Nagoya Math. J., **233** (2019), 170–192
DOI 10.1017/nmj.2017.32

TEST VECTORS FOR LOCAL CUSPIDAL RANKIN–SELBERG INTEGRALS

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Abstract. Let π_1, π_2 be a pair of cuspidal complex, or ℓ-adic, representations of the general linear group of rank n over a nonarchimedean local field F of residual characteristic p, different to ℓ. Whenever the local Rankin–Selberg L-factor $L(X, \pi_1, \pi_2)$ is nontrivial, we exhibit explicit test vectors in the Whittaker models of π_1 and π_2 such that the local Rankin–Selberg integral associated to these vectors and to the characteristic function of $\mathfrak{o}_F^n$ is equal to $L(X, \pi_1, \pi_2)$. As an application we prove that the L-factor of a pair of banal ℓ-modular cuspidal representations is the reduction modulo ℓ of the L-factor of any pair of ℓ-adic lifts.

§1. Introduction

The integral representation of local L-factors, of pairs of complex irreducible representations of general linear groups over a nonarchimedean local field F, was developed in the fundamental paper [5] of Jacquet–Piatetski-Shapiro–Shalika. These L-factors are Euler factors which are the greatest common divisors, in a certain sense, of families of integrals I of Whittaker functions. For $n \geqslant m$, as a by-product of the definition, if π_1 and π_2 are irreducible smooth complex (or ℓ-adic) representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ respectively with Whittaker models $W(\pi_1, \psi)$ and $W(\pi_2, \psi^{-1})$, extended to all irreducible representations via the Langland's classification, then it is known that there is a finite number r of Whittaker functions $W_i \in W(\pi_1, \psi)$ and $W_i' \in W(\pi_2, \psi^{-1})$, and a finite number of Schwartz functions Φ_i on F^n if $n = m$, such that the L-factor $L(X, \pi_1, \pi_2)$ can be expressed as $\sum_{i=1}^{r} I(X, W_i, W_i')$, or $\sum_{i=1}^{r} I(X, W_i, W_i', \Phi_i)$ when $n = m$. A natural question which thus arises is whether one can find an explicit family of such *test vectors*.

A famous instance of test vectors is the essential vectors for generic representations (cf. [4, 6, 9]). It is shown in these references that these

Received July 28, 2016. Revised August 8, 2017. Accepted August 13, 2017.
2010 Mathematics subject classification. 11F70.

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https://doi.org/10.1017/nmj.2017.32 Published online by Cambridge University Press

vectors are test vectors for $L(X, \pi_1, \pi_2)$ when π_1 is a generic representation of $\mathrm{GL}_n(F)$, π_2 is an unramified standard module of $\mathrm{GL}_m(F)$, and $n > m$.

Interesting partial results have been obtained in [7], and, as indicated in [7], the theory of derivatives and its interpretation in terms of restriction of Whittaker functions (cf. [3, 9]) should reduce the general problem to the cuspidal case. Here, we establish the cuspidal case: that for pairs of cuspidal representations π_1 and π_2, we can choose $r = 1$, and moreover, we exhibit explicit test vectors, in the interesting case, whenever $L(X, \pi_1, \pi_2)$ is not equal to one. The fact that r can be chosen to be 1 when $L(X, \pi_1, \pi_2) = 1$, for any pair of irreducible representations π_1 and π_2 of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, is explained in the proof of [5, Theorem 2.7] and follows from standard facts on Kirillov models. We do not provide completely explicit test functions in this case, possibly a quite technical problem, and we in fact do not consider this case in the remainder of this article, as it is not needed for our application to reduction modulo ℓ.

Before we state our main theorem, we explain our normalization of Haar measure (Section 4), as for our application to reduction modulo ℓ some care needs to be taken with the normalization. Let $\mathfrak{o}_F$ denote the ring of integers of F with unique maximal ideal $\mathfrak{p}_F$, and let q denote the cardinality of the residue field $\mathfrak{o}_F/\mathfrak{p}_F$ and p its characteristic. We fix our Haar measure on $\mathrm{GL}_n(F)$ to give the pro-p unipotent radical K_n^1 of $\mathrm{GL}_n(\mathfrak{o}_F)$ volume one. It will turn out that this is a good choice of normalization for reduction modulo ℓ for primes ℓ not equal to p because K_n^1 is a pro-p subgroup. In particular, the volume of any pro-p subgroup of $\mathrm{GL}_n(F)$ which occurs in our computation will be a power of q.

Now we state our main theorem. Let π_1 and π_2 be cuspidal complex, or ℓ-adic, representations of $\mathrm{GL}_n(F)$ such that $L(X, \pi_1, \pi_2)$ is nontrivial, so that $\pi_2 \simeq \chi\pi_1^\vee$ for some unramified character χ of $F^\times$. Let e denote the common ramification index of π_1 and π_2 (see Section 6). We denote by W_1 and W_2 the explicit Whittaker functions for π_1 and π_2, as constructed in [12], with respect to a suitable nondegenerate character of the standard maximal unipotent subgroup of $\mathrm{GL}_n(F)$ and suitable maximal extended simple types in π_1 and π_2.

THEOREM 9.1. *There is an integer r such that*

$$\begin{aligned} I(X, W_1, W_2, 1_{\mathfrak{o}_F^n}) &= (q-1)(q^{n/e}-1)q^r \frac{1}{1-(\nu(\varpi_F)X)^{n/e}} \\ &= (q-1)(q^{n/e}-1)q^r L(X, \pi_1, \pi_2). \end{aligned}$$

https://doi.org/10.1017/nmj.2017.32 Published online by Cambridge University Press

The factor q^r occurs in our computation as a product of volumes, with respect to certain quotient measures, of quotients of pro-p subgroups related to the groups of Bushnell–Kutzko [2] in their explicit construction of π_1, π_2. Clearly, after our computation we could simply renormalize our measure by the factor $(q-1)(q^{n/e}-1)q^r$ and under the new normalization have an equality between the integral and the L-factor, hence $(W_1, W_2, 1_{\mathfrak{o}_F^n})$ is a test vector in the sense described earlier. However, it is important to keep track of these factors for our application to reduction modulo ℓ.

We now describe the proof of this theorem. In Section 7, we carefully choose an appropriate basis of F^n and simple types in our cuspidal representations, so that the subgroup of $\mathrm{GL}_n(F)$ defined by these simple types decomposes well with respect to the Iwasawa decomposition and satisfies some other important properties (see Proposition 7.1). In Section 8, we analyze the support of the explicit Whittaker functions of Paskunas and Stevens in terms of this well chosen group (Proposition 8.4). This preparation, which constitutes a substantial amount of the path to our main result, then allows us to compute the integral in Section 9.

Our interest in test vectors originated in the study of ℓ-modular Rankin–Selberg L-factors, for $\ell \neq p$, as introduced in [8]. Let π_1 and π_2 be integral cuspidal ℓ-adic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, and $\tau_1 = r_\ell(\pi_1)$ and $\tau_2 = r_\ell(\pi_2)$ their reductions modulo ℓ, which are cuspidal ℓ-modular representations. By [8, Theorem 3.13], the local factor $L(X, \tau_1, \tau_2)$ always divides $r_\ell(L(X, \pi_1, \pi_2))$. In particular, $L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2))$ whenever $L(X, \pi_1, \pi_2) = 1$. Hence the interesting case, where a strict division can happen is when $L(X, \pi_1, \pi_2)$ is not equal to 1, and, in particular, $n = m$. In [10], it was shown that for banal representations the ℓ-modular Godement–Jacquet L-factor is equal to the reduction modulo ℓ of the ℓ-adic Godement–Jacquet L-factor. It is thus natural to ask: if π_1 and π_2 are ℓ-adic integral cuspidal representations of $\mathrm{GL}_n(F)$ with banal reductions τ_1 and τ_2, does one have $L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2))$? As a corollary of our main result on test vectors applied to ℓ-adic Rankin–Selberg integrals, we answer this question in the affirmative.

