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STABILITY OF A PROCESSOR-SHARING QUEUE WITH VARYING THROUGHPUT

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Abstract

In this paper we present a stability criterion for processor-sharing queues, in which the throughput may depend on the number of customers in the system (such as in the case of interferences between users). Such a system is represented by a point measure-valued stochastic recursion keeping track of the remaining processing times of the customers.

Keywords: Queueing system; stability; measure-valued process; stochastic recurrence; processor sharing

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1. Introduction

In this paper we address the question of stationarity in the general ergodic framework for processor-sharing queues, in which the throughput (i.e. the quantity of work achieved by the server(s) per unit of time) may depend on the state of the system. More precisely, we assume hereafter that the server(s) (it will be clear in the sequel that the effective number of servers does not really matter, only the quantity of work consumed per unit of time matters) processes all the jobs present in the system simultaneously and fairly. Whenever there are n customers in the system, each of them is thus served at a rate that depends on n, say $r(n)$. The classical case is when $r(n) = 1/n,\ n \geq 1$, so that the total throughput equals $nr(n) = 1$ whenever the system is nonempty: this is the classical processor-sharing queue. Hereafter, we consider a more general context in which the total throughput may decrease with the number of customers in the system (hence, $nr(n) \leq 1$). This is the case, for instance, in a wireless network in which the number of users currently active may decrease the efficiency of the resources. Another case is when the value of n, the number of customers, does *not* change the nominal service rate $r(n)$, say $r(n) = 1$ for all n. This corresponds to the classical queue with infinitely many servers.

In both cases and under general stationary ergodic assumptions, Loynes' stability result does not hold, since this is not a proper G/G/1 queue (the throughput may be less or larger than 1). We address the question of the existence of a stationary version of such queues by representing them with point measure-valued stochastic recursions in the Palm setting, so as to take into account the dependency on the number of customers. These point measures keep track of all the remaining service times of all the customers in the system. Then it is possible to provide conditions for the existence of a stationary version of this sequence, which allows us to explicitly construct stationary queues under these assumptions.

This paper is organized as follows. After some preliminaries in Section 2, we present the queueing models we consider in Section 3. In Section 4 we study the particular case of the

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G/G/∞ queue, and in Section 5 we present a stability criterion for generalized processor queues with state-dependent throughput.

2. Preliminaries

Let M_f^+ and $\mathcal{C}_b$ respectively denote the set of positive finite measures on $\mathbb{R}_+^*$ and the set of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Equipped with the *weak topology* $\sigma(M_f^+, \mathcal{C}_b)$, M_f^+ is a Polish space (see [2]). Let $\tilde{0}$ be the zero measure on $\mathbb{R}$ (i.e. such that $\tilde{0}(\mathfrak{B}) = 0$ for any Borel set $\mathfrak{B}$ on $\mathbb{R}$). For any $\mu \in M_f^+$ and any measurable $f\colon \mathbb{R} \to \mathbb{R}$, we classically write $\langle \mu, f \rangle := \int f \, \mathrm{d}\mu$. For any $y \in \mathbb{R}$ and any measurable $f\colon \mathbb{R} \to \mathbb{R}$, let $\tau_y f(\cdot) = f(\cdot - y)\mathbf{1}_{\{\cdot > y\}}$. Then, for any $\mu \in M_f^+$, $\tau_y \mu$ denotes the only element of M_f^+ such that $\langle \tau_y \mu, f \rangle = \langle \mu, \tau_y f \rangle$.

Let the set M_f^+ be endowed with the *increasing partial integral order*, '$\preceq$': for any two $\mu, \nu \in M_f^+$, $\mu \preceq \nu$ if $\langle \mu, f \rangle \le \langle \nu, f \rangle$ for any measurable nondecreasing function f such that these integrals exist. Of course, $\tilde{0} \preceq \mu$ for any $\mu \in M_f^+$. Furthermore, we note the following lemma.

Lemma 1. *Any sequence of M_f^+ that is '$\preceq$'-increasing and bounded above converges for the weak topology.*

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a '$\preceq$'-increasing sequence of M_f^+ that is bounded above by $\mu \in M_f^+$. Then, as easily seen, the sequence of nonincreasing real functions, $\{\mu_n([\cdot, \infty))\}_{n\in\mathbb{N}}$, tends pointwise and, hence (this is the Diniz theorem), uniformly to a nonincreasing real function f that is right continuous and has a countable number of discontinuities. Moreover, $f(0) \le \mu(\mathbb{R}_+^*) < \infty$, and we can fully characterize a measure $\mu^* \in M_f^+$ setting $\mu^*((0, x)) = f(0) - f(x)$ for all $x \in \mathbb{R}_+^*$. In particular, $\sup_{x\in\mathbb{R}_+^*} |\mu^n((0, x)) - \mu^*((0, x))| \longrightarrow 0$ as $n \to \infty$; hence, μ^n tends to μ^* in total variation. This completes the proof.

