above results for $OT(X)$, we prove the following Theorem.

THEOREM 2.11. *If X is a chain then the semigroup $OP(X)$ is regular.*

PROOF: Let $\alpha \in OP(X)$ and, for each $a \in \operatorname{ran}\alpha$, choose $d_a \in a\alpha^{-1}$. Define a partial transformation β via: $\operatorname{dom}\beta = \operatorname{ran}\alpha$ and $a\beta = d_a$ for each $a \in \operatorname{ran}\alpha$. Then $x(\alpha\beta\alpha) = x\alpha$ for all $x \in \operatorname{dom}\alpha$ (since $d_a\alpha = a$ for all $a \in \operatorname{ran}\alpha$) and in fact

$\operatorname{dom}(\alpha\beta\alpha) = \operatorname{dom}\alpha$. Hence, $\alpha = \alpha\beta\alpha$. Also, if $a < b$ in $\operatorname{dom}\beta = \operatorname{ran}\alpha$ then $d_a < d_b$ by Lemma 2.1, so β is order-preserving, and the result follows. ∎

As usual, if X is a set and $\alpha \in P(X)$, we define the *shift* of α to be $s(\alpha) = |S(\alpha)|$ where

$$S(\alpha) = \{x \in \operatorname{dom}\alpha : x\alpha \neq x\}$$

and we write

$$P(X, \aleph_0) = \{\alpha \in P(X) : s(\alpha) < \aleph_0\}.$$

It is well-known that $P(X, \aleph_0)$ is a semigroup: it is sometimes called the semigroup of *almost identical* partial transformations of X [**15**]. If X is a poset, we let $OP(X, \aleph_0)$ denote the semigroup of all order-preserving partial transformations of X with finite shift, and $OI(X, \aleph_0)$ will denote the semigroup of all one-to-one transformations in $OP(X, \aleph_0)$.

If X is a chain and $\alpha \in OP(X, \aleph_0)$, we can define a map $\beta \in OP(X)$ as in the proof of Theorem 2.11 so that $\alpha = \alpha\beta\alpha$. In fact, since $a\alpha^{-1} = \{a\}$ for all $a \in \operatorname{ran}\alpha \setminus S(\alpha)\alpha$, we have $a\beta = a$ for all $a \in \operatorname{ran}\alpha \setminus S(\alpha)\alpha$ and so $S(\beta) \subseteq S(\alpha)\alpha$ (since $\operatorname{dom}\beta = \operatorname{ran}\alpha$). Hence, $\beta \in OP(X, \aleph_0)$ and we have proved that $OP(X, \aleph_0)$ is regular.

In passing, we note that Lavers [**13**] considered certain subsemigroups of $OP(\mathbf{Z}^+, \aleph_0)$ in a different context, with the aim of giving presentations for them and describing their principal left (right) ideals.

In [**4**, Proposition 1.4], Fernandes noted that $OI(X)$ is regular if X is a finite chain. In fact, following the proof of Theorem 2.11, it is clear that if X is any chain and $\alpha \in OI(X)$ then there exists $\beta \in OI(X)$ with $\alpha = \alpha\beta\alpha$. So, $OI(X)$ is regular for any chain X. Indeed, since the idempotents of $OI(X)$ are simply those transformations which fix a subchain of X pointwise, and hence they commute, we deduce that $OI(X)$ is an inverse semigroup. The significance of $OI(X)$ is illustrated by a result in [**2**]: namely, any set X with $|X| \neq 3$ can be ordered so that $OI(X)$ forms a transversal of the set of $\mathcal{H}$-classes in $I(X)$.

Finally, we consider the semigroup:

$$OT(X, \aleph_0) = \{\alpha \in OT(X) : s(\alpha) < \aleph_0\}$$

and aim to show it is regular if X is a chain. However, for this we need another three Lemmas.

