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Dimension estimates for C^1 iterated function systems and repellers. Part II

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Abstract. This is the second part of our study on the dimension theory of C^1 iterated function systems (IFSs) and repellers on $\mathbb{R}^d$. In the first part [D.-J. Feng and K. Simon. Dimension estimates for C^1 iterated function systems and repellers. Part I. *Preprint*, 2020, arXiv:2007.15320], we proved that the upper box-counting dimension of the attractor of every C^1 IFS on $\mathbb{R}^d$ is bounded above by its singularity dimension, and the upper packing dimension of every ergodic invariant measure associated with this IFS is bounded above by its Lyapunov dimension. Here we introduce a generalized transversality condition (GTC) for parameterized families of C^1 IFSs, and show that if the GTC is satisfied, then the dimensions of the IFS attractor and of the ergodic invariant measures are given by these upper bounds for almost every (in an appropriate sense) parameter. Moreover, we verify the GTC for some parameterized families of C^1 IFSs on $\mathbb{R}^d$.

Key words: iterated function systems, Hausdorff and box-counting dimensions, singularity dimension, Lyapunov dimension, transversality condition
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1. *Introduction*

The present paper is a continuation of our work in [24] for studying the dimension theory of C^1 iterated function systems (IFSs) and repellers.

One of the fundamental problems in fractal geometry and dynamical systems is to compute various fractal dimensions of attractors of IFSs and associated invariant measures.

The corresponding problem has been well understood when the underlying IFSs consist of similitudes or conformal maps satisfying certain separation conditions (see e.g. [7, 25, 26, 30, 39, 41, 44]). The problem becomes substantially more difficult when the underlying IFSs are non-conformal. In the last three decades, much significant progress has been achieved for affine IFSs, see e.g. [2, 5, 11, 13, 19, 23, 27, 32, 34, 37], the references in the survey papers [10, 17] and an upcoming book [3].

In contrast to the extensive studies on affine IFSs, there have been relatively few results on those IFSs which are neither conformal nor affine. In 1994, Falconer [15] introduced a quantity (known as the singularity dimension) in terms of sub-additive topological pressure, and showed that it is an upper bound for the upper box-counting dimension of repellers of C^2 expanding maps satisfying a 'bunching' condition. Later, in 1997, Zhang [50] proved that this upper bound holds for the Hausdorff dimension of repellers of arbitrary C^1 expanding maps. We remark that the results of Falconer and Zhang extend directly to the IFS setting. Recently, Cao, Pesin and Zhao [9] also gave an upper bound for the upper box-counting dimension of repellers of $C^{1+\alpha}$ expanding maps satisfying a certain dominated splitting property. However that upper bound depends on the splitting involved and is usually strictly larger than the singularity dimension. In [24], the authors proved that the singularity dimension is an upper bound of the upper box-counting dimension of the attractor of every C^1 IFS or the repeller of every C^1 expanding map, which improved the aforementioned results in [9, 15, 50]. The authors also established a measure analogue of this result, that is, the upper packing dimension of every ergodic invariant measure associated with a C^1 IFS or repeller is bounded above by its Lyapunov dimension, which improved an earlier result of Jordan and Pollicott [31] for the upper Hausdorff dimension of measures. The reader is referred to §2.2 for the definitions of singularity dimension and Lyapunov dimension.

In [29], Hu computed the box-counting dimension of repellers of C^2 maps on $\mathbb{R}^2$ which have an invariant strong unstable foliation along which they expand more strongly than in the complementary directions. Very recently, Falconer, Fraser and Lee [18] computed the L^q-spectra of Bernoulli measures associated with a class of planar IFSs consisting of $C^{1+\alpha}$ maps for which the Jacobian is a lower triangular matrix subject to a domination condition and satisfying the rectangular open set condition. As a corollary, they obtained a formula for the box-counting dimension of the attractors of such planar IFSs. In another recent paper [33], Jurga and Lee proved that, under slightly stronger assumptions, these Bernoulli measures (and more generally, quasiBernoulli measures) on the attractors are exact dimensional with dimension given by a Ledrappier–Young-type formula. In earlier related works, Bedford and Urbański [6] calculated the box-counting and Hausdorff dimensions of the attractors of a very special class of planar nonlinear triangular $C^{1+\alpha}$ IFSs (of which the attractors are curves), Manning and Simon [36] and Bárány [1] studied the sub-additive pressure associated with nonlinear $C^{1+\alpha}$ IFSs whose maps have triangular Jacobians.

In this paper, we introduce a generalized transversality condition (GTC) for parameterized families of C^1 IFSs on $\mathbb{R}^d$, and show that if the GTC is satisfied, then for almost every (a.e.) (in an appropriate sense) parameter, the Hausdorff and box-counting dimensions of the IFS attractor are indeed given by the singularity dimension, and the dimension of ergodic invariant measures on the attractor is given by its Lyapunov dimension. Moreover, we will verify the GTC for several classes of translational families of C^1 IFSs.

Before formulating our results precisely, we first recall some basic notation and definitions. By a C^1 IFS on a compact set $Z \subset \mathbb{R}^d$, we mean a finite collection $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ of self-maps on Z, such that there exists an open set $U \supset Z$ so that each f_i extends to a C^1-diffeomorphism $f_i : U \to f_i(U) \subset U$ with

$$\rho_i := \sup_{x \in U} \|D_x f_i\| < 1,$$

where $D_x f$ stands for the differential of f at x and $\|\cdot\|$ is the standard matrix norm (that is, $\|A\|$ is the largest singular value of A).

Let K be the attractor of the IFS $\mathcal{F}$, that is, K is the unique non-empty compact subset of Z such that

$$K = \bigcup_{i=1}^{\ell} f_i(K) \tag{1.1}$$

(cf. [30]).

Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$. Let $\Pi : \Sigma \to K$ denote the corresponding coding map associated with the IFS $\mathcal{F}$, that is,

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0), \quad \mathbf{i} = (i_n)_{n=1}^{\infty}. \tag{1.2}$$

It is well known that Π is continuous and surjective [30]. For a σ-invariant Borel probability measure μ on Σ, let $\Pi_*\mu$ denote the push-forward of μ by Π, that is, $\Pi_*\mu(E) = \mu(\Pi^{-1}(E))$ for each Borel subset E of $\mathbb{R}^d$.

For a Borel probability measure ξ on $\mathbb{R}^d$, we call

$$\underline{d}_\xi(x) = \liminf_{r\to 0} \frac{\log \xi(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\xi(x) = \limsup_{r\to 0} \frac{\log \xi(B(x,r))}{\log r}$$

the *lower and upper local dimensions of ξ at x*, where $B(x, r)$ stands for the closed ball centered at x of radius r. Moreover, we call

$$\underline{\dim}_H \xi = \operatorname*{ess\,inf}_{x\in \mathrm{spt}(\xi)} \underline{d}_\xi(x) \quad \text{and} \quad \overline{\dim}_P \xi = \operatorname*{ess\,sup}_{x\in \mathrm{spt}(\xi)} \overline{d}_\xi(x)$$

the *lower Hausdorff dimension and upper packing dimension of ξ*, respectively. If $\underline{\dim}_H \xi = \overline{\dim}_P \xi$, we say that ξ is *exact dimensional* and write $\dim \xi$ or $\dim_H \xi$ for this common value.