COROLLARY 10.1. *Let τ_1 and τ_2 be two banal cuspidal ℓ-modular representations of $\mathrm{GL}_n(F)$, and π_1 and π_2 be any cuspidal ℓ-adic lifts, then*

$$L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2)).$$

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In [8], this corollary plays a key role in the classification of L-factors of generic ℓ-modular representations, and their relationship with ℓ-adic L-factors via reduction modulo ℓ.

It would be interesting to pursue the methods of this paper for integral representations of other L-factors, such as the Asai, exterior square, and symmetric square L-factors.

§2. Notations

Let F be a nonarchimedean local field of residual characteristic p and residual cardinality q. Throughout, R will denote one of the fields $\mathbb{C}$, $\overline{\mathbb{Q}_\ell}$, and $\overline{\mathbb{F}_\ell}$ and we assume that $\ell \neq p$. For E any extension of F, we denote by $\mathfrak{o}_E$ the ring of integers of E; by ϖ_E a uniformizer of E; by $\mathfrak{p}_E = (\varpi_E)$ the unique maximal ideal of $\mathfrak{o}_E$; by q_E the residual cardinality of E; and let $||_{E,R} : E^\times \to R^\times$ denote the unramified character defined by $|\varpi_E|_{E,R} = q_E^{-1}$, thus $||_{E,\mathbb{C}}$ is the restriction of the absolute value to $E^\times$ normalized in the usual way. When the field R considered is clear we remove the index R from $||_{E,R}$, and when $E = F$ we remove the index F as well.

Let $G_n = \mathrm{GL}_n(F)$, $K_n = \mathrm{GL}_n(\mathfrak{o}_F)$, $K_n^1 = 1 + \mathrm{Mat}_{n,n}(\mathfrak{p}_F)$, and let Z_n be the center of G_n. For g in G_n, by abuse of notation, we denote by $|g|$ the quantity $|\det(g)|$. Put $\eta_n = (0 \ \cdots \ 0 \ 1) \in \mathrm{Mat}_{1,n}(F)$, and let P_n be the standard mirabolic subgroup of G_n, that is, the set of all matrices g in G_n such that $\eta_n g = \eta_n$. Let N_n be the unipotent radical of the standard Borel subgroup of upper triangular matrices in G_n. For $k \in \mathbb{Z}$, let $G_n^{(k)} = \{g \in G_n : \ |g|_F = q^{-k}\}$. For any subset X of G_n, let $X^{(k)} = X \cap G_n^{(k)}$, and let $\mathbf{1}_X$ denote the characteristic function of X.

Let $\overline{\mathbb{Q}_\ell}$ denote an algebraic closure of the ℓ-adic numbers, $\overline{\mathbb{Z}_\ell}$ denote its ring of integers, and $\overline{\mathbb{F}_\ell}$ denote its residue field which is an algebraic closure of the finite field of ℓ-elements.

§3. Representations with coefficients in R

We only consider smooth R-representations, that is smooth representations with coefficients in R, and we use $\vee$ as an exponent to denote the contragredient. We call a representation on a $\overline{\mathbb{Q}_\ell}$-vector space an *$\ell$-adic representation*, and a representation on an $\overline{\mathbb{F}_\ell}$-vector space an *$\ell$-modular representation*. Let $(\pi, \mathcal{V})$ be an irreducible ℓ-adic representation of G_n. We call π *integral* if $\mathcal{V}$ contains a G_n-stable $\overline{\mathbb{Z}_\ell}$-lattice. Notice that for an ℓ-adic character $\nu : G_n \to \overline{\mathbb{Q}_\ell}^\times$ this just means that ν takes values in $\overline{\mathbb{Z}_\ell}^\times$.

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An R-representation is called *cuspidal* if it is irreducible and never appears as a quotient of a properly parabolically induced representation. By [13, II 4.12], a cuspidal ℓ-adic representation is integral if and only if its central character is integral, hence the contragredient of a cuspidal ℓ-adic representation π is integral if and only if π is integral. Let π be an integral cuspidal ℓ-adic representation and $\mathfrak{L}$ be a G_n-stable $\overline{\mathbb{Z}_\ell}$-lattice in the space of π. Let $r_{\mathfrak{L}}(\pi)$ be the ℓ-modular representation induced on the space $\mathfrak{L} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{F}_\ell}$. This ℓ-modular representation is also cuspidal (and irreducible) by [13, III 5.10], and hence independent of the choice of the lattice $\mathfrak{L}$ by the Brauer–Nesbitt principle [14, Theorem 1], we thus write $r_\ell(\pi)$ for $r_{\mathfrak{L}}(\pi)$ and call $r_\ell(\pi)$ the *reduction modulo* ℓ of π. We also say that π *lifts* $r_\ell(\pi)$, and it follows from [13, III 5.10] that all cuspidal ℓ-modular representations lift to cuspidal ℓ-adic representations. Following [11, Remark 8.15], we call a cuspidal ℓ-modular representation τ *banal* if $\tau \not\simeq \tau \otimes |\,|_F$ (notice that the definition in [11, Remark 8.15] refers to a condition given in Proposition 8.9 of this reference, which in the cuspidal case reduces to the condition we give here). For H a closed subgroup of G, we write Ind_H^G for the functor of smooth induction taking representations of H to representations of G, and write ind_H^G for the functor of smooth induction with compact support.

§4. Normalization of Haar measures

We now discuss our normalization of Haar measures. The basic reference for R-Haar measures is [14, I 2], but we also refer the reader to [8, Section 2.2] for more details on the splitting of Haar measures with respect to standard decompositions. Let dg be the Haar measure on G_n normalized to give K_n^1 volume 1.

We normalize the right Haar measure on P_n so that $dp(P_n \cap K_n^1) = 1$, on N_n so that $dn(N_n \cap K_n^1) = 1$, and on Z_n so that $dz(Z_n \cap K_n^1) = 1$. For the remainder of this section, let G denote a closed subgroup of G_n with Haar measure $d_G g$. For any open subgroup U of G, we define the Haar measure $d_U g$ on U as the restriction of $d_G g$, in particular $d_U g$ is normalized as soon as $d_G g$ is.

If H is a closed subgroup of G with right Haar measure $d_H h$, and such that the modulus character of G restricts to H as the modulus character of H, we descend $d_G g$ to a right-invariant measure $d_{H\backslash G} g$ on $H\backslash G$ as explained in [14, I 2.8]. For f a smooth map from G to R with compact support,

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denoting by f^H the map on $H\backslash G$ defined by

$$f^H(g) = \int_H f(hg)\, d_H h,$$

the usual relation is satisfied:

$$\int_{H\backslash G} f^H(g)\, d_{H\backslash G} g = \int_G f(g)\, d_G g.$$

This implies that $d_{H\backslash G} g$ is normalized as soon as $d_G g$ and $d_H g$ are.

Indeed, if K is a compact subgroup of G, applying the equality above to $f = \mathbf{1}_K$, so that

$$f^H = d_H(K \cap H)\mathbf{1}_{H\backslash HK}$$

gives the relation

$$d_G(K) = d_{H\backslash G}(H\backslash HK) d_H(K \cap H). \tag{1}$$

This gives for example the normalization

$$d_{H\backslash G}(H\backslash HK_n^1) = d_H(H \cap K_n^1)\backslash d_G(G \cap K_n^1).$$

With these normalizations, we have the splitting

$$dg = |p|_F^{-1}\, dp\, dz\, dk.$$

This splitting descends on $N_n\backslash G_n$, in which case dg denotes the normalized right-invariant measure on $N_n\backslash G_n$ and dp the right-invariant measure on $N_n\backslash P_n$. Notice that with such normalizations, the volume of all pro-p subgroups of G_n, of P_n and of Z_n will be (positive or negative) powers of q. Moreover, for such choices, reduction modulo ℓ commutes with integration (cf. [8, Remark 2.1]), that is, if $f \in \mathcal{C}_c^\infty(X, \overline{\mathbb{Z}_\ell})$ for X equal to G_n or any of the homogeneous spaces $K\backslash L$ with L a subgroup of G_n considered above, then $\int_X f(x)\, dx \in \overline{\mathbb{Z}_\ell}$, and

$$r_\ell\left(\int_X f(x)\, dx\right) = \int_X r_\ell(f(x))\, dx.$$

For the rest of this section, we suppose that R has characteristic zero, and we recall some classical equalities, which all follow from Relation (1).