Lemma 2.12. *Let X be a poset, $\alpha \in OP(X)$ and $a \in \operatorname{dom}\alpha$. Then*

$$\{x \in \operatorname{dom}\alpha : a\alpha < x < a\} \subseteq S(\alpha) \text{ and } \{x \in \operatorname{dom}\alpha : a < x < a\alpha\} \subseteq S(\alpha).$$

Proof: If $x \in \operatorname{dom}\alpha$ and $a\alpha < x < a$ then $x\alpha \leqslant a\alpha$. Thus, if $x\alpha = x$, we have $x \leqslant a\alpha$, a contradiction; so, $x \in S(\alpha)$ as required. The other containment follows similarly. ∎

LEMMA 2.13. *Let X be a poset, $\alpha \in OP(X)$ and $A \subseteq \operatorname{ran}\alpha$. If $\max(A)$ and $\max(A\alpha^{-1})$ exist then $\max(A) = [\max(A\alpha^{-1})]\alpha$.*

PROOF: Since $\max(A) \in A \subseteq \operatorname{ran}\alpha$, there exists $x \in \operatorname{dom}\alpha$ such that $\max(A) = x\alpha$. Then $x \in A\alpha^{-1}$, so $x \leqslant \max(A\alpha^{-1})$ and, since α is order-preserving, we deduce that $\max(A) \leqslant [\max(A\alpha^{-1})]\alpha$. Since $\max(A\alpha^{-1}) \in A\alpha^{-1}$ and $A \subseteq \operatorname{ran}\alpha$, we know $[\max(A\alpha^{-1})]\alpha \in A$ and this implies $[\max(A\alpha^{-1})]\alpha \leqslant \max(A)$. Hence, equality holds as required. ∎

LEMMA 2.14. *Let X be a poset, $\alpha \in OP(X)$ and $A, B \subseteq \operatorname{ran}\alpha$, and suppose $\max A, \max B, \max(A\alpha^{-1})$ and $\max(B\alpha^{-1})$ exist.*

(1) *If $\max A = \max B$ then $\max(A\alpha^{-1}) = \max(B\alpha^{-1})$.*
(2) *If X is a chain and $\max A < \max B$ then $\max(A\alpha^{-1}) < \max(B\alpha^{-1})$.*

PROOF: By Lemma 2.13, $[\max(A\alpha^{-1})]\alpha = \max A$. Therefore, if $\max A = \max B$, we have $[\max(A\alpha^{-1})]\alpha \in B$ and hence $\max(A\alpha^{-1}) \in B\alpha^{-1}$. Consequently, $\max(A\alpha^{-1}) \leqslant \max(B\alpha^{-1})$, and a dual argument establishes equality in (1). On the other hand, if $\max A < \max B$ then Lemma 2.13 implies $[\max(A\alpha^{-1})]\alpha < [\max(B\alpha^{-1})]\alpha$. Therefore, if X is a chain, we must have $\max(A\alpha^{-1}) < \max(B\alpha^{-1})$ (since α is order-preserving). ∎

We can now prove the following result.

THEOREM 2.15. *If X is a chain then the semigroup $OT(X, \aleph_0)$ is regular.*

PROOF: Let $\alpha \in OT(X, \aleph_0)$ and note that $\operatorname{dom}\alpha = X$. For each $x \in X$, we define $d_x \in X$ as follows.

CASE I. $x \in \operatorname{ran}\alpha$. Since $x\alpha^{-1}$ is nonempty and finite for each $x \in \operatorname{ran}\alpha$, and X is a chain, we know $\max(x\alpha^{-1})$ always exists in this case. So, we put

$$d_x = \max(x\alpha^{-1}) \quad \text{if } x \in \operatorname{ran}\alpha.$$

CASE II. $x \notin \operatorname{ran}\alpha$. In this case, $x \in S(\alpha)$ and this implies $x\alpha < x$ or $x < x\alpha$ since X is a chain. Then, from Lemma 2.12, we deduce that $\{z \in X : x\alpha < z < x\}$ and $\{z \in X : x < z < x\alpha\}$ are finite sets. In turn this implies $\{y \in \operatorname{ran}\alpha : x\alpha \leqslant y < x\} = C$ say, and $\{y \in \operatorname{ran}\alpha : x < y \leqslant x\alpha\} = D$ say, are finite sets. Therefore, since $y\alpha^{-1}$ is finite for each $y \in \operatorname{ran}\alpha$, the inverse images of C and D are also finite. Hence, if $x\alpha < x$ then $C \neq \emptyset$ and $\max(C\alpha^{-1})$ exists; and if $x < x\alpha$ then $D \neq \emptyset$ and $\min(D\alpha^{-1})$ exists. So, in this case, we put