To introduce the notion of GTC, let $\ell \geq 2$ and let $\mathcal{F}^t = \{f_1^t, \ldots, f_\ell^t\}$, $t \in \Omega$, be a parameterized family of C^1 IFSs defined on a common compact subset Z of $\mathbb{R}^d$, where (Ω, ρ) is a separable metric space, such that the following two conditions hold.

(C1) The maps f_i^t have a common Lipschitz constant $\theta \in (0, 1)$, that is,

$$|f_i^t(x) - f_i^t(y)| \leq \theta |x - y| \tag{1.3}$$

for all $1 \leq i \leq \ell$, $t \in \Omega$ and $x, y \in Z$.

(C2) The mapping $t \mapsto f_i^t(x)$ is continuous over Ω for every given $x \in Z$ and $1 \leq i \leq \ell$.

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For each $t \in \Omega$, let K^t denote the attractor of $\mathcal{F}^t$, and let $\Pi^t : \Sigma \to \mathbb{R}^d$ denote the coding map associated with the IFS $\mathcal{F}^t$. Due to the conditions (C1) and (C2), the mapping $(t, \mathbf{i}) \mapsto \Pi^t(\mathbf{i})$ is continuous over the product space $\Omega \times \Sigma$.

For $t \in \Omega$, $r > 0$ and $\mathbf{i} \in \Sigma_* := \bigcup_{n=0}^{\infty}\{1, \ldots, \ell\}^n$, set

$$Z_{\mathbf{i}}^t(r) = \inf_{x\in\Sigma} \min\left\{\frac{r^k}{\phi^k(D_{\Pi^t x} f_{\mathbf{i}}^t)} : k = 0, 1, \ldots, d\right\}, \tag{1.4}$$

where $f_{\mathbf{i}}^t := f_{i_1}^t \circ \cdots \circ f_{i_n}^t$ for $\mathbf{i} = i_1 \ldots i_n$, f_ε^t denotes the identity map on $\mathbb{R}^d$ and $\phi^s(\cdot)$ stands for the singular value function (see (2.5) for the definition).

Definition 1.1. Let η be a locally finite Borel measure on Ω. We say that the family $\mathcal{F}^t$, $t \in \Omega$, satisfies a GTC with respect to η if there exist $\delta_0 > 0$ and a function $\psi : (0, \delta_0) \to [0, \infty)$ with $\lim_{\delta\to 0} \psi(\delta) = 0$ such that the following statement holds: for every $t_0 \in \Omega$ and every $0 < \delta < \delta_0$, there exists a constant $C = C(t_0, \delta) > 0$ such that for all distinct $\mathbf{i}, \mathbf{j} \in \Sigma$ and $r > 0$,

$$\eta\{t \in B(t_0, \delta) : |\Pi^t(\mathbf{i}) - \Pi^t(\mathbf{j})| < r\} \le Ce^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} Z_{\mathbf{i}\wedge\mathbf{j}}^{t_0}(r), \tag{1.5}$$

where $B(t_0, \delta)$ denotes the closed ball in Ω of radius δ centered at t, $\mathbf{i} \wedge \mathbf{j}$ denotes the common initial segment of $\mathbf{i}$ and $\mathbf{j}$, and $|\mathbf{i} \wedge \mathbf{j}|$ is the length of the word $\mathbf{i} \wedge \mathbf{j}$.

The introduction of the GTC is inspired by the work of Jordan, Pollicott and Simon [32], who defined the self-affine transversality condition for certain translational families of affine IFSs. The new feature here is that the upper bound term on the right-hand side of (1.5) depends upon t_0, δ and $|\mathbf{i} \wedge \mathbf{j}|$, while in the setting of [32], the corresponding upper bound term is independent of these parameters and is determined by the linear parts of one pre-given affine IFS.

For $t \in \Omega$ and a σ-invariant measure μ on Σ, we write

$$d(t) := \dim_S(\mathcal{F}^t), \quad d_\mu(t) := \dim_{L,\mathcal{F}^t} \mu \tag{1.6}$$

for the singularity dimension of $\mathcal{F}^t$ and the Lyapunov dimension of μ with respect to $\mathcal{F}^t$, respectively; see Definitions 2.3–2.4. For $E \subset \mathbb{R}^d$, let $\dim_H E$ denote the Hausdorff dimension of E, and let $\overline{\dim}_B E$, $\underline{\dim}_B E$ denote the upper and lower box-counting dimensions of E, respectively (cf. [16]). When $\overline{\dim}_B E = \underline{\dim}_B E$, the common value is said to be the box-counting dimension of E and is denoted by $\dim_B E$.

The first result of the present paper is the following.

THEOREM 1.2. *Let $\mathcal{F}^t = \{f_1^t, \ldots, f_\ell^t\}$, $t \in \Omega$, be a parameterized family of C^1 IFSs defined on a common compact subset Z of $\mathbb{R}^d$, such that the conditions (C1)–(C2) hold. Let η be a locally finite Borel measure on Ω. Assume that $(\mathcal{F}^t)_{t\in\Omega}$ satisfies the GTC with respect to η. Then the following properties hold.*

(i) *Let μ be a σ-invariant ergodic measure on Σ. For η-a.e. $t \in \Omega$, $\Pi_*^t\mu$ is exact dimensional and*

$$\dim_H \Pi_*^t\mu = \min\{d, d_\mu(t)\}.$$

Moreover, $\Pi^t_*\mu \ll \mathcal{L}_d$ for η-a.e. $t \in \{t' \in \Omega : d_\mu(t') > d\}$, where $\mathcal{L}_d$ denotes the Lebesgue measure on $\mathbb{R}^d$.

(ii) *For η-a.e. $t \in \Omega$,*

$$\dim_H K^t = \dim_B K^t = \min\{d, d(t)\}.$$

Moreover, $\mathcal{L}_d(K^t) > 0$ for η-a.e. $t \in \{t' \in \Omega : \ d(t') > d\}$.

The above theorem is a nonlinear analogue of the results of Jordan, Pollicott and Simon [32, Theorems 4.2 and 4.3] for affine IFSs. We emphasize that in the nonlinear case, the singularity and Lyapunov dimensions depend on the parameter t, while in the affine case, the corresponding quantities are constant. This is a key difference between the affine case and the nonlinear case. We remark that Theorem 1.2 also extends and generalizes the corresponding results of Simon, Solomyak and Urbański ([45, Theorem 3.1], [46, Theorem 2.3]) for $C^{1+\alpha}$ conformal IFSs on $\mathbb{R}$.