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For a finite set A, we let $|A|$ denote its cardinality in R. Suppose that $G=K$ compact, and U is an open subgroup of K, then

$$d_{U\backslash K}(U\backslash K)=\frac{d_K(K)}{d_K(U)}=|U\backslash K|\in R. \tag{2}$$

Finally, if V is a closed subgroup of K (using the fact that K is unimodular, hence that $d_K(UV)=d_K(V^{-1}U^{-1})=d_K(VU)$), one obtains

$$\begin{aligned} d_{V\backslash K}(V\backslash VU) &= \frac{d_K(VU)}{d_V(V)}=\frac{d_K(UV)}{d_K(U)}\frac{d_V(V\cap U)}{d_V(V)}\frac{d_K(U)}{d_V(V\cap U)} \\ &= \frac{|U\backslash UV|}{|V\cap U\backslash V|}\frac{d_K(U)}{d_V(V\cap U)}=\frac{d_K(U)}{d_V(V\cap U)}=d_{V\cap U\backslash U}(V\cap U\backslash U). \end{aligned} \tag{3}$$

By convention, from now on, we use the same letter for the measure on G and its descent to $H\backslash G$ (and when the context is clear for its restriction to an open subgroup as well).

§5. Rankin–Selberg integrals and local factors

Let ψ be an additive character of F which is trivial on $\mathfrak{p}_F$, but nontrivial on $\mathfrak{o}_F$. By abuse of notation, also denote by ψ the nondegenerate character of N_n defined for $x=(x_{i,j})\in N_n$ by

$$\psi(x)=\psi\left(\sum_{i=1}^{n-1}x_{i,i+1}\right),$$

which is necessarily integral in the ℓ-adic case because N_n is exhausted by its pro-p subgroups. If π is a cuspidal representation of G_n, then it is *generic* (cf. [1] in the complex or ℓ-adic case, and [13, III 5.10] for $R=\overline{\mathbb{F}_\ell}$), meaning $\dim(\mathrm{Hom}_{N_n}(\pi,\psi))=1$, and hence it has a unique *Whittaker model* $W(\pi,\psi)$, equal to the image of π in $\mathrm{Ind}_{N_n}^{G_n}(\psi)$. Suppose that π is an integral cuspidal ℓ-adic representation of G_n, then the $\overline{\mathbb{Z}_\ell}$-submodule $W_e(\pi,\psi)$ of $W(\pi,\psi)$ consisting of all functions in $W(\pi,\psi)$ which take values in $\overline{\mathbb{Z}_\ell}$ is a G_n-stable lattice in π (cf. [14, Theorem 2]). Then by definition $r_\ell(\pi)\simeq W_e(\pi,\psi)\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{F}_\ell}$, which is irreducible and cuspidal (cf. Section 2.1 and the references given there). Thus $W_e(\pi,\psi)\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{F}_\ell}$ is a space of Whittaker functions for π with values in $\overline{\mathbb{F}_\ell}$, hence equal to $W(r_\ell(\pi),r_\ell(\psi))$. For $W\in W_e(\pi,\psi)$, we write $r_\ell(W)$ for the image of W in $W(r_\ell(\pi),r_\ell(\psi))$.

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Finally, we recall the definition of the Rankin–Selberg local L-factors for a pair of cuspidal R-representations of G_n. The construction is originally due to Jacquet–Piatetski-Shapiro–Shalika [5] for complex representations, and works equally well for $\overline{\mathbb{Q}_\ell}$-representations. This construction was extended to a construction for representations over any algebraically closed field of characteristic prime to p in [8]. As we are ultimately interested in $\mathbb{C}, \overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{F}_\ell}$ representations we give precise references to the construction in [8].

Let π_1 and π_2 be cuspidal representations of G_n, $W_1 \in W(\pi_1, \psi)$, $W_2 \in W(\pi_2, \psi^{-1})$, and $\Phi \in \mathcal{C}_c^\infty(F^n)$ be a locally constant function from F^n to R with compact support. By [8, Proposition 3.3], for $k \in \mathbb{Z}$, the coefficients

$$c_k(W_1, W_2, \Phi) = \int_{N_n \backslash G_n^{(k)}} W_1(g) W_2(g) \Phi(\eta_n g)\, dg$$

are well defined and vanish for k sufficiently negative. In fact, these coefficients vanish for k sufficiently negative because both W_1 and W_2 vanish on $P_n^{(k)}$ for such k, as a consequence of [5, Proposition 2.2]. Hence the *local Rankin–Selberg integral*

$$I(X, W_1, W_2, \Phi) = \sum_{k \in \mathbb{Z}} c_k(W_1, W_2, \Phi) X^k$$

is a formal Laurent series with coefficients in R. In fact, by [8, Theorem 3.5], $I(X, W_1, W_2, \Phi) \in R(X)$ is a rational function, and as W_1 varies in $W(\pi_1, \psi)$, W_2 varies in $W(\pi_2, \psi^{-1})$, and Φ varies in $\mathcal{C}_c^\infty(F^n)$, the R-submodule of $R(X)$ spanned by $I(X, W_1, W_2, \Phi)$ is a fractional ideal of $R[X^{\pm 1}]$, and has a unique generator $L(X, \pi_1, \pi_2)$ which is an Euler factor. We call $L(X, \pi_1, \pi_2)$ the *local Rankin–Selberg L-factor*, and note that it does not depend on the choice of the character ψ. If $R = \overline{\mathbb{Q}_\ell}$, it is shown in [8, Corollary 3.6] that the L-factor is the inverse of a polynomial in $\overline{\mathbb{Z}_\ell}[X]$, and thus it makes sense to talk of its reduction modulo ℓ. Moreover, it follows from [8, Theorem 3.13], that if π_1 and π_2 are two integral cuspidal ℓ-adic representations of G_n, then one has

$$L(X, r_\ell(\pi_1), r_\ell(\pi_2)) | r_\ell(L(X, \pi_1, \pi_2)).$$

Now by [5, Proposition 8.1, (ii)], the L-factor $L(X, \pi_1, \pi_2)$ is equal to 1 unless $\pi_2 \simeq \chi \pi_1^\vee$ for some unramified character χ of $F^\times$. Hence if $\pi_2 \not\simeq \chi \pi_1^\vee$ then $L(X, r_\ell(\pi_1), r_\ell(\pi_2)) = r_\ell(L(X, \pi_1, \pi_2)) = 1$.

For our computations to come, we use a decomposition of the Rankin–Selberg integral in the special case where $\pi_2 \simeq \pi_1^\vee$, in particular their central

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characters are inverse of each other. Thus we assume this is the case for the rest of this section. For $k \in \mathbb{Z}$, we set

$$b_k(W_1, W_2) = \int_{N_n \backslash P_n^{(k)}} W_1(p) W_2(p)\, dp,$$

which, similarly to c_k, vanishes for k sufficiently negative, and we put

$$I_{(0)}(X, W_1, W_2) = \sum_{k \in \mathbb{Z}} b_k(W_1, W_2) q^k X^k.$$

Let $\Phi \in \mathcal{C}_c^\infty(F^n)$ be a K_n-invariant function, for $i \in \mathbb{Z}$, we set

$$a_{ni}(\Phi) = \int_{z \in G_1^{(ni)}} \Phi(\eta_n z)\, dz,$$

which vanishes for i sufficiently negative, and we put

$$Z(X, \Phi) = \sum_{i \in \mathbb{Z}} a_{ni}(\Phi) X^{ni}.$$

As $G_n^{(k)} = \coprod_{i \in \mathbb{Z}} P_n^{(k-ni)} Z_n^{(ni)} K_n$, from the splitting of Section 4 we find

$$c_k(W_1, W_2, \Phi) = \sum_{i \in \mathbb{Z}} a_{ni}(\Phi) q^{k-ni} \int_{(K_n \cap P_n) \backslash K_n} b_{k-ni}(\rho(k) W_1, \rho(k) W_2)\, dk,$$

from which we deduce

$$I(X, W_1, W_2, \Phi) = Z(X, \Phi) \left(\int_{(K_n \cap P_n) \backslash K_n} I_{(0)}(X, \rho(k) W_1, \rho(k) W_2)\, dk \right).$$

Taking Φ equal to the characteristic function $\mathbf{1}_{\mathfrak{o}_F^n}$, we obtain the formula

$$I(X, W_1, W_2, \mathbf{1}_{\mathfrak{o}_F^n}) = \frac{q-1}{1-X^n} \int_{(K_n \cap P_n) \backslash K_n} I_{(0)}(X, \rho(k) W_1, \rho(k) W_2)\, dk. \tag{4}$$

The equality $Z(X, \mathbf{1}_{\mathfrak{o}_F^n}) = (q - 1/1 - X^n)$ is standard (cf. [10, Theorem 3.1]) except that in our setting, we get the extra constant $q - 1$ from our choice of normalization on Z_n, as we set $dz(Z_n \cap K_n^1) = 1$ instead of the usual $dz(Z_n \cap K_n) = 1$.