$$d_x = \begin{cases} \max(\{y \in \operatorname{ran}\alpha : x\alpha \leqslant y < x\}\alpha^{-1}) & \text{if } x\alpha < x, \\ \min(\{y \in \operatorname{ran}\alpha : x < y \leqslant x\alpha\}\alpha^{-1}) & \text{if } x < x\alpha. \end{cases}$$

Now let $\beta : X \to X, x \to d_x$ and note that $\alpha = \alpha\beta\alpha$ as in the proof of Theorem 2.11. Also note that

$$\left\{x \in \operatorname{ran}\alpha : \max\left(x\alpha^{-1}\right) \neq x\right\} \subseteq \left\{x \in \operatorname{ran}\alpha : x\alpha^{-1} \neq \{x\}\right\}$$

and the second set in this display is finite since $S(\alpha)$ is finite. By definition of β, this means $\{x \in \operatorname{ran}\alpha : x\beta \neq x\}$ is finite. But we have:

$$S(\beta) \subseteq (X \setminus \operatorname{ran}\alpha) \cup \{x \in \operatorname{ran}\alpha : x\beta \neq x\}$$

where both sets in this union are finite. Hence, $\beta \in T(X, \aleph_0)$.

To show β is order-preserving, suppose $a < b$ in X and consider four cases.

CASE 1. $a, b \in \operatorname{ran}\alpha$. Then, by definition, $d_a \in a\alpha^{-1}$ and $d_b \in b\alpha^{-1}$, so $d_a < d_b$ by Lemma 2.1.

CASE 2. $a \in \operatorname{ran}\alpha$ and $b \notin \operatorname{ran}\alpha$. Then $b\alpha \neq b$, so $b\alpha < b$ or $b < b\alpha$ since X is a chain, and we consider two possibilities.

(i) Suppose $b\alpha < b$. If $a \in \{y \in \operatorname{ran}\alpha : b\alpha \leqslant y < b\} = B$ say, then $\max\left(a\alpha^{-1}\right) \leqslant \max\left(B\alpha^{-1}\right)$ which implies $d_a \leqslant d_b$. On the other hand, if $a \notin B$ then, since $a \in \operatorname{ran}\alpha$ and $a < b$ by supposition, we must have $a < b\alpha$ and so $a < y$ for all $y \in B$. Hence, Lemma 2.1 implies $u < v$ for all $u \in a\alpha^{-1}$ and $v \in B\alpha^{-1}$, and so $\max\left(a\alpha^{-1}\right) < \max\left(B\alpha^{-1}\right)$: that is, $d_a < d_b$.

(ii) Suppose $b < b\alpha$. Then $a < b < b\alpha$, so $a < y$ for all $y \in \operatorname{ran}\alpha$ such that $b < y \leqslant b\alpha$. Hence, Lemma 2.1 implies

$$\max\left(a\alpha^{-1}\right) < \min\left(\{y \in \operatorname{ran}\alpha : b < y \leqslant b\alpha\}\alpha^{-1}\right)$$

and we have shown $d_a < d_b$.

CASE 3. $a \notin \operatorname{ran}\alpha$ and $b \in \operatorname{ran}\alpha$. In this case, $a\alpha < a$ or $a < a\alpha$. Since $a < b$, the first possibility leads to

$$\max\left(\{y \in \operatorname{ran}\alpha : a\alpha \leqslant y < a\}\alpha^{-1}\right) < \max\left(b\alpha^{-1}\right)$$

and so $d_a < d_b$. If the second possibility occurs then $a < b \leqslant a\alpha$ or $a\alpha < b$, and an argument similar to that in the first paragraph of Case 2 leads to

$$\min\left(\{y \in \operatorname{ran}\alpha : a < y \leqslant a\alpha\}\alpha^{-1}\right) \leqslant \max\left(b\alpha^{-1}\right)$$

and it follows that $d_a \leqslant d_b$.