Now let $\mathcal{M} \subset M_f^+$ be the subset of finite (simple) counting measures on $\mathbb{R}_+^*$. Any $\mu \in \mathcal{M} \setminus \{\tilde{0}\}$ reads $\mu = \sum_{i=1}^{N(\mu)} \delta_{\alpha_i(\mu)}$, where $N(\mu) := \mu(\mathbb{R}_+^*)$ is the number of atoms of μ, δ_x is the Dirac measure at $x \in \mathbb{R}+$, and $\alpha_1(\mu) < \alpha_2(\mu) < \cdots < \alpha_{N(\mu)}(\mu)$. Then, $\tau_y(\mu) = \sum_{i=1}^{N(\mu)} \delta_{\alpha_i(\mu)-y} \mathbf{1}_{\{\alpha_i(\mu)>y\}}$ and, for any two $\mu, \nu \in \mathcal{M} \setminus \{\tilde{0}\}$, $\mu \preceq \nu$ whenever

(i) $N(\mu) \le N(\nu)$,

(ii) for all $i = 0, \ldots, N(\mu) - 1$, $\alpha_{N(\mu)-i}(\mu) \le \alpha_{N(\nu)-i}(\nu)$.

For any $\mu \in \mathcal{M} \setminus \{\tilde{0}\}$, let $Z(\mu) = \alpha_{N(\mu)}(\mu)$, the largest atom of μ. Finally, we write $x^+ = \max\{x, 0\}$ for any real number x, and $\sum_{i=j}^{k} \cdot \equiv 0$ whenever $k < j$ and $\max\{\varnothing\} \equiv 0$.

3. The model

Let us first introduce our definitions and assumptions on the queueing systems we will consider in the sequel. Let $(\Omega, \mathcal{F}, \mathrm{P}, \theta_t)$ be a probability space furnished with a bijective flow $(\theta_t)_{t\ge0}$, under which P is stationary and ergodic. Define on Ω the θ_t-compatible simple point process $(A_t)_{t\in\mathbb{R}}$ of points $\cdots < T_{-2} < T_{-1} < T_0 \le 0 < T_1 < T_2 < \cdots$, which represent the arrival times of the customers in a queue without a buffer. The process $(A_t)_{t\in\mathbb{R}}$ is marked by a sequence $\{\sigma_n\}_{n\in\mathbb{Z}}$, where, for all $n \in \mathbb{Z}$, σ_n is the service duration requested by customer C_n who arrived at time T_n. Also, for all $n \in \mathbb{Z}$, let $\xi_n = T_{n+1} - T_n$, and suppose that the generic random variables (RVs) σ and ξ are integrable. We consider servers that follow a generalized processor-sharing discipline. By this we mean that all present customers are taken care of simultaneously at a rate r, which is equal for all customers. An example is of

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TABLE 1.

Number of customers	Nominal service rate	Throughput
1	1.000	1.00
2	0.495	0.99
3	0.300	0.90
⋮	⋮	⋮
100	0.008	0.80

course provided by the classical processor-sharing queue, but it will be shown in the subsequent sections that significant results can also be obtained for a wider class of systems. Indeed, in many cases, it is plausible to assume that the amount of work in the system might affect the throughput, considering, for instance, the working cost induced by the switching mechanism in the processor or the interferences between the users of a wireless network. In both cases, it is then natural to assume that the rate r is a nonincreasing function of the service profile, i.e. $\mu \preceq \nu$ implies that $r(\mu) \geq r(\nu)$. Hereafter, for the sake of simplicity, we will restrict our attention to the subcase, where r is a nonincreasing function of the number of customers in the system, although it should be clear that all the results below also hold when r is a function of the whole service profile. In other words, at any t, each customer is allocated a quantity of work, $r(Q_t)$, per unit of time, where $Q(t)$ denotes the number of customers in the system at t, that is, $r(i) \geq r(j)$ for all $i, j \in \mathbb{N}^*$ such that $i \leq j$. In Table 1 we illustrate, through a naive example, the effect of a large number of customers on the throughput.

Provided that C_n is in the system at t, his *remaining processing time* at this instant is the time before his service completion. The *service profile* of the system at t is the $\mathcal{M}$-valued process keeping track of the remaining processing times of all the customers in the system at t:

$$\mu(t) = \sum_{i=1}^{Q(t)} \delta_{\alpha_i(\mu(t))},$$

where $\alpha_1(\mu(t)) \leq \alpha_2(\mu(t)) \leq \cdots \leq \alpha_{Q(t)}(\mu(t))$ denote the remaining processing times of the customers in the system at t, ranked in decreasing order. Let $W(t)$ denote the workload at t. Then the workload and the congestion processes can be recovered easily from the service profile process by writing, for all t,

$$Q(t) = N(\mu(t)), \qquad W(t) = \langle \mu(t), I \rangle,$$

where I is the identity function. The processes μ, Q, and W have càdlàg paths (i.e. paths that are continuous from the right with left limits), and, for all t, let $\mu(t-) = \lim_{s \uparrow\uparrow t} \mu(s)$ (and similarly for $Q(t-)$ and $W(t-)$). For all $n \in \mathbb{N}$, we respectively denote by $\mu_n = \mu(T_n-)$, $Q_n = Q(T_n-)$, and $W_n = X(T_n-)$ the service profile, the congestion, and the workload just before the arrival of customer C_n.