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§6. Simple types and reduction modulo ℓ

For the beginning of this section we assume that $R=\mathbb{C}$ *or* $\overline{\mathbb{Q}_\ell}$. Let V be an n-dimensional F-vector space, let $\mathrm{End}_F(V)$ denote the F-algebra $\mathrm{End}_F(V)$ of F-endomorphisms of V and let G denote the group $\mathrm{Aut}_F(V)$ of F-automorphisms of V. Hence G identifies with G_n as soon as we choose a basis of V. In [2], every cuspidal R-representation of G is constructed explicitly as $\mathrm{ind}_{\mathbf{J}}^G(\Lambda)$, where $\mathbf{J}$ is an open and compact-mod-center subgroup of G, and Λ is an irreducible representation of $\mathbf{J}$ of finite dimension. The pairs $(\mathbf{J},\Lambda)$ are called *extended maximal simple types*, and for any such pair $\mathrm{ind}_{\mathbf{J}}^G(\Lambda)$ is (irreducible and) cuspidal by [2, Chapter 6]. We briefly explain the construction of the group $\mathbf{J}$, focusing on the properties which we shall use.

An $\mathfrak{o}_F$-lattice chain $\mathcal{L}$ in V is a nonempty set of $\mathfrak{o}_F$-lattices $\{L_i : i\in\mathbb{Z}\}$ such that, for all $i\in\mathbb{Z}$, $L_{i+1}\subsetneq L_i$ and there exists $e(\mathcal{L})\in\mathbb{Z}$ such that $L_{i+e(\mathcal{L})}=\varpi_F L_i$. The construction of [2], starts with data $(\beta,\mathcal{L})$ called *maximal simple strata* consisting of

(1) an element $\beta\in\mathrm{End}_F(V)$ which generates a simple field extension $E=F[\beta]$;
(2) an $\mathfrak{o}_F$-*lattice chain* $\mathcal{L}$ in V such that $E^\times\mathcal{L}\subset\mathcal{L}$ (i.e., for any $x\in E^\times$ and $L\in\mathcal{L}$ we have $xL\in\mathcal{L}$); in particular $\mathcal{L}$ is an $\mathfrak{o}_E$-*lattice chain*, and it is required (as $(\beta,\mathcal{L})$ is maximal) that $L_{i+1}=\varpi_E L_i$;

which satisfy a technical condition (cf. [2, 1.5.5] where the simple strata we consider are among those denoted $[\mathfrak{A},-,0,\beta]$).

Let $(\beta,\mathcal{L})$ be a maximal simple strata. We denote by $\mathfrak{A}=\mathfrak{A}(\mathcal{L})$ the $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ and $\mathfrak{B}=\mathfrak{B}(\beta,\mathcal{L})$ the $\mathfrak{o}_E$-order in $\mathrm{End}_E(V)$ defined by $\mathcal{L}$,

$$\mathfrak{A}=\mathrm{End}_{\mathfrak{o}_F}(\mathcal{L})=\bigcap_k \mathrm{End}_{\mathfrak{o}_F}(L_k),\qquad \mathfrak{B}=\mathfrak{B}(\beta,\mathcal{L})=\mathrm{End}_{\mathfrak{o}_E}(\mathcal{L})=\mathrm{End}_{\mathfrak{o}_E}(L_0).$$

In [2, 3.1] Bushnell–Kutzko define compact open subgroups of G denoted by $H^1=H^1(\beta,\mathcal{L})$, $J^1=J^1(\beta,\mathcal{L})$, and $J=J(\beta,\mathcal{L})$. The properties we need are:

(1) the groups $H^1\leqslant J^1$ are pro-p (by definition), are normalized by $E^\times$ and are normal subgroups of J by [2, 3.1.15], moreover $J\subset\mathrm{Aut}_{\mathfrak{o}_F}(L_0)$ (by definition).
(2) Put $m=n/[E:F]$, by [2, 3.1.15] we have

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$$J = \mathfrak{B}^\times J^1, \qquad \mathfrak{B}^\times \cap J^1 = 1 + \varpi_E \mathfrak{B} \qquad \text{and}$$
$$J/J^1 \simeq \mathfrak{B}^\times/(1+\varpi_E\mathfrak{B}) \simeq G_m(k_E).$$

We then set $\mathbf{J} = \mathbf{J}(\beta, \mathcal{L}) = E^\times J$, in particular $\mathbf{J}$ is compact mod $E^\times$ and hence compact mod $F^\times$. Notice that if $\pi \simeq \mathrm{ind}_{\mathbf{J}}^G \Lambda$, the center $F^\times$ of G acts by the central character ω_π of π through Λ. Finally, we note that the construction of Λ depends on our fixed additive character ψ (cf. [2, 3.2]).

The definitions above do not include the groups of the maximal simple types for level zero cuspidal representations (see [2, 5.5.10(b)]), although these can be considered formally as part of the construction described above for the *maximal zero strata* $(0, \mathcal{L})$ with $\beta = 0$ and $e(\mathcal{L}) = 1$. In this case, we put $J = \mathfrak{A}^\times$, $\mathbf{J} = F^\times J$, $H^1 = J^1 = 1 + \varpi_F \mathfrak{A}$, and $J/J^1 = \mathfrak{A}^\times/(1+\varpi_F\mathfrak{A}) \simeq G_n(k_F)$.

Now we consider $\overline{\mathbb{F}_\ell}$-representations. It follows from [13, Chapitre IV] that the Bushnell–Kutzko classification of cuspidal $\overline{\mathbb{Q}_\ell}$-representations adapts well to $\overline{\mathbb{F}_\ell}$-representations. We only need to know the following facts:

Let τ be a cuspidal ℓ-modular representation of G. As we recalled in Section 3, there exists an integral cuspidal ℓ-adic representation π such that $\tau = r_\ell(\pi)$. Choose an extended maximal simple type $(\mathbf{J}, \Lambda)$ such that $\pi \simeq \mathrm{ind}_{\mathbf{J}}^G(\Lambda)$, as in the beginning of this section. A cuspidal ℓ-adic representation is integral if and only if its central character ω_π is integral, by [13, II 4.13] (the direction integral implies integral central character being clear). We recall why this is true. First as $\mathbf{J}$ is compact mod $F^\times$, we claim that the irreducible representation Λ is integral if and only if ω_π is integral. Again, one direction is clear. For the other, suppose that ω_π is integral and choose a random not necessarily $\mathbf{J}$-stable lattice $\mathfrak{L}_0$ in the space V_Λ of Λ. It is stabilized by a compact open subgroup U of $\mathbf{J}$, and choosing representatives $c_1, \ldots, c_r$ of $\mathbf{J}/F^\times U$, one has $\Lambda(\mathbf{J})(\mathfrak{L}_0) = \sum_{i=1}^r \Lambda(c_i)(\mathfrak{L}_0)$, hence $\mathfrak{L}_\Lambda = \Lambda(\mathbf{J})(\mathfrak{L}_0)$ is a $\mathbf{J}$-stable lattice in V_Λ by [13, 9.3]. The induced $\overline{\mathbb{Z}_\ell}$-representation $\mathrm{ind}_{\mathbf{J}}^G(\mathfrak{L}_\Lambda)$ is then a lattice in π by [13, 9.3]. Moreover $\tau = r_\ell(\pi) \simeq \mathrm{ind}_{\mathbf{J}}^G(r_\ell(\Lambda))$, and $r_\ell(\Lambda)$ is an irreducible representation of $\mathbf{J}$ by irreducibility of τ.