Now a natural question arises of how to verify the GTC for a parameterized family of C^1 IFSs. In what follows, we investigate this question for certain translational families of C^1 IFSs.

First we introduce some definitions.

Definition 1.3. Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on a compact set $Z \subset \mathbb{R}^d$ such that $f_i(Z) \subset \mathrm{int}(Z)$ for each i. Set

$$f_i^{\mathbf{t}} := f_i + \mathbf{t}_i, \quad i = 1, \ldots, \ell, \tag{1.7}$$

where $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \mathbb{R}^{\ell d}$ with $\mathbf{t}_i \in \mathbb{R}^d$. By continuity, there is a small $r_0 > 0$ such that $f_i^{\mathbf{t}}(Z) \subset \mathrm{int}(Z)$ for every $\mathbf{t}$ with $|\mathbf{t}| < r_0$ and every i, where $|\cdot|$ is the Euclidean norm. Set $\mathcal{F}^{\mathbf{t}} = \{f_i^{\mathbf{t}}\}_{i=1}^{\ell}$ for each $\mathbf{t}$ with $|\mathbf{t}| < r_0$. We call $(\mathcal{F}^{\mathbf{t}})_{\mathbf{t}\in\Delta}$, where $\Delta := \{\mathfrak{s} \in \mathbb{R}^{\ell d} : \ |\mathfrak{s}| < r_0\}$, a *translational family of IFSs generated by $\mathcal{F}$*.

Definition 1.4. Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on a compact set $Z \subset \mathbb{R}^d$. We say that $\mathcal{F}$ is *dominated lower triangular*, if for each $z \in Z$ and $i \in \{1, \ldots, \ell\}$, the Jacobian $D_z f_i$ of f_i at z is a lower triangular matrix such that

$$|(D_z f_i)_{11}| \geq |(D_z f_i)_{22}| \geq \cdots \geq |(D_z f_i)_{dd}|.$$

We remark that in the above definition, the condition for an IFS to be dominated lower triangular is slightly weaker than that required in [18, 33].

Definition 1.5. Let $\ell \in \mathbb{N}$ with $\ell \geq 2$. Assume for $j = 1, \ldots, n$, $\mathcal{F}_j = \{f_{i,j}\}_{i=1}^{\ell}$ is an IFS on $Z_j \subset \mathbb{R}^{q_j}$. Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be an IFS on $Z_1 \times \cdots \times Z_n \subset \mathbb{R}^{q_1} \times \cdots \times \mathbb{R}^{q_n}$ given by

$$f_i(x_1, \ldots, x_n) = (f_{i,1}(x_1), \ldots, f_{i,n}(x_n)), \quad i = 1, \ldots, \ell, \ x_k \in Z_k \text{ for } 1 \leq k \leq n.$$

We say that $\mathcal{F}$ is the *direct product* of $\mathcal{F}_1, \ldots, \mathcal{F}_n$, and write $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$.

Now we are ready to state the second main result of the paper.

THEOREM 1.6. *Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on a compact set $Z \subset \mathbb{R}^d$ such that $f_i(Z) \subset \mathrm{int}(Z)$ for each i. Suppose either one of the following three conditions holds.*

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(i) *$\mathcal{F}$ is dominated lower triangular on Z satisfying*

$$\max_{i\neq j}\Big(\sup_{y,z\in Z}\|D_y f_i\|+\|D_z f_j\|\Big)<1, \tag{1.8}$$

and Z is convex.

(ii) *$\mathcal{F}$ is a C^1 conformal IFS on Z satisfying (1.8), and Z is connected.*

(iii) *$\mathcal{F}=\mathcal{F}_1\times\cdots\times\mathcal{F}_n$, where for each $k\in\{1,\ldots,n\}$, $\mathcal{F}_k$ is a C^1 IFS on a compact $Z_k\subset\mathbb{R}^{d_k}$ satisfying either (i) or (ii), in which $\mathcal{F}$ and Z are replaced by $\mathcal{F}_k$ and Z_k, respectively.*

Then there is a small $r_0>0$ such that the translational family

$$\mathcal{F}^{\mathbf{t}}=\{f_i+\mathbf{t}_i\}_{i=1}^{\ell},\quad \mathbf{t}=(\mathbf{t}_1,\ldots,\mathbf{t}_\ell)\in\Delta:=\{\mathfrak{s}\in\mathbb{R}^{\ell d}:\ |\mathfrak{s}|<r_0\},$$

satisfies the GTC with respect to the Lebesgue measure $\mathcal{L}_{\ell d}$ on Δ. As a consequence, the conclusions of Theorem 1.2 hold for the family $(\mathcal{F}^{\mathbf{t}})_{\mathbf{t}\in\Delta}$.

The above theorem is a (partial) nonlinear extension of the corresponding results in [13, 32, 47] for affine IFSs. Recall that in the case when $\mathcal{F}=\{f_i(x)=A_ix+a_i\}_{i=1}^{\ell}$ is an affine IFS on $\mathbb{R}^d$, under the assumption that

$$\max_{1\le i\le\ell}\|A_i\|<1/3, \tag{1.9}$$

Falconer [13] proved that the dimension of the attractor of $\mathcal{F}^{\mathbf{t}}=\{f_i+\mathbf{t}_i\}_{i=1}^{\ell}$ is equal to its affinity dimension for $\mathcal{L}_{\ell d}$-a.e. $\mathbf{t}=(\mathbf{t}_1,\ldots,\mathbf{t}_\ell)\in\mathbb{R}^{\ell d}$. Later, Solomyak [47] pointed out that the bound $1/3$ in (1.9) can be replaced by $1/2$. By an observation of Edgar [12], $1/2$ is optimal. Under the same assumption that

$$\max_{1\le i\le\ell}\|A_i\|<1/2, \tag{1.10}$$

Jordan, Pollicott and Simon [32] showed that the translational family $(\mathcal{F}^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^{\ell d}}$ satisfies the self-affine transversality condition. It was pointed out in [3, Theorem 9.1.2] that the assumption (1.10) can by further replaced by a slightly more general condition $\max_{i\neq j}(\|A_i\|+\|A_j\|)<1$.

We remark that Theorem 1.6 also extends the results of Simon, Solomyak and Urbański ([45, Proposition 7.1], [46, Corollary 7.3]) for translational families of $C^{1+\alpha}$ conformal IFSs on $\mathbb{R}$. It is worth pointing out that for every C^1 conformal IFS satisfying the open set condition (or C^1 conformal expanding map), the dimension of its attractor (or repeller) satisfies the Bowen–Ruelle formula, and is equal to the singularity dimension; meanwhile, the dimension of ergodic invariant measures on the attractor (repeller) is given by the Lyapunov dimension (see [7, 25, 39, 44]).