CASE 4. $a \notin \operatorname{ran}\alpha$ and $b \notin \operatorname{ran}\alpha$. Then $a\alpha < a$ or $a < a\alpha$, and similarly for b. So, we put

$$A = \{y \in \operatorname{ran}\alpha : a\alpha \leqslant y < a\} \quad \text{and} \quad B = \{y \in \operatorname{ran}\alpha : b\alpha \leqslant y < b\}$$

and consider four possibilities.

(i) Suppose $a\alpha < a$ and $b\alpha < b$. Since X is a chain, we have

$$a\alpha \leqslant y < b \quad \text{if and only if} \quad b\alpha \leqslant y < b \quad \text{or} \quad a\alpha \leqslant y < b\alpha.$$

Moreover, $\max B$ exists (as in the definition of d_x in Case II) and this is greater than all $y \in \operatorname{ran}\alpha$ such that $a\alpha \leqslant y < b\alpha$. Hence, $\max\{y \in \operatorname{ran}\alpha : a\alpha \leqslant y < b\}$ exists and it equals $\max B$. Then $a < b$ implies

$$\max A \leqslant \max\{y \in \operatorname{ran}\alpha : a\alpha \leqslant y < b\} = \max B$$

and so Lemma 2.14 implies $\max\left(A\alpha^{-1}\right) \leqslant \max\left(B\alpha^{-1}\right)$, so $d_a \leqslant d_b$ as required.

(ii) Suppose $a\alpha < a$ and $b < b\alpha$. Then $a\alpha < a < b < b\alpha$ and so $u < u'$ for all $u \in A$ and $u' \in \{y \in \operatorname{ran}\alpha : b < y \leqslant b\alpha\}$. Hence, $v < v'$ for all $v \in A\alpha^{-1}$ and $v' \in \{y \in \operatorname{ran}\alpha : b < y \leqslant b\alpha\}\alpha^{-1}$, and it follows that

$$\max\left(A\alpha^{-1}\right) < \min\left(\{y \in \operatorname{ran}\alpha : b < y \leqslant b\alpha\}\alpha^{-1}\right).$$

Thus, $d_a < d_b$ for this possibility.

(iii) Suppose $a < a\alpha$ and $b\alpha < b$. Then $a < a\alpha \leqslant b\alpha < b$ and so $u \leqslant u'$ for all $u \in \{y \in \operatorname{ran}\alpha : a < y \leqslant a\alpha\}$ and $u' \in B$. Hence,

$$\max\{y \in \operatorname{ran}\alpha : a < y \leqslant a\alpha\} \leqslant \max B$$

and, using Lemma 2.14, we obtain

$$\min\left(\{y \in \operatorname{ran}\alpha : a < y \leqslant a\alpha\}\alpha^{-1}\right) \leqslant \max\left(B\alpha^{-1}\right).$$

Thus, $d_a \leqslant d_b$ for this possibility.

(iv) Suppose $a < a\alpha$ and $b < b\alpha$. An argument dual to that in (i), which uses the dual of Lemma 2.14, shows that $d_a \leqslant d_b$ for this possibility.

Hence, $a < b$ implies $a\beta \leqslant b\beta$ in all possible cases, and the Theorem is completely proved. ∎

3. Order-preserving transformations of non-chains

Very little (if any) research appears to have been done on semigroups of order-preserving transformations of arbitrary posets. In this section, we suppose X is not a chain and determine when $OP(X)$ is regular, and then we state conditions under which $OT(X)$ is regular. We need two preliminary results, the first of which we are unable to find in the literature. If X is a poset, we say $a \in X$ is *isolated* if for every $x \in X, x \leqslant a$ or $x \geqslant a$ implies $x = a$, and X is *isolated* if all its elements are isolated.