Finally, we give another characterization of banal cuspidal representations: recall, from Section 3, by definition τ is banal if and only if the cardinality of the cuspidal line $\mathbb{Z}_\tau = \{||^k \tau,\ k \in \mathbb{Z}\}$ is greater than 1. By [11, Lemme 5.3], this cardinality is the same as the integer $o(\tau)$ introduced in [11, Section 5.2, (5.4)]. From [11, Section 5.2, (5.4)], $o(\tau)$ is the order of $q^{n/e}$ in $\overline{\mathbb{F}_\ell}^\times$, where $e = e(E/F)$ is the ramification index attached to $(\mathbf{J}, \Lambda)$ which

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in particular does not depend on the choice of extended maximal simple type. Hence τ is banal if and only if $q^{n/e}-1\neq 0$ in $\overline{\mathbb{F}_\ell}$.

§7. The modified Paskunas–Stevens basis

For this section $R=\mathbb{C}$ *or* $\overline{\mathbb{Q}_\ell}$. Let π be a cuspidal R-representation of G and $(\mathbf{J}=\mathbf{J}(\beta,\mathcal{L}),\Lambda)$ be an extended maximal simple type in π. According to [12, Corollaries 3.4 and 4.13], there exists an F-basis $\mathcal{B}=(v_1,\ldots,v_n)$ of V particularly suited to relating the Whittaker model of π and the model $\mathrm{ind}_{\mathbf{J}}^G(\Lambda)$ defined via type theory. In particular, $\mathcal{B}$ splits $\mathcal{L}$, that is, $L_k=\bigoplus_{i=1}^n \mathfrak{p}_F^{a_i(k)}v_i$ with $a_i(k)\in\mathbb{Z}$ for all $k\in\mathbb{Z}$, and is such that if N is the maximal unipotent subgroup of G attached to the maximal flag defined by $\mathcal{B}$, and if ψ, by abuse of notation, denotes the nondegenerate character of N defined for $x\in N$ by

$$\psi(x)=\psi\left(\sum_{i=1}^{n-1}\mathrm{Mat}_{\mathcal{B}}(x)_{i,i+1}\right),$$

where $\mathrm{Mat}_{\mathcal{B}}(x)$ denotes the matrix of x with respect to the basis $\mathcal{B}$, then the triple (J,Λ,ψ) satisfies

$$\mathrm{Hom}_{N\cap J}(\psi,\Lambda)\neq 0.$$

Let P be the *mirabolic subgroup* defined by

$$P=\{g\in G,\,(g-\mathrm{Id})V\subset \mathrm{Vect}_F(v_1,\ldots,v_{n-1})\}.$$

We put $\mathcal{M}=(P\cap J)J^1$, which is a group as J^1 is normal in J. It follows from [12] that the image of $\mathcal{M}$ in $J/J^1\simeq G_m(k_E)$ is isomorphic to $P_m(k_E)$. We now explain how to extract this from [12]: in the notation of [12], our group P is denoted $\mathcal{M}_F$ and [12, Corollary 4.8] shows that

$$\mathcal{M}=(P\cap\mathfrak{B}^\times)J^1. \tag{5}$$

In [12, Section 4.1], Paskunas–Stevens introduce another mirabolic group they denote by $\mathcal{M}_E$ which satisfies $P\cap\mathfrak{B}^\times=\mathcal{M}_E\cap\mathfrak{B}^\times$ by the equality just before [12, Corollary 4.7], and they also denote by $\mathcal{M}_{\mathfrak{B}}$ the group $(\mathcal{M}_E\cap\mathfrak{B}^\times)(1+\varpi_E\mathfrak{B})$. Hence Equation (5) gives $\mathcal{M}=\mathcal{M}_{\mathfrak{B}}J^1$ as $(1+\varpi_E\mathfrak{B})=\mathfrak{B}^\times\cap J^1$. Finally, from the discussion after the proof of [12, Lemma 4.10], the image of $\mathcal{M}_{\mathfrak{B}}$ in $\mathfrak{B}^\times/1+\varpi_E\mathfrak{B}\simeq G_m(k_E)$ is isomorphic to $P_m(k_E)$, hence the same is true for the image of $\mathcal{M}$ in $J/J^1\simeq\mathfrak{B}^\times/1+\varpi_E\mathfrak{B}\simeq G_m(k_E)$.

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In particular, the following index will appear in our computation:

$$|J/\mathcal{M}| = |G_m(k_E)/P_m(k_E)| = q_E^m - 1 = q^{n/e} - 1.$$

For $i \in \{1, \ldots, n\}$, the functions $a_i : \mathbb{Z} \to \mathbb{Z}$ satisfy the relation $a_{i+e}(k) = a_i(k) + 1$. In particular, this holds for $i = n$, and the map $k \mapsto a_n(k)$ is increasing with values in $\mathbb{Z}$, so there is k_0 between 1 and e such that $a_n(k_0) = a_n(k_0 - 1) + 1$, and then $a_n(k_0 + i) = a_n(k_0)$ for $i \in \{0, \ldots, e-1\}$. Hence by reindexing the lattice chain $\mathcal{L}$ if necessary, by a translation, $k \mapsto k - k_0$, we can suppose that

$$a_n(0) = a_n(-1) + 1 = 0, \qquad \text{and} \qquad a_n(1) = \cdots = a_n(e-1) = 0.$$

We recall that $L_0 = \bigoplus_{i=1}^n \mathfrak{p}_F^{a_i(0)} v_i$, and we set $\mathcal{B}' = (\varpi_F^{a_1(0)} v_1, \ldots, \varpi_F^{a_n(0)} v_n)$, which we write as $\mathcal{B}' = (w_1, \ldots, w_n)$.

We use this basis to identify G with G_n. With this choice, one has $J \subset K_n$ because $J \subset \mathrm{Aut}_{\mathfrak{o}_F}(L_0)$. The group P identifies with P_n, the group N identifies with N_n, and the character ψ of N_n identifies with

$$\psi_t : n \mapsto \psi\left(\sum_{i=1}^{n-1} t_i n_{i,i+1}\right),$$

where $t_i = \varpi_F^{a_i(0) - a_{i+1}(0)}$.

For our computation to come, it will be useful to notice the following property of $\mathcal{B}'$: one has

$$L_0 = \bigoplus_{i=1}^{n} \mathfrak{o}_F w_i, \qquad L_k = \bigoplus_{i=1}^{n-1} \mathfrak{p}_F^{a_i(k) - a_i(0)} w_i \oplus \mathfrak{o}_F w_n,$$

for $k \in \{1, \ldots, e-1\}$. As $\varpi_E L_k = L_{k+1}$ for any $k \in \mathbb{Z}$, the properties above and the fact that $L_{k+e} = \varpi_F L_k$, imply that the last row of $\varpi_E^i \in G_n$ belongs to $(\mathfrak{o}_F)^n - (\mathfrak{p}_F)^n$ for $i = 0, \ldots, e-1$, and more generally that it belongs to $(\mathfrak{p}_F^l)^n - (\mathfrak{p}_F^{l+1})^n$ if $i = le + r$, with $r \in \{0, \ldots, e-1\}$. As an immediate consequence, if we write an Iwasawa decomposition of ϖ_E^i,

$$\varpi_E^i = p_i z_i k_i, \qquad p_i \in P_n, \; z_i \in Z_n, \; k_i \in K_n,$$

we can choose $z_i = I_n$ for $i = 0, \ldots, e-1$, and more generally $z_i = \varpi_F^l I_n$ for $i = le + r$, with $r \in \{0, \ldots, e-1\}$. In particular $|p_i| = q^{-in/e}$, for $i = 0, \ldots, e-1$.

For clarity, we list the properties of the data (J, Λ, ψ_t) that we use.