The paper is organized as follows. In §2, we give some preliminaries, including the variational principle for sub-additive topological pressure, and the definitions and properties of singularity dimension and Lyapunov dimension. In §3, we prove Theorem 1.2. The proof of Theorem 1.6 is rather long and will be given in §§4–7, where we divide the whole proof into three different parts, by considering the conditions (i)–(iii) in Theorem 1.6 separately.

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2. *Preliminaries*

2.1. *Variational principle for sub-additive pressure.* In order to define the singularity and Lyapunov dimensions, we require some elements from the sub-additive thermodynamic formalism.

Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$. That is, $\Sigma = \{1, \ldots, \ell\}^{\mathbb{N}}$, which is endowed with the product topology, and $\sigma : \Sigma \to \Sigma$ is the left shift defined by $(x_i)_{i=1}^{\infty} \mapsto (x_{i+1})_{i=1}^{\infty}$. Write $\Sigma_n = \{1, \ldots, \ell\}^n$ for $n \geq 0$, with the convention $\Sigma_0 = \{\varepsilon\}$, where ε stands for the empty word. Set $\Sigma_* = \bigcup_{n=0}^{\infty} \Sigma_n$. For $x = (x_i)_{i=1}^{\infty} \in \Sigma$ and $n \in \mathbb{N}$, write $x|n = x_1 \ldots x_n$.

Let $C(\Sigma)$ denote the set of real-valued continuous functions on Σ. Let $\mathcal{G} = \{g_n\}_{n=1}^{\infty}$ be a *sub-additive potential* on Σ, that is, $g_n \in C(\Sigma)$ for all $n \geq 1$ such that

$$g_{m+n}(x) \leq g_n(x) + g_m(\sigma^n x) \quad \text{for all } x \in \Sigma \text{ and } n, m \in \mathbb{N}. \tag{2.1}$$

The *topological pressure of* $\mathcal{G}$ is defined by

$$P(\Sigma, \sigma, \mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log\bigg(\sum_{I \in \Sigma_n} \sup_{x\in[I]} \exp(g_n(x))\bigg), \tag{2.2}$$

where $[I] := \{x \in \Sigma : x|n = I\}$ for $I \in \Sigma_n$. The limit can be seen to exist by using a standard sub-additivity argument.

If the potential $\mathcal{G}$ is additive, that is, $g_n = \sum_{k=0}^{n-1} g \circ \sigma^k$ for some $g \in C(\Sigma)$, then $P(\Sigma, \sigma, \mathcal{G})$ recovers the classical topological pressure $P(\Sigma, \sigma, g)$ of g (see e.g. [49]).

Let $\mathcal{M}(\Sigma, \sigma)$ denote the set of σ-invariant Borel probability measures on Σ. For $\mu \in \mathcal{M}(\Sigma, \sigma)$, let $h_\mu(\sigma)$ denote the measure-theoretic entropy of μ (cf. [49]). Moreover, for $\mu \in \mathcal{M}(\Sigma, \sigma)$, by sub-additivity,

$$\mathcal{G}_*(\mu) := \lim_{n\to\infty} \frac{1}{n} \int g_n \, d\mu = \inf_n \frac{1}{n} \int g_n \, d\mu \in [-\infty, \infty). \tag{2.3}$$

See e.g. [49, Theorem 10.1]. We call $\mathcal{G}_*(\mu)$ the *Lyapunov exponent* of $\mathcal{G}$ with respect to μ.

The following variational principle for the topological pressure of sub-additive potentials generalizes the classical variational principle for additive potentials [43, 48].

THEOREM 2.1. [8] *Let $\mathcal{G} = \{g_n\}_{n=1}^{\infty}$ be a sub-additive potential on (Σ, σ). Then*

$$P(\Sigma, \sigma, \mathcal{G}) = \sup\{h_\mu(\sigma) + \mathcal{G}_*(\mu) : \mu \in \mathcal{M}(\Sigma, \sigma)\}. \tag{2.4}$$

Although in [8] this is proved for sub-additive potentials on an arbitrary continuous dynamical system on a compact space, we state it only for shift spaces. Particular cases of the above result, under stronger assumptions on the potentials, were previously obtained by many authors, see for example [4, 14, 20, 22, 34, 38] and references therein.

Measures that achieve the supremum in (2.4) are called *equilibrium measures* for the potential $\mathcal{G}$. There exists at least one ergodic equilibrium measure; see e.g. [21, Proposition 3.5] and the remark there.

2.2. *Singularity dimension and Lyapunov dimension with respect to C^1 IFSs.* In this subsection, we define the singularity and Lyapunov dimensions with respect to C^1 IFSs.

Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on a compact set $Z \subset \mathbb{R}^d$ and let K denote the attractor of $\mathcal{F}$ (cf. §1). Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$ and let $\Pi : \Sigma \to K$ be the coding map defined as in (1.2).

Let $\mathbb{R}^{d\times d}$ denote the collection of $d \times d$ real matrices. For $T \in \mathbb{R}^{d\times d}$, let

$$\alpha_1(T) \geq \cdots \geq \alpha_d(T)$$

denote the singular values of T. Following [13], for $s \geq 0$, we define the *singular value function* $\phi^s : \mathbb{R}^{d\times d} \to [0, \infty)$ as

$$\phi^s(T) = \begin{cases} \alpha_1(T)\cdots\alpha_k(T)\alpha_{k+1}^{s-k}(T) & \text{if } 0 \leq s < d, \\ \det(T)^{s/d} & \text{if } s \geq d, \end{cases} \tag{2.5}$$

where $k = [s]$ is the integral part of s. Here we make the convention that $0^0 = 1$. The following result on ϕ^s is well known; see e.g. [13].

LEMMA 2.2.
(i) $\phi^s(ST) \leq \phi^s(S)\phi^s(T)$ *for all* $S, T \in \mathbb{R}^{d\times d}$ *and* $s \geq 0$.
(ii) $\phi^{s+t}(T) \leq \phi^s(T)\|T\|^t$ *for all* $T \in \mathbb{R}^{d\times d}$, $s, t \geq 0$.

For a differentiable mapping $f : U \subset \mathbb{R}^d \to \mathbb{R}^d$, let $D_z f$ denote the differential of f at $z \in U$. Sometimes we also write $f'(z)$ for $D_z f$, and also call $D_z f$ the Jacobian matrix of f at z. Below we introduce the concepts of singularity and Lyapunov dimensions.

Definition 2.3. The *singularity dimension of* $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$, written as $\dim_S \mathcal{F}$, is the unique non-negative value s for which

$$P(\Sigma, \sigma, \mathcal{G}^s) = 0,$$

where $\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$ is the sub-additive potential on Σ defined by

$$g_n^s(x) = \log \phi^s(D_{\Pi\sigma^n x} f_{x|n}), \quad x \in \Sigma, \tag{2.6}$$

with $f_{x|n} := f_{x_1} \circ \cdots \circ f_{x_n}$ for $x = (x_n)_{n=1}^{\infty}$.