Let $(\Omega, \mathcal{F}, \mathrm{P}^0)$ be the Palm space of A, let $\theta := \theta_{T_1}$, let θ^{-1} be its measurable inverse, and, for all $n \in \mathbb{Z}$, let $\theta^n = \theta \circ \theta \circ \cdots \circ \theta$ and $\theta^{-n} = \theta^{-1} \circ \theta^{-1} \circ \cdots \circ \theta^{-1}$. Note that P^0 is stationary and ergodic under θ, i.e. for all $\mathfrak{A} \in \mathcal{F}$, $\mathrm{P}^0[\theta^{-1}\mathfrak{A}] = \mathrm{P}^0[\mathfrak{A}]$ and $\theta\mathfrak{A} = \mathfrak{A}$ implies that $\mathrm{P}^0[\mathfrak{A}] = 0$ or 1 and that all θ-contracting events (such that $\mathrm{P}^0[\mathfrak{A}^c \cap \theta^{-1}\mathfrak{A}] = 0$) are θ-invariant. Letting $\xi := \xi_0$ and $\sigma := \sigma_0$, we have, for all $n \in \mathbb{Z}$, $\xi_n := \xi \circ \theta^n$ and $\sigma_n := \sigma \circ \theta^n$.

We say that the E-valued random sequence $\{X_n\}_{n\in\mathbb{N}}$ is a stochastically recursive sequence (SRS) whenever, for some random mapping $\phi\colon E \to E$,

$$X_{n+1} = \phi \circ \theta^n(X_n), \qquad n \in \mathbb{N}, \mathrm{P}^0\text{-almost surely } (\mathrm{P}^0\text{-a.s.}).$$

For any E-valued RV Y, let $\{X_n^{[Y]}\}_{n\in\mathbb{N}}$ be the SRS $\{X_n\}_{n\in\mathbb{N}}$ such that $X_0^{[Y]} = Y$, P^0-a.s. We follow the formalism of [1] and formulate the question of stationarity for the SRS $\{X_n\}_{n\in\mathbb{N}}$ in the following terms. There exists a *stationary version* of $\{X_n\}_{n\in\mathbb{N}}$ whenever, for some Y and all n, $X_n^{[Y]} = Y \circ \theta^n$, P^0-a.s., or, in other words, provided that the equation

$$Y \circ \theta = \phi(Y)$$

admits a solution that is an E-valued RV. We say that two sequences of RVs $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ couple provided that

$$\mathrm{P}^0[\text{there exists } N(\omega),\ X_n(\omega) = Y_n(\omega) \text{ for all } n \geq N(\omega)] = 1$$

and that there is strong backwards coupling from $\{X_n\}_{n\in\mathbb{N}}$ with the stationary sequence $\{Y \circ \theta^n\}$ whenever

$$\mathrm{P}^0[\text{there exists } N'(\omega),\ X_n \circ \theta^{-n}(\omega) = Y(\omega) \text{ for all } n \geq N'(\omega)] = 1.$$

Lemma 2. *The sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is stochastically recursive for any rate function r: letting, for all $\mu \in \mathcal{M}$ and $x \in \mathbb{R}_+^*$,*

- *for all $i \leq N(\mu)$,*

$$\gamma_i^r(\mu, x) = r(N(\mu) - i + 1)\bigg(x - \sum_{j=1}^{i-1} \alpha_j(\mu)\bigg(\frac{1}{r(N(\mu) - j + 1)} - \frac{1}{r(N(\mu) - j)}\bigg)\bigg),$$

- $i^r(\mu, x) = \max\{i \leq N(\mu);\, \alpha_i(\mu) \leq \gamma_i^r(\mu, x)\}$,
- $\gamma^r(\mu, x) := \gamma^r_{(i^r(\mu,x)+1)\wedge 1}(\mu, x)$,
- $\Phi^r(\mu, x) = \tau_{\gamma^r(\mu,x)}\mu$,