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PROPOSITION 7.1. *With the above choice of basis we have:*

(1) *The inclusion* $J \subset K_n$.
(2) *The space* $\mathrm{Hom}_{N_n \cap J}(\psi_t, \Lambda) \neq 0$.
(3) *Set* $\mathcal{M} = (P_n \cap J)J^1$, *then* $|J/\mathcal{M}| = q^{n/e} - 1$.
(4) *The element* $\varpi_E^i \in P_n K_n$ *if and only if* $i \in \{0, \ldots, e-1\}$ *and, in this case, if we choose* $p_i \in P_n$ *and* $k_i \in K_n$, *such that* $\varpi_E^i = p_i k_i$, *then we have* $|p_i| = |\varpi_E^i| = q^{-in/e}$.

For the remainder, we consider the $k_i \in K_n$ and $p_i \in P_n$ chosen in Proposition 7.1 Statement (4) as fixed.

As $P_n \cap J^1$ is a pro-p subgroup of P_n, and J^1 is a pro-p subgroup of G_n, the volume

$$dk(P_n \cap J^1 \backslash J^1) = \frac{dk(J^1)}{dp(P_n \cap J^1)}$$

is a power of q thanks to our normalization of measures, and we write

$$dk(P_n \cap J^1 \backslash J^1) = q^{r_1}.$$

A certain volume will appear in our later computation, we compute it in the next lemma.

LEMMA 7.2. *For any* $i \in \{0, \ldots, e-1\}$, *we have*

$$dk((P_n \cap K_n) \backslash (P_n \cap K_n) k_i J) = q^{r_1}(q^{n/e} - 1) q^{-in/e}.$$

Proof. We have

$$\begin{aligned} dk((P_n \cap K_n) \backslash (P_n \cap K_n) k_i J) &= dk((P_n \cap K_n) \backslash (P_n \cap K_n) k_i J k_i^{-1}) \\ &= dk((P_n \cap k_i J k_i^{-1}) \backslash k_i J k_i^{-1}), \end{aligned}$$

the last equality thanks to Relation (3). Now, $dk(k_i J k_i^{-1}) = dk(J)$. We also notice that

$$p_i (P_n \cap k_i J k_i^{-1}) p_i^{-1} = P_n \cap \varpi_E^i J \varpi_E^{-i} = P_n \cap J,$$

hence

$$P_n \cap k_i J k_i^{-1} = p_i^{-1} (P_n \cap J) p_i.$$

As for any compact open subset A of P_n, one has $dp(pAp^{-1}) = |p| dp(A)$, as is easily seen by writing $dp = dg du$, with dg on G_{n-1} and du on U_n, we

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obtain the relation

$$dp(P_n \cap k_iJk_i^{-1}) = |p_i|^{-1}dp(P_n \cap J) = q^{in/e}dp(P_n \cap J).$$

We then obtain from Relations (1) and (2):

$$\begin{aligned} dk((P_n \cap k_iJk_i^{-1})\backslash k_iJk_i^{-1}) &= \frac{dk(k_iJk_i^{-1})}{dp(P_n \cap k_iJk_i^{-1})} \\ &= q^{-in/e}\frac{dk(J)}{dp(P_n \cap J)} \\ &= q^{-in/e}dk((P_n \cap J)\backslash J). \end{aligned}$$

Now by Relations (1) and (2) again, one has

$$\begin{aligned} dk((P_n \cap J)\backslash J)) &= \frac{dk(J)}{dp(P_n \cap J)} = \frac{dk(J)}{dk(\mathcal{M})}\frac{dk(\mathcal{M})}{dp(P_n \cap J)} \\ &= |J\backslash \mathcal{M}|dk(P_n \cap J\backslash \mathcal{M}). \end{aligned}$$

Finally, because $\mathcal{M} = (P_n \cap J)J^1$, applying Relation (3) gives:

$$dk((P_n \cap J)\backslash J)) = |J\backslash \mathcal{M}|dk(P_n \cap J^1\backslash J^1) = q^{r_1}(q^{n/e} - 1)$$

by Proposition 7.1(3) and our definition of r_1. This concludes the proof. ∎

§8. Explicit Whittaker functions of Paskunas–Stevens

In this section we continue to assume that $R = \mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$. We now recall the definition and some properties of the explicit Whittaker functions of [12]. We set

$$\mathcal{U} = (N_n \cap J)H^1.$$

We extend ψ_t to the group $\mathcal{U}$ as in [12, Definition 4.2], and, by abuse of notation, denote this extension by ψ_t. We fix a normal compact open subgroup $\mathcal{N}$ of $\mathcal{U}$ contained in $\ker(\psi_t)$. We also denote by ρ the trace character of Λ and $\rho^\vee$ that of $\Lambda^\vee$.

DEFINITION 8.1. (*Bessel functions*) For $j \in \mathbf{J}$, we define

$$\mathcal{J}(j) = |\mathcal{N}\backslash \mathcal{U}|^{-1} \sum_{\mathcal{N}\backslash \mathcal{U}} \psi_t(u)^{-1}\rho(ju), \qquad \text{and}$$

$$\mathcal{J}^\vee(j) = |\mathcal{N}\backslash \mathcal{U}|^{-1} \sum_{\mathcal{N}\backslash \mathcal{U}} \psi_t(u)\rho^\vee(ju).$$

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The Bessel functions enjoy the following properties:

PROPOSITION 8.2.

(1) *We have the equality* $\mathcal{J}(1) = 1$.
(2) $\mathcal{J}(uj) = \mathcal{J}(ju) = \psi_t(u)\mathcal{J}(j)$ *for* $u \in \mathcal{U}$ *and* $j \in \mathbf{J}$.
(3) *For all* $j \in \mathbf{J}$, *we have the relation*

$$\mathcal{J}^\vee(j) = \mathcal{J}(j^{-1}).$$

(4) *For all* j_1 *and* j_2 *in* $\mathbf{J}$, *we have*

$$\sum_{m \in \mathcal{U} \backslash \mathcal{M}} \mathcal{J}(j_1 m^{-1})\mathcal{J}(m j_2) = \mathcal{J}(j_1 j_2).$$

Proof. See [12, Proposition 5.3 and Theorem 5.6]. The third property follows from a simple change of variables, and the relation $\rho^\vee(ab) = \rho(b^{-1}a^{-1})$ for any a and b in $\mathbf{J}$. The final property follows from [12, Proposition 5.3, Property (v)], thanks to the bijection $m \leftrightarrow m^{-1}$ between $\mathcal{M}/\mathcal{U}$ and $\mathcal{U}\backslash\mathcal{M}$. ∎

We can now define the explicit Whittaker functions W and $W^\vee$ of Paskunas–Stevens following [12, Section 5.2] and recall a first property.

DEFINITION 8.3. Both W and $W^\vee$ are supported on $N_n\mathbf{J}$, and

$$W(nj) = \psi_t(n)\mathcal{J}(j)$$

for $n \in N_n$ and $j \in \mathbf{J}$, whereas

$$W^\vee(nj) = \psi_t^{-1}(n)\mathcal{J}^\vee(j) = \psi_t^{-1}(n)\mathcal{J}(j^{-1})$$

for $n \in N_n$ and $j \in \mathbf{J}$. Moreover, W belongs to $W(\pi, \psi_t)$ and $W^\vee$ belongs to $W(\pi^\vee, \psi_t^{-1})$.

We now prove further properties of W and $W^\vee$.

PROPOSITION 8.4. *For* $l \geqslant 0$, *let* $W_l = \mathbf{1}_{G_n^{(l)}} W$, *and* $W_l^\vee = \mathbf{1}_{G_n^{(l)}} W^\vee$.

(1) *The functions* $(W_l)\,|_{P_n K_n}$ *and* $(W_l)^\vee\,|_{P_n K_n}$ *are zero unless* $l = in/e$ *for some* $i \in \{0, \ldots, e-1\}$, *and in this case*

$$(W_l)\,|_{P_n K_n} = \mathbf{1}_{N_n \varpi_E^i J} W\,|_{P_n K_n}, \qquad (W_l^\vee)\,|_{P_n K_n} = \mathbf{1}_{N_n \varpi_E^i J} W^\vee\,|_{P_n K_n}.$$

(2) *If* $W_{in/e}(pk) \neq 0$, *then* $i \in \{0, \ldots, e-1\}$, $k \in P_n \varpi_E^i J$, *and, in fact,* $k \in (P_n \cap K_n) k_i J$.
(3) *If* $W_{in/e}(p\varpi_E^i j) \neq 0$ *with* $p \in P_n$ *and* $j \in J$, *then* $p \in N_n(P_n \cap J)$.