Definition 2.4. Let μ be a σ-invariant Borel probability measure on Σ. The *Lyapunov dimension of* μ *with respect to* $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$, written as $\dim_{\mathrm{L},\mathcal{F}} \mu$, is the unique non-negative value s for which

$$h_\mu(\sigma) + \mathcal{G}_*^s(\mu) = 0,$$

where $\mathcal{G}^s = \{g_n^s\}_{n=1}^{\infty}$ is defined as in (2.6) and $\mathcal{G}_*^s(\mu) := \lim_{n\to\infty}(1/n)\int g_n^s \, d\mu$.

Remark 2.5.
(i) It is not hard to show that there exist $a < b < 0$ such that

$$nsa \leq g_n^s(x) \leq nsb, \quad g_n^{s+t}(x) \leq g_n^s(x) + ntb$$

for all $x \in \Sigma$, $n \in \mathbb{N}$ and $s, t \geq 0$, where $g_n^s(x)$ is defined as in (2.6). The existence and uniqueness of s in Definitions 2.3–2.4 just follow from this fact.

(ii) The concept of singularity dimension was first introduced by Falconer [13, 15]; see also [35]. It is also called *affinity dimension* in the case when the IFS $\{f_i\}_{i=1}^{\ell}$ is affine, that is, each map f_i is affine.

(iii) The definition of Lyapunov dimension of invariant measures with respect to an IFS presented above was adopted from [31]. It is a generalization of that given in [32] for affine IFSs.

The following result describes the relation between the singularity dimension and the Lyapunov dimension.

LEMMA 2.6. *Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be a C^1 IFS on a compact subset Z of $\mathbb{R}^d$. Suppose $\theta \in (0, 1)$ is a common Lipschitz constant for $f_1, \ldots, f_\ell$. That is,*

$$|f_i(x) - f_i(y)| \le \theta |x - y| \quad \textit{for all } 1 \le i \le \ell,\ x, y \in Z.$$

Then the following properties hold.

(i) $\dim_S \mathcal{F} = \sup\{\dim_{L,\mathcal{F}} \mu : \mu \in \mathcal{M}(\Sigma, \sigma)\}$. *The supremum is attained by at least one ergodic measure.*

(ii) $\dim_S \mathcal{F} \le (\log \ell / \log(1/\theta))$.

Proof. Since θ is a common Lipschitz constant for $f_1, \ldots, f_\ell$, $\|D_z f_i\| \le \theta$ for each $1 \le i \le \ell$ and $z \in Z$. It follows from Lemma 2.2(ii) that for $s_2 > s_1 \ge 0$,

$$\phi^{s_2}(D_{\Pi\sigma^n x} f_{x|n}) \le \phi^{s_1}(D_{\Pi\sigma^n x} f_{x|n}) \theta^{n(s_2 - s_1)} \quad \text{for all } x \in \Sigma,\ n \in \mathbb{N},$$

from which we see that

$$P(\Sigma, \sigma, \mathcal{G}^{s_2}) \le P(\Sigma, \sigma, \mathcal{G}^{s_1}) - (s_2 - s_1) \log(1/\theta).$$

Hence $P(\Sigma, \sigma, \mathcal{G}^s)$ is strictly decreasing in s.

Now let $\mu \in \mathcal{M}(\Sigma, \sigma)$. Write $s = \dim_{L,\mathcal{F}} \mu$. Then $h_\mu(\sigma) + \mathcal{G}_*^s(\mu) = 0$. Applying Theorem 2.1 to the sub-additive potential $\mathcal{G}^s$ yields that $P(\Sigma, \sigma, \mathcal{G}^s) \ge 0$. Hence $\dim_S \mathcal{F} \ge s = \dim_{L,\mathcal{F}} \mu$. It follows that

$$\dim_S \mathcal{F} \ge \sup\{\dim_{L,\mathcal{F}} \mu : \mu \in \mathcal{M}(\Sigma, \sigma)\}.$$

To show that equality holds, write $s' = \dim_S \mathcal{F}$. Let ν be an ergodic equilibrium measure for the potential $\mathcal{G}^{s'}$. Then

$$0 = P(\Sigma, \sigma, \mathcal{G}^{s'}) = h_\nu(\sigma) + \mathcal{G}_*^{s'}(\nu),$$

which implies that $\dim_{L,\mathcal{F}} \nu = s'$. That is, $\dim_{L,\mathcal{F}} \nu = \dim_S \mathcal{F}$. This completes the proof of (i).

To see (ii), notice that $\phi^{s'}(D_{\Pi\sigma^n x} f_{x|n}) \le \theta^{ns'}$ for all $x \in \Sigma$ and $n \in \mathbb{N}$. It follows from the definition of $P(\Sigma, \sigma, \mathcal{G}^{s'})$ that

$$0 = P(\Sigma, \sigma, \mathcal{G}^{s'}) \le \lim_{n\to\infty} \frac{1}{n} \log(\ell^n \theta^{ns'}) = \log \ell + s' \log \theta,$$

from which we obtain $s' \le (\log \ell / \log(1/\theta))$. This completes the proof of (ii). □

For a σ-invariant ergodic measure μ on Σ, let $\Pi_*\mu$ denote the push-forward of μ by Π. In the following, we present the main result obtained in the first part [24] of our study on the dimension of C^1 IFSs: the upper box-counting dimension of the attractor of $\mathcal{F}$ is bounded above by the singularity dimension of $\mathcal{F}$, whilst the upper packing dimension of $\Pi_*\mu$ is bounded above by the Lyapunov dimension of μ.

THEOREM 2.7. [24] *Let $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ be a C^1 IFS with attractor K, and let μ be a σ-invariant ergodic measure on Σ. Then the following properties hold.*

(i) $\overline{\dim}_B K \le \dim_S \mathcal{F}$.

(ii) $\overline{\dim}_P \Pi_*\mu \le \dim_{L,\mathcal{F}}(\mu)$.

3. *The proof of Theorem 1.2*

In this section, we prove Theorem 1.2. A key part of the proof is the following proposition.

PROPOSITION 3.1. *Assume that $\mathcal{F}^t$, $t \in \Omega$, satisfies the GTC with respect to a locally finite Borel measure η on Ω. Let μ be a σ-invariant ergodic measure on Σ. Let $t_0 \in \Omega$ and $0 < \delta < \delta_0$, where δ_0 is given as in Definition 1.1, Then the following properties hold.*

(i) *For η-a.e. $t \in B(t_0, \delta)$,*

$$\underline{\dim}_H \Pi^t_*\mu \ge \min\{d, d_\mu(t_0)\} - \frac{\psi(\delta)}{\log(1/\theta)}, \tag{3.1}$$

where $\psi(\cdot)$ is given as in Definition 1.1, and θ is given as in (1.3).

(ii) *If $d_\mu(t_0) > d + (\psi(\delta)/\log(1/\theta))$, then $\Pi^t_*\mu \ll \mathcal{L}_d$ for η-a.e. $t \in B(t_0, \delta)$.*

The proof of the above proposition is adapted from an argument used in [32, Propositions 4.3 and 4.4]. For the reader's convenience, we include a full proof. We begin with the following.