Lemma 3.1. *Suppose X is a poset which is not a chain. If X is not isolated then it contains a subposet with one of the following forms:*

$$\Pi_1 = \{a, b, c : \{a, c\} \text{ and } \{b, c\} \text{ are isolated, and } a < b\}$$
$$\Pi_2 = \{a, b, c : \{b, c\} \text{ is isolated, and } a < b,\ a < c\},$$
$$\Pi_3 = \{a, b, c : \{b, c\} \text{ is isolated, and } b < a,\ c < a\}.$$

Proof: Since X is not isolated, it contains a non-isolated element a, say. Then there exists $b \in X$ with $a < b$ or $b < a$ and, without loss of generality, we assume $a < b$. By Zorn's Lemma, there is a maximal chain M in X containing a and b; and, since X is not a chain, we can choose $c \in X \setminus M$. If c is not comparable with any element of M, we obtain Π_1. On the other hand, suppose $d < c$ for some $d \in M$ and note that, in this case, if $y \in M$ and $y < d$ then $y < c$. Therefore, if for every $x \in M$, $d < x$ implies $c < x$ or $x < c$, we deduce that $M \cup \{c\}$ is a chain, contradicting the maximality of M. Hence, there exists $e \in M$ such that $d < e$ and $c \not< e$ and $e \not< c$, and we obtain Π_2. Finally, if $c < d$ for some $d \in M$, the dual of the above argument gives us Π_3. ∎

Lemma 3.2. *Suppose X is a poset which contains a subposet of the form Π_1, Π_2 or Π_3. If S denotes one of the semigroups $OP(X), OI(X), OP(X, \aleph_0)$ or $OI(X, \aleph_0)$ then S is not regular.*

Proof: We consider three cases. ∎

Case 1. X contains Π_1. Let $\alpha \in P(X)$ satisfy: $\operatorname{dom}\alpha = \{a, c\}$ and $a\alpha = b, c\alpha = a$. Then, $\alpha \in S$ since $\{a, c\}$ is isolated. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in S$. Then $b = a\alpha = (b\beta)\alpha$ and $a = c\alpha = (a\beta)\alpha$ which implies $b\beta = a$ and $a\beta = c$. But, since $a < b$ and $c \not\leqslant a$, this means β is not order-preserving, a contradiction. Hence, α is not a regular element of S.

Case 2. X contains Π_2. Let $\alpha \in P(X)$ satisfy: $\operatorname{dom}\alpha = \{b, c\}$ and $b\alpha = b,\ c\alpha = a$. Then, $\alpha \in S$ since $\{b, c\}$ is isolated. Suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in S$. Then $b = b\alpha = (b\beta)\alpha$ and $a = c\alpha = (a\beta)\alpha$ which implies $b\beta = b$ and $a\beta = c$. But, since $a < b$ and $c \not\leqslant b$, this means β is not order-preserving, a contradiction. Hence, α is not a regular element of S.

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CASE 3. X contains Π_3. The result follows in this case by recalling the notation at the start of Section 2 and applying Case 2 to $S(X, \leqslant_{\text{opp}})$. ∎

The next result now follows easily from the above Lemmas.

THEOREM 3.3. *Let X be a poset which is not a chain and suppose S is one of the semigroups $OP(X), OI(X), OP(X, \aleph_0)$ or $OI(X, \aleph_0)$. Then S is regular if and only if X is isolated.*

In [1, pp.27-33], Changphas provides various conditions under which $OT(X)$ is or is not regular when X is not a chain. We summarise some of that work in the following three results without proof.

THEOREM 3.4. *Suppose X is a poset. Then $OT(X)$ is not regular if X contains a subposet of the form*

$$\{a, b, c, d : \{a, b\} \text{ is isolated, and } d < c < a \text{ and } d < c < b\}$$

or

$$\{a, b, c, d : \{a, b\} \text{ and } \{b, c\} \text{ are isolated, and } d < c < a \text{ and } d < b\}.$$

THEOREM 3.5. *Suppose X is a poset and let $m(X)$ $[M(X)]$ denote the set of all minimal [maximal] elements of X. Then $OT(X)$ is regular if $X = m(X) \cup M(X)$ and $x < y$ for all $x \in m(X)$ and $y \in M(X)$.*

THEOREM 3.6. *Suppose X is a poset with a minimum element 0 and a maximum element 1. Then $OT(X)$ is regular if $\{x, y\}$ is isolated for all distinct $x, y \in X \setminus \{0, 1\}$.*

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