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Proof. The first statement follows from the fact that W is supported on $N_n\mathbf{J} = \coprod_{i\in\mathbb{Z}} N_n\varpi_E^i J$, this is a disjoint union because the absolute value of the determinant on $N_n\varpi_E^i J$ is $q^{-ni/e}$, and Statement (4) of Proposition 7.1. Hence, if $W_{in/e}(pk) \neq 0$, then $W(pk) \neq 0$, so $pk \in N_n\varpi_E^l J$ for a unique $l \in \{0, \ldots, e-1\}$, but this l must be equal to i, and this gives the first assertion of the second statement. In particular $k \in p^{-1}N_n\varpi_E^i J \subset P_n\varpi_E^i J$. But $P_n\varpi_E^i J = P_n p_i k_i J = P_n k_i J$, hence $k \in P_n k_i J \cap K_n = (P_n \cap K_n)k_i J$. This proves the second statement. For the third, we observe that if $W_{in/e}(p\varpi_E^i j) \neq 0$, then $p\varpi_E^i j \in N_n\varpi_E^i J$, hence $p \in N_n\varpi_E^i J j^{-1}\varpi_E^{-i} = N_n J$, which implies that $p \in N_n(P_n \cap J)$. ∎

§9. Test vectors

Again, we assume that $R = \mathbb{C}$, or $\overline{\mathbb{Q}_\ell}$, and π_1 and π_2 are cuspidal R-representations of G_n. We denote by $(\mathbf{J}, \Lambda)$ the extended maximal simple type of π_1, by $e = e(E/F)$ the ramification index of the field extension associated to $(\mathbf{J}, \Lambda)$, and by $W, W^\vee$ the explicit Whittaker functions associated to π_1 (see Definition 8.3). This section is dedicated to proving our main result on test vectors.

THEOREM 9.1. *Suppose that $L(X, \pi_1, \pi_2)$ is nontrivial, so that $\pi_2 \simeq \chi\pi_1^\vee$ for some unramified character χ of $F^\times$. Then there is an integer r such that*

$$I(X, W, \chi W^\vee, 1_{\mathfrak{o}_F^n}) = \frac{q^r(q-1)(q^{n/e}-1)}{1-(\chi(\varpi_F)X)^{n/e}} = q^r(q-1)(q^{n/e}-1)L(X, \pi_1, \pi_2).$$

We are now ready to prove the following crucial proposition. We recall that for all integers $l \geqslant 0$, the restriction W_l has been defined in Proposition 8.4.

PROPOSITION 9.2. *Let $F_l : (K_n \cap P_n)\backslash K_n/J^1 \to R$ be defined by*

$$F_l(k) = \int_{j\in J^1}\int_{N_n\backslash P_n} W_l(pkj)W_l^\vee(pkj)\,dp\,dj.$$

Then F_l is nonzero if an only if $l = in/e$ and $i \in \{0, \ldots, e-1\}$, and in this case, it is supported on $(K_n \cap P_n)k_i J$. Moreover, for $i \in \{0, \ldots, e-1\}$, and for $k \in (K_n \cap P_n)k_i J$, there is an integer r_2 independent of i such that

$$F_{in/e}(k) = q^{r_2}.$$

Proof. If $F_l(k)$ is nonzero, then $W_l(pkj)$ is nonzero at least for some $p \in P_n$ and $j \in J$, but then according to Statements (1) and (2)

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of Proposition 8.4, this implies that l is of the form $l = in/e$ with $i \in \{0, \ldots, e-1\}$, and $k \in (K_n \cap P_n)k_i J$. Moreover, from Statement (2) of the same proposition, we can write $k = p_0 \varpi_E^i j_0$ for $p_0 \in P_n$ and $j_0 \in J$. But now notice that for such a k, we have

$$\begin{aligned} F_l(k) &= \int_{j \in J^1} \int_{N_n \backslash P_n} W_l(pp_0 \varpi_E^i j_0 j) W_l^\vee(pp_0 \varpi_E^i j_0 j)\, dp\, dj \\ &= \int_{j \in J^1} \int_{N_n \backslash P_n} W_l(p \varpi_E^i j_0 j) W_l^\vee(p \varpi_E^i j_0 j)\, dp\, dj. \end{aligned}$$

Hence by Statement (3) of Proposition 8.4

$$\begin{aligned} F_l(k) &= \int_{j \in J^1} \int_{N_n \backslash N_n(P_n \cap J)} W_l(p \varpi_E^i j_0 j) W_l^\vee(p \varpi_E^i j_0 j)\, dp\, dj \\ &= \int_{j \in J^1} \int_{N_n \cap J \backslash P_n \cap J} W_l(m \varpi_E^i j_0 j) W_l^\vee(m \varpi_E^i j_0 j)\, dm\, dj \\ &= \int_{j \in J^1} \int_{N_n \cap J \backslash P_n \cap J} \mathcal{J}(m \varpi_E^i j_0 j) \mathcal{J}(j^{-1} j_0^{-1} \varpi_E^{-i} m^{-1})\, dm\, dj, \end{aligned}$$

the last equality according to Proposition 8.2(3). Now, as $\mathbf{J}$ normalizes J^1, and as for any $t \in G_n$ normalizing J^1, the automorphism $j \mapsto tjt^{-1}$ of J^1 has modulus character equal to $\mathbf{1}$, because J^1 is an open subgroup of the unimodular group G_n, we have

$$\begin{aligned} F_l(k) &= \int_{j \in J^1} \int_{N_n \cap J \backslash P_n \cap J} \mathcal{J}(mj \varpi_E^i j_0) \mathcal{J}(j_0^{-1} \varpi_E^{-i} (mj)^{-1})\, dm\, dj \\ &= \int_{N_n \cap J \backslash \mathcal{M}} \mathcal{J}(m \varpi_E^i j_0) \mathcal{J}(j_0^{-1} \varpi_E^{-i} m^{-1})\, dm. \end{aligned}$$

We write

$$dm(N_n \cap J \backslash (N_n \cap J)H^1) = dm(N_n \cap H^1 \backslash H^1) = q^{r_2},$$

which is indeed a power of q as H^1 is pro-p. Moreover, as H^1 is normal in J, and as the integrand is invariant under $\mathcal{U}$ thanks to Property (2) in Proposition 8.2

$$F_l(k) = q^{r_2} \int_{\mathcal{U} \backslash \mathcal{M}} \mathcal{J}(m \varpi_E^i j) \mathcal{J}(j^{-1} \varpi_E^{-i} m^{-1})\, dm = q^{r_2},$$

the last equality thanks to Statement (4) of Proposition 8.2. ∎

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Proposition 9.3. *The coefficient*

$$b_l = \int_{P_n \cap K_n \backslash K_n} \int_{N_n \backslash P_n} W_l(pk) W_l^\vee(pk)\, dp\, dk$$

is zero unless $l = in/e$ *for some* $i \in \{0, \ldots, e-1\}$*, in which case there is an integer* r *such that*

$$b_l = q^r (q^{n/e} - 1) q^{-in/e}.$$

Proof. By definition, b_l is equal to

$$\int_{P_n \cap K_n \backslash K_n / J^1} F_l(k)\, dk = q^{r_3} \int_{P_n \cap K_n \backslash K_n} F_l(k)\, dk$$

with $dk(J^1) = q^{r_3}$ (J^1 is pro-p). So according to Proposition 9.2, this is zero if $l \neq in/e$ for $i \in \{0, \ldots, e-1\}$, and if $l = in/e$ for $i \in \{0, \ldots, e-1\}$, it is equal to

$$\begin{aligned} q^{r_3} \int_{P_n \cap K_n \backslash (P_n \cap K_n) k_i J} F_l(k)\, dk &= q^{r_2 + r_3} dk(P_n \cap K_n \backslash (P_n \cap K_n) k_i J) \\ &= q^r (q^{n/e} - 1) q^{-in/e}, \end{aligned}$$

where we write $r = r_1 + r_2 + r_3$, from Lemma 7.2. ∎

If π is a cuspidal R-representation of G_n of ramification index e, we denote by $R(\pi)$ its *ramification group*, that is the group of unramified characters ν of $F^\times$ which satisfy $\nu\pi \simeq \pi$. It follows from [2, 6.2.5], that $R(\pi)$ is isomorphic to the group of n/eth roots of unity in $R^\times$, via $\nu \mapsto \nu(\varpi_F)$.