LEMMA 3.2. *Assume that $(\mathcal{F}^t)_{t\in\Omega}$ satisfies the GTC with respect to a locally finite Borel measure η on Ω. Let s be non-integral with $0 < s < d$. Let $t_0 \in \Omega$ and $0 < \delta < \delta_0$, where δ_0 is given as in Definition 1.1. Then there exists a number $c > 0$, dependent on s and δ, such that for all distinct $\mathbf{i}, \mathbf{j} \in \Sigma$,*

$$\int_{B(t_0,\delta)} |\Pi^t(\boldsymbol{i}) - \Pi^t(\mathbf{j})|^{-s}\, d\eta(t) \le c e^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \Big(\max_{x\in\Sigma} \phi^s(D_{\Pi^{t_0}x} f^{t_0}_{\mathbf{i}\wedge\mathbf{j}}) \Big)^{-1}, \tag{3.2}$$

where $\psi(\cdot)$ is given as in Definition 1.1.

Proof. Take $y \in \Sigma$ so that $\phi^s(D_{\Pi^{t_0}y} f^{t_0}_{\mathbf{i}\wedge\mathbf{j}}) = \max_{x\in\Sigma} \phi^s(D_{\Pi^{t_0}x} f^{t}_{\mathbf{i}\wedge\mathbf{j}})$. Let k be the unique integer such that $s \in (k, k+1)$. Clearly $k \in \{0, 1, \ldots, d-1\}$. For convenience, write

$$a := \phi^k(D_{\Pi^{t_0}y} f^{t_0}_{\mathbf{i}\wedge\mathbf{j}}), \quad b := \phi^{k+1}(D_{\Pi^{t_0}y} f^{t_0}_{\mathbf{i}\wedge\mathbf{j}}),$$

where $\phi^s(\cdot)$ stands for the singular value function (see (2.5) for the definition). A direct check shows that

$$\phi^s(D_{\Pi^{t_0}y} f^{t}_{\mathbf{i}\wedge\mathbf{j}}) = a^{k+1-s} b^{s-k}. \tag{3.3}$$

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Observe that

$$\begin{aligned}
&\int_{B(t_0,\delta)} |\Pi^t(\mathbf{i}) - \Pi^t(\mathbf{j})|^{-s}\, d\eta(t) \\
&\quad = s\int_0^\infty r^{-s-1}\eta\{t \in B(t_0,\delta) : |\Pi^t(\mathbf{i}) - \Pi^t(\mathbf{j})| < r\}\, dr \\
&\quad \le sCe^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \int_0^\infty r^{-s-1} Z^{t_0}_{\mathbf{i}\wedge\mathbf{j}}(r)\, dr \quad \text{(by (1.5))} \\
&\quad \le sCe^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \int_0^\infty r^{-s-1} \min\left\{\frac{r^k}{a}, \frac{r^{k+1}}{b}\right\} dr \quad \text{(by (1.4))} \\
&\quad \le sCe^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \int_0^{b/a} \frac{r^{k-s}}{b}\, dr + \int_{b/a}^\infty \frac{r^{k-s-1}}{a}\, dr \\
&\quad = sCe^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \left(\frac{1}{k+1-s} + \frac{1}{s-k}\right) a^{s-k-1} b^{k-s} \\
&\quad = sC\left(\frac{1}{k+1-s} + \frac{1}{s-k}\right) e^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} (\phi^s(D_{\Pi^{t_0}y} f^{t_0}_{\mathbf{i}\wedge\mathbf{j}}))^{-1} \quad \text{(by (3.3))}.
\end{aligned}$$

This proves (3.2) by setting $c = sC((1/(k+1-s)) + (1/(s-k)))$. □

Proof of Proposition 3.1. Fix $t_0 \in \Omega$ and $\delta \in (0, \delta_0)$. We first prove part (i). Let $\epsilon > 0$ and let s be non-integral so that

$$0 < s < \min\{d, d_\mu(t)\} - \frac{\psi(\delta)}{\log(1/\theta)} - 2\epsilon. \tag{3.4}$$

To show that (3.1) holds for η-a.e. $t \in B(t_0, \delta)$, it suffices to show that

$$\underline{\dim}_H \Pi^t_*\mu \ge s \quad \text{for } \eta\text{-a.e. } t \in B(t_0,\delta). \tag{3.5}$$

For this purpose, we write

$$\varphi^s(I) = \max_{x\in\Sigma} \phi^s(D_{\Pi^{t_0}_x} f^{t_0}_I), \qquad I \in \Sigma_*. \tag{3.6}$$

We first prove that

$$\lim_{n\to\infty} \frac{\mu([\mathbf{i}|n])}{\varphi^s(\mathbf{i}|n)\exp(-n\psi(\delta))\theta^{n\epsilon}} = 0 \quad \text{for } \mu\text{-a.e. } \mathbf{i} \in \Sigma. \tag{3.7}$$

To see this, according to the definition of $d_\mu(t_0)$ (cf. (1.6) and Definition 2.4),

$$h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log \phi^{d_\mu(t_0)}(D_{\Pi^{t_0}\sigma^n\mathbf{i}} f^{t_0}_{\mathbf{i}|n})\, d\mu(\mathbf{i}) = 0. \tag{3.8}$$

It follows from (3.8), the Shannon–McMillan–Breiman theorem and Kingman's sub-additive ergodic theorem (see [49, pp. 93 and 231]) that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\mu([\mathbf{i}|n])}{\phi^{d_\mu(t_0)}(D_{\Pi^{t_0}\sigma^n\mathbf{i}} f^{t_0}_{\mathbf{i}|n})} = 0 \quad \text{for } \mu\text{-a.e. } \mathbf{i} \in \Sigma. \tag{3.9}$$

Observe that for each $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$,

$$\begin{aligned}\phi^{d_\mu(t_0)}(D_{\Pi^{t_0}\sigma^n\mathbf{i}} f_{\mathbf{i}|n}^{t_0}) &\le \phi^s(D_{\Pi^{t_0}\sigma^n\mathbf{i}} f_{\mathbf{i}|n}^{t_0}) \| D_{\Pi^{t_0}\sigma^n\mathbf{i}} f_{\mathbf{i}|n}^{t_0}\|^{d_\mu(t_0)-s} \quad \text{(by Lemma 2.2(ii))} \\ &\le \varphi^s(\mathbf{i}|n)\theta^{n(d_\mu(t_0)-s)} \qquad \text{(by (3.6) and (1.3))} \\ &\le \varphi^s(\mathbf{i}|n)\exp(-n\psi(\delta))\theta^{2n\epsilon} \quad \text{(by (3.4))}.\end{aligned}$$

Combining the above inequality with (3.9) yields (3.7).