we have, for any initial profile μ_0 and all $n \in \mathbb{N}$,

$$\mu_{n+1} = \Phi^r(\mu_n + \delta_{\sigma_n}, \xi_n). \tag{1}$$

Proof. Just after the arrival of C_n, the service profile reads $\mu := \mu_n + \delta_{\sigma_n}$. Set $T_0' := T_n$ and $\alpha_0(\mu) = 0$. For any $i \in \{1, \dots, N(\mu)\}$, let T_i' be the theoretical departure of customer $\tilde{C}_i$ whose remaining service time at T_n is $\alpha_i(\mu)$. The remaining service time of $\tilde{C}_i$ at T_{i-1}' is $\alpha_i(\mu) - \alpha_{i-1}(\mu)$, and between T_{i-1}' and T_i', $\tilde{C}_i$ is served at rate $r(N(\mu) - i + 1)$. Hence, we have the induction formula

$$T_i' = T_{i-1}' + \frac{\alpha_i(\mu) - \alpha_{i-1}(\mu)}{r(N(\mu) - i + 1)}, \qquad i \in \{1, \dots, N(\mu)\}, \tag{2}$$

from which we deduce, for all $i \in \{1, \dots, N(\mu)\}$,

$$T_i' = T_n + \frac{\alpha_i(\mu)}{r(N(\mu) - i + 1)} + \sum_{j=1}^{i-1} \alpha_j(\mu)\bigg(\frac{1}{r(N(\mu) - j + 1)} - \frac{1}{r(N(\mu) - j)}\bigg). \tag{3}$$

For any i, customer $\tilde{C}_i$ leaves the system before T_{n+1} provided that $T'_i - T_n \leq \xi_n$, which is equivalent to $\alpha_i(\mu) \leq \gamma_i^r(\mu, \xi_n)$ in view of (3). In particular, $i^r(\mu, \xi_n)$ denotes the index of the last customer leaving the system before T_{n+1} (or 0 if there is no departure between T_n and T_{n+1}). Then the system is not empty at $T_{n+1}-$ provided that $i^r(\mu, \xi_n) < N(\mu)$, and in this case, $\{\tilde{C}_i,\ i \in \{i^r(\mu, \xi_n) + 1, N(\mu)\}\}$ is the set of customers present in the system at $T_{n+1}-$. For such $i > i^r(\mu, \xi_n)$, the remaining service time of $\tilde{C}_i$ at T_{n+1} is given by

$$\alpha_i(\mu) - \alpha_{i^r(\mu,\xi_n)}(\mu) - r(N(\mu) - i^r(\mu, \xi_n))(T_{n+1} - T'_{i^r(\mu,\xi_n)}) = \alpha_i(\mu) - \gamma^r(\mu, \xi_n).$$

Thus, functional mapping of the profile at T_n onto the profile at $T_{n+1}-$ reads

$$\Phi^r(\cdot, \xi_n) \colon \mu \longmapsto \sum_{i=i^r(\mu,\xi_n)+1}^{N(\mu)} \delta_{\alpha_i(\mu)-\gamma^r(\mu,\xi_n)}.$$

To obtain the announced result, we note that, for any $\mu \in \mathcal{M}$, $x \in \mathbb{R}_+^*$, and any $i < N(\mu)$, we have

$$\gamma_{i+1}^r(\mu, x) - \gamma_i^r(\mu, x) = \frac{r(N(\mu) - i) - r(N(\mu) - i + 1)}{r(N(\mu) - i + 1)}(\gamma_i^r(\mu, x) - \alpha_i(\mu)),$$

which is nonnegative if and only if $i \leq i^r(\mu, x)$. Hence,

$$\gamma^r(\mu, x) = \max_{1 \leq i \leq N(\mu)} \gamma_i^r(\mu, x) \tag{4}$$

and, in particular, $\Phi^r(\mu, \xi_n) = \tau_{\gamma^r(\mu,\xi_n)}\mu$, P^0-a.s.

For a fixed $x \in \mathbb{R}+$, the two following monotonicity properties of the mappings $\Phi^r(\cdot, x)$ hold, as shown in Appendix A.

Lemma 3. *For any $x \in \mathbb{R}+$ and any rate function r, the mapping $\Phi^r(\cdot, x)$ is nondecreasing from $\mathcal{M}$ into itself.*

Lemma 4. *For any $x \in \mathbb{R}+$ and any $\mu \in \mathcal{M}$, and for any two rate functions r and $\tilde{r}$ such that $r(i) \leq \tilde{r}(i)$ for all $i \in \mathbb{N}^*$, $\Phi^r(\mu, x) \succeq \Phi^{\tilde{r}}(\mu, x)$.*

4. The pure delay system

Let us first consider the case where the rate function is constant with respect to the size of the system, say $r(i) = 1$ for any $i \geq 1$. This corresponds to the classical 'pure delay' G/G/∞ queue: all present customers are simultaneously served at unit rate and, hence, spend a time equal to their service duration in the system, which is equivalent to saying that there is an infinite number of servers. In this case, the recursive equation (1) driving the service profile sequence (for which a diffusion approximation is given in [5] in the M/GI/∞ case) specializes to

$$\mu_{n+1} = \tau_{\xi_n}(\mu_n + \delta_{\sigma_n})$$

and a stationary service profile for the queue is a solution to the equation

$$\mu \circ \theta = \tau_\xi(\mu + \delta_\sigma). \tag{5}$$

The following lemma (see [7]) will be used in the sequel.