Proof of Theorem 9.1. We first suppose that $\pi_2 \simeq \pi_1^\vee$. By Equation (4), the integral $I(X, W, W^\vee, \mathbf{1}_{\mathfrak{o}_F^n})$ is equal to

$$\frac{q-1}{1 - X^n} \int_{(K_n \cap P_n) \backslash K_n} I_{(0)}(X, \rho(k) W, \rho(k) W^\vee)\, dk.$$

Now, as $WW^\vee = \sum_{l \in \mathbb{Z}} W_l W_l^\vee$, by Statement (1) of Propositions 8.4 and 9.3, we have

$$\begin{aligned} \int_{(K_n \cap P_n) \backslash K_n} I_{(0)}(X, \rho(k) W, \rho(k) W^\vee)\, dk &= \sum_{i=0}^{e-1} b_{in/e} q^{in/e} X^{in/e} \\ &= q^r (q^{n/e} - 1) \sum_{i=0}^{e-1} X^{in/e} = q^r (q^{n/e} - 1) \frac{1 - X^n}{1 - X^{n/e}}. \end{aligned}$$

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This gives the equality

$$I(X, W, W^\vee, \mathbf{1}_{\mathfrak{o}_F^n}) = (q-1)(q^{n/e}-1)\frac{q^r}{1-X^{n/e}}.$$

On the other hand, and by [5, Proposition 8.1], the factor $L(X, \pi, \pi^\vee)$ is equal to

$$L(X, \pi, \pi^\vee) = \prod_{\nu \in R(\pi)} \frac{1}{1-\nu(\varpi_f)X} = \frac{1}{1-X^{n/e}}.$$

Now in general, as we supposed that $L(X, \pi_1, \pi_2)$ is not equal to 1, we have $\pi_2 \simeq \chi\pi_1^\vee$ for some unramified character χ of $F^\times$. However, we have

$$L(X, \pi_1, \pi_2) = L(X, \pi_1, \chi\pi_1^\vee) = L(\chi(\varpi_F)X, \pi_1, \pi_1^\vee).$$

On the other hand, we have

$$\begin{aligned} I(X, W, \chi W^\vee, \mathbf{1}_{\mathfrak{o}_F^n}) &= I(\chi(\varpi_F)X, W, W^\vee, \mathbf{1}_{\mathfrak{o}_F^n}) \\ &= (q-1)(q^{n/e}-1)\frac{q^r}{1-(\chi(\varpi_F)X)^{n/e}}. \end{aligned}$$

However,

$$L(X, \pi_1, \pi_2) = L(\chi(\varpi_F)X, \pi, \pi^\vee) = \frac{1}{1-(\chi(\varpi_F)X)^{n/e}},$$

and we are done. $\square$

§10. L-factors of banal cuspidal ℓ-modular representations

In this section, we consider the cases $R = \overline{\mathbb{F}_\ell}$, and $R = \overline{\mathbb{Q}_\ell}$. In the $\overline{\mathbb{Q}_\ell}$ setting, we continue with the notations of the last section, and note that as ψ is integral, so are ψ_t and ψ_t^{-1}. Our main theorem has the following interesting corollary.

Corollary 10.1. *Let τ_1 and τ_2 be two banal cuspidal ℓ-modular representations of G_n, and π_1 and π_2 be any cuspidal ℓ-adic lifts, then*

$$L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2)).$$

Proof. We already noticed in Section 5 that if $L(X, \pi_1, \pi_2)$ is equal to 1, then

$$L(X, \tau_1, \tau_2) = r_\ell(L(X, \pi_1, \pi_2)) = 1,$$

whether τ_1 and τ_2 are banal or not. Hence we only need to focus on the case when $L(X,\pi_1,\pi_2)$ is not equal to 1, that is $\pi_2 \simeq \chi\pi_1^\vee$ for some unramified character χ. Let W be the Stevens–Paskunas explicit Whittaker function associated to an extended maximal simple type of π_1 as in the statement of Theorem 9.1.

LEMMA 10.2. *The explicit Whittaker functions W and $\chi W^\vee$ lie in the $\overline{\mathbb{Z}_\ell}$-submodules $W_e(\pi_1,\psi_t)$ and $W_e(\pi_2,\psi_t^{-1})$ respectively.*

Proof. As in the proof of Theorem 9.1, the representation π_1 contains an extended maximal simple type $(\mathbf{J}_1,\Lambda_1)$ and W is chosen to be the Paskunas–Stevens Whittaker function of Definition 8.3 relative to this data. As π_1 is integral, Λ_1 is integral by the end of Section 6. This implies that the trace character ρ_{Λ_1} of Λ_1 has values in $\overline{\mathbb{Z}_\ell}$. In particular the Bessel function $\mathcal{J}_1$ (see Definition 8.1) associated to the pair $(\mathbf{J}_1,\Lambda_1)$ takes values in $\overline{\mathbb{Z}_\ell}$. Hence, as ψ_t is integral, $W\in W_e(\pi_1,\psi_t)$ (see Definition 8.3). Now, π_2 is of the form $\chi\pi_1^\vee$ with χ an unramified character of $F^\times$ (which is integral as χ is unramified), so Proposition 8.2(3) implies that the Bessel function $\chi\mathcal{J}_1^\vee$ is integral. We conclude that $\chi W^\vee$ belongs to $W_e(\pi_2,\psi_t^{-1})$ (see Definition 8.3 again). ∎

Granted $W\in W_e(\pi_1,\psi_t)$ and $\chi W^\vee\in W_e(\pi_2,\psi_t)$, we have

$$\begin{aligned} r_\ell(q^r(q^{n/e}-1))r_\ell(L(X,\pi_1,\pi_2)) &= r_\ell(I(X,W,\chi W^\vee,\mathbf{1}_{\mathfrak{o}_F^n})) \\ &= I(X,r_\ell(W),r_\ell(\chi W^\vee),r_\ell(\mathbf{1}_{\mathfrak{o}_F^n})). \end{aligned}$$

Notice that $r_\ell(q^r(q-1)(q^{n/e}-1))$ is nonzero if and only if π_1 (hence π_2) is banal by the end of Section 6. As the integral $I(X,r_\ell(W),r_\ell(\chi W^\vee),r_\ell(\mathbf{1}_{\mathfrak{o}_F^n}))$ belongs to the fractional ideal $(L(X,\tau_1,\tau_2))$ of $\overline{\mathbb{F}_\ell}[X^{\pm1}]$, we deduce that $r_\ell(L(X,\pi_1,\pi_2))$ divides $L(X,\tau_1,\tau_2)$. As in any case, thanks to [8, Theorem 3.13], the L-factor $L(X,\tau_1,\tau_2)$ divides $r_\ell(L(X,\pi_1,\pi_2))$, we deduce the desired equality. ∎

REMARK 10.3. As noticed in the introduction and Section 5, the analogue of Corollary 10.1 is also true when π_1 and π_2 are cuspidal representations of general linear groups of different ranks as the L-factors are all trivial.

Acknowledgments. We thank Paul Broussous and Shaun Stevens for stimulating discussions, and Jim Cogdell for bringing the test vector problem to our attention. We thank the referee for a very thorough review which improved greatly the presentation and clarified certain parts of

https://doi.org/10.1017/nmj.2017.32 Published online by Cambridge University Press

the arguments. This work has benefited from the support of the Heilbronn Institute for Mathematical Research, GDRI 571 "French–British–German network in Representation Theory", and from the grant ANR-13-BS01-0012 FERPLAY.

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https://doi.org/10.1017/nmj.2017.32 Published online by Cambridge University Press

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https://doi.org/10.1017/nmj.2017.32 Published online by Cambridge University Press