By (3.7), we may find a countable disjoint collection of Borel subsets E_j of Σ with $\mu(\Sigma \setminus \bigcup_{j=1}^\infty E_j) = 0$ and numbers $c_j > 0$ such that

$$\mu_j([\mathbf{i}|n]) \le c_j \varphi^s(\mathbf{i}|n)\exp(-n\psi(\delta))\theta^{n\epsilon} \quad \text{for all } \mathbf{i} \in \Sigma,\ n \in \mathbb{N}, \tag{3.10}$$

where μ_j stands for the restriction of μ to E_j defined by $\mu_j(A) = \mu(E_j \cap A)$. Clearly,

$$\mu = \sum_{j=1}^\infty \mu_j. \tag{3.11}$$

Hence, to prove (3.5), it suffices to show that for each j,

$$\underline{\dim}_H \Pi_*^t \mu_j \ge s \quad \text{for } \eta\text{-a.e. } t \in B(t_0, \delta). \tag{3.12}$$

By the potential theoretic characterization of the Hausdorff dimension (see e.g. [16, Theorem 4.13]), it is enough to show that for each j and η-a.e. $t \in B(t_0, \delta)$, $\Pi_*^t \mu_j$ has finite s-energy:

$$I_s(\Pi_*^t\mu_j) := \iint \frac{d\Pi_*^t\mu_j(x) d\Pi_*^t\mu_j(y)}{|x-y|^s} < \infty.$$

Integrating over $B(t_0, \delta)$ with respect to η and using Fubini's theorem,

$$\begin{aligned}\int_{B(t_0,\delta)} I_s(\Pi_*^t\mu_j)\, d\eta(t) &= \int_{B(t_0,\delta)} \iint \frac{d\Pi_*^t\mu_j(x) d\Pi_*^t\mu_j(y)}{|x-y|^s} d\eta(t) \\ &= \int_{B(t_0,\delta)} \iint \frac{d\mu_j(\mathbf{i})\, d\mu_j(\mathbf{j})}{|\Pi^t(\mathbf{i}) - \Pi^t(\mathbf{j})|^s} d\eta(t) \\ &= \iint \int_{B(t_0,\delta)} \frac{d\eta(t)}{|\Pi^t(\mathbf{i}) - \Pi^t(\mathbf{j})|^s}\, d\mu_j(\mathbf{i})\, d\mu_j(\mathbf{j}) \\ &\le \iint c e^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} (\varphi^s(\mathbf{i}\wedge\mathbf{j}))^{-1} d\mu_j(\mathbf{i})\, d\mu_j(\mathbf{j}) \quad \text{(by (3.2), (3.6))} \\ &\le c \int \sum_{n=0}^\infty e^{n\psi(\delta)} (\varphi^s(\mathbf{j}|n))^{-1} \mu_j([\mathbf{j}|n]) d\mu_j(\mathbf{j}) \\ &\le c c_j \int \sum_{n=0}^\infty \theta^{n\epsilon} d\mu_j(\mathbf{j}) \quad \text{(by (3.10))} \\ &< \infty.\end{aligned}$$

It follows that $I_s(\Pi_*^t\mu_j) < \infty$ for η-a.e. $t \in B(t_0, \delta)$. This completes the proof of part (i).

Next we prove part (ii). Take a small $\epsilon > 0$ so that

$$d_\mu(t_0) > d + \frac{\psi(\delta)}{\log(1/\theta)} + 2\epsilon. \tag{3.13}$$

Then for every $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$,

$$\begin{aligned}\phi^{d_\mu(t_0)}(D_{\Pi^{t_0}\sigma^n\mathbf{i}} f^{t_0}_{\mathbf{i}|n}) &\le \phi^d(D_{\Pi^{t_0}\sigma^n\mathbf{i}} f^{t_0}_{\mathbf{i}|n}) \|D_{\Pi^{t_0}\sigma^n\mathbf{i}} f^{t_0}_{\mathbf{i}|n}\|^{d_\mu(t_0)-d} \quad \text{(by Lemma 2.2(ii))}\\ &\le \varphi^d(\mathbf{i}|n)\theta^{n(d_\mu(t_0)-d)} \quad \text{(by (3.6) and (1.3))}\\ &\le \varphi^d(\mathbf{i}|n)\exp(-n\psi(\delta))\theta^{2n\epsilon} \quad \text{(by (3.13))}.\end{aligned}$$

Combining the above inequality with (3.9) yields that

$$\lim_{n\to\infty} \frac{\mu([\mathbf{i}|n])}{\varphi^d(\mathbf{i}|n)\exp(-n\psi(\delta))\theta^{n\epsilon}} = 0 \quad \text{for } \mu\text{-a.e. } \mathbf{i} \in \Sigma.$$

Hence, there exist finite positive measures ν_j and numbers a_j $(j \ge 1)$ such that $\mu = \sum_{j=1}^\infty \nu_j$ and

$$\nu_j([\mathbf{i}|n]) \le a_j \varphi^d(\mathbf{i}|n)\exp(-n\psi(\delta))\theta^{n\epsilon} \quad \text{for all } \mathbf{i} \in \Sigma,\ n \in \mathbb{N}. \tag{3.14}$$

Since $\mu = \sum_{j=1}^\infty \nu_j$, to show that $\Pi^t_*\mu \ll \mathcal{L}_d$ for η-a.e. $t \in B(t_0, \delta)$, it suffices to show that for each j,

$$\Pi^t_*\nu_j \ll \mathcal{L}_d \quad \text{for } \eta\text{-a.e. } t \in B(t_0, \delta). \tag{3.15}$$

To this end, fix j. We will follow a standard approach (introduced by Peres and Solomyak in [40]). In particular, it suffices to show that

$$I := \int_{B(t_0,\delta)} \int \liminf_{r\to 0} \frac{\Pi^t_*\nu_j(B_{\mathbb{R}^d}(x,r))}{r^d}\, d\Pi^t_*\nu_j(x)\, d\eta(t) < \infty,$$

where $B_{\mathbb{R}^d}(x, r)$ stands for the closed ball in $\mathbb{R}^d$ centered at x of radius r. Observe that by (1.4) and (3.6),

$$Z^{t_0}_{\boldsymbol{\omega}}(r) \le \inf_{x\in\Sigma} \frac{r^d}{\phi^d(D_{\Pi^{t_0}x} f^t_{\boldsymbol{\omega}})} = \frac{r^d}{\varphi^d(\boldsymbol{\omega})} \quad \text{for } \boldsymbol{\omega} \in \Sigma_*,\ r > 0. \tag{3.16}$$