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Lemma 5. *There exists a unique* P^0*-a.s. finite solution to the equation*

$$L \circ \theta = [\max\{L, \sigma\} - \xi]^+ \tag{6}$$

given by

$$L := \left[\sup_{j \in \mathbb{N}^*} \left(\sigma_{-j} - \sum_{i=1}^{j} \xi_{-i}\right)\right]^+.$$

Proof. Existence. Loynes' theorem for stochastic recurrences (see [1, p. 107] and [6]) can be applied since the mapping $x \mapsto [\max\{x, \sigma\} - \xi]^+$ is P^0-a.s. continuous and nondecreasing. The minimal solution L to (6) classically reads as the P^0-almost sure limit of Loynes's sequence $\{L_n^{[0]} \circ \theta^{-n}\}_{n \in \mathbb{N}}$, where $\{L_n^{[0]}\}_{n \in \mathbb{N}}$ is the initially null SRS that is defined by

$$L_{n+1}^{[0]} = [\max\{L_n^{[0]}, \sigma_n\} - \xi_n]^+ \quad \text{for all } n \in \mathbb{N}.$$

It is routine to check from Birkhoff's ergodic theorem (and the fact that σ is not identically 0) that L is P^0-a.s. finite.

Uniqueness. Let $\tilde{L}$ be a solution to (6). First, note that if $\tilde{L} > \sigma$, P^0-a.s. would imply that, on a P^0-a.s. event,

$$\tilde{L} \circ \theta > 0 \quad \Longleftrightarrow \quad \tilde{L} \circ \theta = \tilde{L} - \xi,$$

a contradiction to the ergodic lemma. Hence, in view of the minimality of L we have

$$\mathrm{P}^0[\tilde{L} = L] = \mathrm{P}^0[\tilde{L} \circ \theta \leq L \circ \theta] \geq \mathrm{P}^0[\tilde{L} \leq \sigma] > 0,$$

which implies that $\{\tilde{L} = L\}$ is P^0-almost sure since it is θ-contracting.

We can now state the following result.

Theorem 1. *Equation (5) admits a finite solution, given by*

$$\mu^\infty = \sum_{i=1}^{\infty} \delta_{(\sigma_{-i} - \sum_{j=1}^{i} \xi_{-j})} \mathbf{1}_{\{\sigma_{-i} \geq \sum_{j=1}^{i} \xi_{-j}\}}.$$

Moreover, provided that

$$\mathrm{P}^0[L \leq 0] > 0, \tag{7}$$

this solution is unique and, for all ζ *such that* $Z(\zeta) \leq L$, P^0*-a.s., the sequence* $\{\mu_n^{[\zeta]}\}_{n \in \mathbb{N}}$ *converges with strong backwards coupling to* μ^∞.

Proof. Existence. It is a straightforward consequence of Birkhoff's ergodic theorem that

$$\mathrm{P}^0[\mu^\infty \in \mathcal{M}] = \mathrm{P}^0\left[\operatorname{card}\left\{i \in \mathbb{N}^*,\ \sigma_{-i} - \sum_{j=1}^{i} \xi_{-j} \geq 0\right\} < \infty\right] > 0.$$

This θ-contracting event is thus P^0-almost sure. On the other hand, in view of Lemma 3, the mapping $\mu \mapsto \tau_\xi(\mu + \delta_\sigma)$ is P^0-a.s. nondecreasing from $\mathcal{M}$ into itself. Furthermore, it

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is continuous for the weak topology, as easily checked from the fact that, for any $\mathcal{M}$-valued sequence $\{\nu_n\}_{n\in\mathbb{N}}$ tending weakly to ν, any $x, s \in \mathbb{R}+$, and any $\phi \in \mathcal{C}_\mathrm{b}$,

$$\begin{aligned}\langle \tau_x \nu_n + \delta_s, \phi\rangle &= \int \phi(y-x)\,\mathrm{d}\nu_n(y) + \phi(s)\\ &\to \int \phi(y-x)\,\mathrm{d}\nu(y) \quad \text{as } n\to\infty\\ &= \langle \tau_x \nu + \delta_s, \phi\rangle.\end{aligned}$$

Thus, we can follow the steps of Loynes' construction (Lemma 1) to conclude that μ^∞ is the '$\preceq$'-minimal solution of (5) since it is the P^0-almost sure limit of the sequence given, for all $n \in \mathbb{N}$, by

$$\mu_n^{[\tilde{0}]} \circ \theta^{-n} = \sum_{i=1}^{\infty} \delta_{(\sigma_{-i} - \sum_{j=1}^{i} \xi_{-j})} \mathbf{1}_{\{\sigma_{-i} \geq \sum_{j=1}^{i} \xi_{-j}\}}.$$

Uniqueness. It is easily checked that, for any solution μ of (5),

$$Z(\mu)\circ\theta = Z(\tau_\xi(\mu+\delta_\sigma)) = [\max\{Z(\mu),\sigma\} - \xi]^+;$$

hence, $Z(\mu) = L$, P^0-a.s. Moreover, since μ^∞ is the minimal solution of (5), we have

$$\{\mu = \mu^\infty\} \supseteq \{\mu = \tilde{0}\} = \{Z(\mu) = 0\} = \{L = 0\}.$$

Hence, whenever (7) holds, the event $\{\mu = \mu^\infty\}$ has a positive probability. Since it is θ-invariant, it is P^0-almost sure.

Coupling. Let ζ be an $\mathcal{M}$-valued RV such that $Z(\zeta) \leq L$, P^0-a.s. It is easy to construct another $\mathcal{M}$-valued RV $\tilde{\zeta}$ such that $\zeta \preceq \tilde{\zeta}$ and $Z(\tilde{\zeta}) = L$, P^0-a.s. by setting, for example, $\tilde{\zeta} = \sum_{i=1}^{N(\zeta)-1} \delta_i(\zeta) + \delta_L$. From Lemma 3, it follows by induction that $\mu_n^{[\zeta]} \preceq \mu_n^{[\tilde{\zeta}]}$, P^0-a.s. for all $n \in \mathbb{N}$. Now note that, for all $n \in \mathbb{N}$, $Z(\mu_n^{[\tilde{\zeta}]}) = L \circ \theta^n$, as is easily checked by induction. Hence, for all $n \in \mathbb{N}$, we have

$$\mathcal{E}_n := \{L\circ\theta^n = 0\} = \{Z(\mu_n^{[\tilde{\zeta}]}) = 0\} = \{\mu_n^{[\tilde{\zeta}]} = \tilde{0}\} \subseteq \{\mu_n^{[\zeta]} = \tilde{0}\}.$$

Therefore, $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a stationary sequence of renovating events of length 1 for $\{\mu_n^{[\zeta]}\}_{n\in\mathbb{N}}$ (see [3] and [4]) for any ζ such that $Z(\zeta) \leq L$, P^0-a.s. Assumption (7) implies the coupling property for such an initial condition in view of Corollary 2.5.1 of [1].

As simple consequences of the latter result, let us note the following coupling properties.

Corollary 1. *Under condition (7), for any ζ such that $Z(\zeta) \leq L$, P^0-a.s.,*

(i) $\{X_n^{[N(\zeta)]}\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to $N(\mu^\infty)$,*

(ii) $\{W_n^{[\langle\zeta,I\rangle]}\}_{n\in\mathbb{N}}$ *converges with strong backwards coupling to $\langle\mu^\infty, I\rangle$.*

5. Processor-sharing queues

We will now consider the case where the rate function depends on the number of customers in the system at the current time. We assume hereafter that the nondecreasing function r is such that

$$\sup_{n\in\mathbb{N}^*} nr(n) \leq 1, \tag{8}$$

$$K_r = \inf_{n\in\mathbb{N}^*} nr(n) > 0. \tag{9}$$

Assumption (8) amounts to saying that there is a single server, since the throughput at time t, given by $Q(t)r(Q(t))$, may not exceed 1. A typical case is the classical processor-sharing queue: assume that $r(n) = n^{-1}$ for any n (and, hence, $K_r = 1$), meaning that all customers are served at a rate that is inversely proportional to the number of customers. In this case the server works at unit rate whatever the number of customers in the system. Whenever $K_r < 1$, the number of customers affects the velocity of service, so that the total throughput may be less than 1. Nevertheless, we assume in (9) that a minimal throughput K_r is granted for a given r, i.e. the server always achieves at least K_r units of work per unit of time. An example is provided by the following idealistic scenario: the server works at unit rate whenever there is only one customer in the system ($r(1) = 1$), and, when there are several customers in service at the same time, the interferences (or operating cost) decrease the efficiency of the server by half, so that $r(i) = 1/2i$ for any $i \geq 2$, which implies in particular that (9) is satisfied for $K_r = \frac{1}{2}$.

In view of Lemma 2, a stationary service profile is a solution to the equation

$$\mu \circ \theta = \Phi^r(\mu + \delta_\sigma, \xi). \tag{10}$$

We have the following result.

Theorem 2. *Let r be a rate function satisfying assumptions (8) and (9). Then, provided that*

$$\mathrm{E}^0[\sigma] < K_r\, \mathrm{E}^0[\xi], \tag{11}$$

(10) admits a unique finite solution μ^r. Moreover, for any $\mathcal{M}$-valued RV ζ such that $\langle \zeta, I\rangle \leq W^{K_r}$, P^0-a.s. (where W^{K_r} is the unique solution of (12), below), the sequence $\{\mu_n^{[\zeta]}\}_{n\in\mathbb{N}}$ converges with strong backwards coupling to μ^r.

Proof. Existence. Fix r to satisfy assumptions (8) and (9). From Loynes's fundamental stability result, the equation

$$W \circ \theta = [W + \sigma - K_r\xi]^+ \tag{12}$$

admits a unique P^0-a.s. finite solution, say W^{K_r}, provided that (11) holds. Let $\tilde{r}$ be the rate function such that, for all $\mu \in \mathcal{M}$, $\tilde{r}(\mu) = K_r/N(\mu)$, so that the throughput under $\tilde{r}$ always equals K_r whenever the system is nonempty. Let ζ be an $\mathcal{M}$-valued RV such that $\langle \zeta, I\rangle \leq W^{K_r}$ and

$$\tilde{\zeta} = \zeta + \delta_{W^{K_r} - \langle\zeta, I\rangle}\mathbf{1}_{\{W^{K_r} > \langle\zeta, I\rangle\}}.$$

It is then clear that $\langle\tilde{\zeta}, I\rangle = W^{K_r}$. Moreover, we have, P^0-a.s. for all $n \in \mathbb{N}$,

$$\langle\mu_{n+1}^{\tilde{r},[\tilde{\zeta}]}, I\rangle = [\langle\mu_n^{\tilde{r},[\tilde{\zeta}]}, I\rangle + \sigma_n - K_r\xi_n]^+,$$

as the throughput equals K_r at any time (as is easily checked from Lemma 2), so that $\langle\mu_{n+1}^{\tilde{r},[\tilde{\zeta}]}, I\rangle = W^{K_r} \circ \theta^n$ for all $n \in \mathbb{N}$. On the other hand, $\zeta \preceq \tilde{\zeta}$; hence, in view of Lemmas 3 and 4, an immediate induction shows that $\mu_n^{r,[\zeta]} \preceq \mu_n^{\tilde{r},[\tilde{\zeta}]}$ for all $n \in \mathbb{N}$, which in turn implies that

$$\langle\mu_n^{r,[\zeta]}, I\rangle \leq \langle\mu_n^{\tilde{r},[\tilde{\zeta}]}, I\rangle = W^{K_r} \circ \theta^n \qquad \text{for all } n \in \mathbb{N}.$$

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Therefore, for all $n \in \mathbb{N}$, on $\mathfrak{A}_n := \{W^{K_r} \circ \theta^n = 0\}$, we have $\langle \mu_n^{r,[\zeta]}, I \rangle = 0$; hence,

$$\mu_n^{r,[\zeta]} = \tilde{0} \quad \text{and} \quad \mu_{n+1}^{r,[\zeta]} = \Phi^r(\delta_{\sigma_n}, \xi_n).$$

Therefore, $\{\mu_n^{r,[\zeta]}\}_{n \in \mathbb{N}}$ admits $\{\mathfrak{A}_n\}_{n \in \mathbb{N}}$ as a stationary sequence of renovating events of length 1. Furthermore, the event $\mathfrak{A}_0 = \{W^{K_r} = 0\}$ has a strictly positive probability, since the contrary would imply that

$$\mathrm{E}^0[W^{K_r} \circ \theta - W^{K_r}] = \mathrm{E}^0[\sigma - K_r \xi] < 0,$$

an absurdity in view of the ergodic lemma. Then it follows from [1, Theorem 2.5.3] that there is strong backwards coupling of $\mu_n^{r,[\zeta]}$ with the stationary sequence $\{\mu^r \circ \theta^n\}_{n \in \mathbb{N}}$, where μ^r is a proper solution to (10).

Uniqueness. Fix r and $\tilde{r}$ to be as above. There exists a solution $\mu^{\tilde{r}}$ to (10). Then we have, P^0-a.s.,

$$\langle \mu^{\tilde{r}}, I \rangle \circ \theta = \langle \Phi^{\tilde{r}}(\mu^{\tilde{r}} + \delta_\sigma, \xi), I \rangle = [\langle \mu^{\tilde{r}}, I \rangle + \sigma - K_r \xi]^+;$$

hence, $\langle \mu^{\tilde{r}}, I \rangle$ equals W^{K_r}, P^0-a.s. Moreover, on $\{\langle \mu^r, I \rangle \leq W^{K^r}\}$, we have, in view of Lemma 2,

$$\langle \mu^r, I \rangle \circ \theta \leq \langle \Phi^{\tilde{r}}(\mu^r + \delta_\sigma, \xi), I \rangle = [\langle \mu^r, I \rangle + \sigma - K_r \xi]^+ \leq W^{K_r} \circ \theta \quad \mathrm{P}^0\text{-a.s.};$$

thus, the event $\{\langle \mu^r, I \rangle \leq W^{K_r}\}$ is θ-contracting. Moreover,

$$\mathrm{P}^0[\langle \mu^r, I \rangle \leq W^{K_r}] \geq \mathrm{P}^0[\langle \mu^r, I \rangle = 0] > 0,$$

as another consequence of (11) and the ergodic lemma. Therefore, $\langle \mu^r, I \rangle \leq W^{K_r}$, P^0-a.s., so that

$$\mathfrak{A}_n \subseteq \{\langle \mu^r, I \rangle \circ \theta^n = 0\} = \{\mu^r \circ \theta^n = \tilde{0}\}.$$

Consequently, $\{\mathfrak{A}_n\}_{n \in \mathbb{N}}$ is a stationary sequence of renovating events of length 1 for $\{\mu^r \circ \theta^n\}_{n \in \mathbb{N}}$ for any solution μ^r of (10) associated to the rate r. Since $\mathrm{P}^0[\mathfrak{A}_0] > 0$, there exists a unique solution to (10) in view of Remark 2.5.3 of [1].

In particular, we have the following corollary.

Corollary 2. *Under condition (11), for any ζ such that $\langle \zeta, I \rangle \leq W^{K_r}$, P^0-a.s.,*

(i) $\{X_n^{[N(\zeta)]}\}_{n \in \mathbb{N}}$ *converges with strong backwards coupling to* $N(\mu^r)$,

(ii) $\{W_n^{[\langle \zeta, I \rangle]}\}_{n \in \mathbb{N}}$ *converges with strong backwards coupling to* $\langle \mu^r, I \rangle$.

Appendix A. Proofs of monotonicity

For easy checking, in this appendix we present the details of the proofs of Lemmas 3 and 4.

Proof of Lemma 3. We again fix $x \in \mathbb{R}+$ and $\mu, \nu \in \mathcal{M}$ such that $\mu \preceq \nu$. Whenever $i^r(\mu, x) < N(\mu)$ (otherwise $\Phi^r(\mu, x) = \tilde{0}$), we have

$$\sum_{j=1}^{N(\nu)-N(\mu)+i^r(\mu,x)} \alpha_j(\nu) \left(\frac{1}{r(N(\nu) - j + 1)} - \frac{1}{r(N(\nu) - j)} \right)$$
$$\geq \sum_{j=1}^{i^r(\mu,x)} \alpha_j(\mu) \left(\frac{1}{r(N(\mu) - j + 1)} - \frac{1}{r(N(\mu) - j)} \right),$$

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which implies that

$$\begin{aligned}\alpha_{N(\nu)-N(\mu)+i^r(\mu,x)+1}(\nu) &\geq \alpha_{i^r(\mu,x)+1}(\mu)\\ &\geq r(N(\mu)-i^r(\mu,x))\\ &\quad\times\bigg(x-\sum_{j=1}^{i^r(\mu,x)}\alpha_j(\mu)\bigg(\frac{1}{r(N(\mu)-j+1)}-\frac{1}{r(N(\mu)-j)}\bigg)\bigg)\\ &\geq \gamma^r_{N(\nu)-N(\mu)+i^r(\mu,x)+1}(\nu,x).\end{aligned}$$

This means that $i_0(\nu,x) \leq N(\nu)-N(\mu)+i_0(\mu,x)$, i.e. $N(\Phi^r(\mu,x)) \leq N(\Phi^r(\nu,x))$. Hence, in view of (4) we have

$$\begin{aligned}\gamma(\mu,\xi) &= \gamma^r_{i^r(\mu,\xi)+1}(\mu,x)\\ &\geq \gamma^r_{(i^r(\nu,\xi)+N(\mu)-N(\nu))^++1}(\mu,x)\\ &\geq r(N(\nu)-i^r(\nu,x))\bigg(x-\sum_{j=1}^{i^r(\nu,x)}\alpha_j(\nu)\bigg(\frac{1}{r(N(\nu)-j+1)}-\frac{1}{r(N(\nu)-j)}\bigg)\bigg)\\ &= \gamma^r(\nu,x),\end{aligned}$$

which clearly implies that $\Phi^r(\mu,x) \preceq \Phi^r(\nu,x)$.

Proof of Lemma 4. We now fix $\mu \in \mathcal{M}$ and $x \in \mathbb{R}_+$. For any two rate functions r and $\tilde{r}$ such that $r(i) \leq \tilde{r}(i)$ for any $i \in \mathbb{N}^*$, the induction formula, (2), straightforwardly shows that $i^r(\mu,x) \geq i^{\tilde{r}}(\mu,x)$, i.e. $N(\Phi^r(\mu,x)) \leq N(\Phi^{\tilde{r}}(\mu,x))$. Hence, as in the previous proof, $\gamma^r(\mu,x) \leq \gamma^{\tilde{r}}(\mu,x)$.

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