Applying Fatou's Lemma and Fubini's Theorem,

$$\begin{aligned} I &\le \liminf_{r\to 0} \frac{1}{r^d} \int_{B(t_0,\delta)} \int \Pi^t_*\nu_j(B_{\mathbb{R}^d}(x,r))\, d\Pi^t_*\nu_j(x)\, d\eta(t)\\ &= \liminf_{r\to 0} \frac{1}{r^d} \int_{B(t_0,\delta)} \iint \mathbf{1}_{\{(x,y):\ |x-y|\le r\}}\, d\Pi^t_*\nu_j(x)\, d\Pi^t_*\nu_j(y)\, d\eta(t)\\ &= \liminf_{r\to 0} \frac{1}{r^d} \int_{B(t_0,\delta)} \iint \mathbf{1}_{\{(\mathbf{i},\mathbf{j}):\ |\Pi^t(\mathbf{i})-\Pi^t(\mathbf{j})|\le r\}}\, d\nu_j(\mathbf{i})\, d\nu_j(\mathbf{j})\, d\eta(t)\\ &= \liminf_{r\to 0} \frac{1}{r^d} \iint \int \mathbf{1}_{\{t\in B(t_0,\delta):\ |\Pi^t(\mathbf{i})-\Pi^t(\mathbf{j})|\le r\}}\, d\eta(t)\, d\nu_j(\mathbf{i})\, d\nu_j(\mathbf{j})\\ &= \liminf_{r\to 0} \frac{1}{r^d} \iint \eta\{t \in B(t_0,\delta) :\ |\Pi^t(\mathbf{i}) - \Pi^t(\mathbf{j})| \le r\}\, d\nu_j(\mathbf{i})\, d\nu_j(\mathbf{j}). \end{aligned}$$

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By (1.5) and (3.16), we obtain that

$$
\begin{aligned}
I &\le \liminf_{r\to 0} \frac{1}{r^d} \iint C e^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} Z^{t_0}_{\mathbf{i}\wedge\mathbf{j}}(r)\, d\nu_j(\mathbf{i})\, d\nu_j(\mathbf{j}) \\
&\le \iint \frac{C e^{|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)}}{\varphi^d(\mathbf{i}\wedge\mathbf{j})}\, d\nu_j(\mathbf{i})\, d\nu_j(\mathbf{j}) \\
&\le \int \sum_{n=0}^{\infty} \frac{C e^{n\psi(\delta)}}{\varphi^d(\mathbf{j}|n)} \nu_j([\mathbf{j}|n])\, d\nu_j(\mathbf{j}) \\
&\le a_j C \int \sum_{n=0}^{\infty} \theta^{n\epsilon}\, d\nu_j(\mathbf{j}) \quad \text{(by (3.14))} \\
&< \infty,
\end{aligned}
$$

which completes the proof of part (ii). □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2(i). Let μ be a σ-invariant ergodic measure on Σ. We first show that for η-a.e. $t \in \Omega$, $\Pi^t_*\mu$ is exact dimensional with dimension equal to $\min\{d, d_\mu(t)\}$. Recall that $\overline{\dim}_P \Pi^t_*\mu \le \min\{d, d_\mu(t)\}$ for each $t \in \Omega$ (see Theorem 2.7(ii)). Hence, it is sufficient to show that for η-a.e. $t \in \Omega$, $\underline{\dim}_H \Pi^t_*\mu \ge \min\{d, d_\mu(t)\}$. Suppose on the contrary that this is false. Then there exist $k \in \mathbb{N}$ and $A \subset \Omega$ with $\eta(A) > 0$ such that

$$\underline{\dim}_H \Pi^t_*\mu < \min\{d, d_\mu(t)\} - \frac{2}{k} \quad \text{for all } t \in A. \tag{3.17}$$

Take a number $\delta \in (0, \delta_0)$ small enough such that

$$\frac{\psi(\delta)}{\log(1/\theta)} < \frac{1}{k}. \tag{3.18}$$

Since Ω is a separable metric space, it has a countable dense subset

$$Y = \{y_n : n \in \mathbb{N}\}.$$

Notice that by (1.3) and Lemma 2.6,

$$0 \le d_\mu(t) \le d(t) \le \frac{\log \ell}{\log(1/\theta)} \quad \text{for each } t \in \Omega.$$

Due to this fact, for each $n \in \mathbb{N}$, we may pick $y^*_n \in B(y_n, \delta/2)$ so that

$$d_\mu(y^*_n) \ge \sup_{t\in B(y_n,\delta/2)} d_\mu(t) - \frac{1}{k}. \tag{3.19}$$

Let $Y^* = \{y^*_n : n \in \mathbb{N}\}$. Clearly, Y^* is countable. We claim that

$$\sup_{y^*\in B(t,\delta)\cap Y^*} d_\mu(y^*) \ge d_\mu(t) - \frac{1}{k} \quad \text{for all } t \in \Omega. \tag{3.20}$$

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To see this, let $t \in \Omega$. Since Y is dense in Ω, there exists an integer m such that $\rho(y_m, t) \le \delta/2$. Then

$$y_m^* \in B(y_m, \delta/2) \subset B(t, \delta). \tag{3.21}$$

Meanwhile, by (3.19),

$$d_\mu(y_m^*) \ge \sup_{t' \in B(y_m, \delta/2)} d_\mu(t') - \frac{1}{k} \ge d_\mu(t) - \frac{1}{k},$$

where in the last inequality, we use the fact that $t \in B(y_m, \delta/2)$ (since $\rho(y_m, t) \le \delta/2$). This proves (3.20), since $y_m^* \in B(t, \delta) \cap Y^*$ by (3.21).

Set for $n \in \mathbb{N}$,

$$\Omega_n = \{t \in B(y_n^*, \delta) : \ d_\mu(y_n^*) \ge d_\mu(t) - 1/k\}. \tag{3.22}$$

By (3.20), $\Omega = \bigcup_{n=1}^\infty \Omega_n$. Define

$$E_n = \{t \in B(y_n^*, \delta) : \ \underline{\dim}_H \Pi_*^t \mu < \min\{d, d_\mu(y_n^*)\} - 1/k\}, \quad n \in \mathbb{N}.$$

By (3.17), we see that $A \cap \Omega_n \subset E_n$ for each $n \in \mathbb{N}$. However, by Proposition 3.1(i) and (3.18), for each $n \in \mathbb{N}$ and η-a.e. $t \in B(y_n^*, \delta)$,

$$\underline{\dim}_H \Pi_*^t \mu \ge \min\{d, d_\mu(y_n^*)\} - \frac{\psi(\delta)}{\log(1/\theta)} > \min\{d, d_\mu(y_n^*)\} - \frac{1}{k}.$$

It follows that $\eta(E_n) = 0$ for each $n \in \mathbb{N}$. Hence

$$\eta(A) = \eta\bigg(A \cap \bigg(\bigcup_{n=1}^\infty \Omega_n\bigg)\bigg) \le \sum_{n=1}^\infty \eta(A \cap \Omega_n) \le \sum_{n=1}^\infty \eta(E_n) = 0,$$

leading to a contradiction. This proves the statement that for η-a.e. $t \in \Omega$, $\Pi_*^t \mu$ is exact dimensional with dimension $\min\{d, d_\mu(t)\}$.

Next we prove that $\Pi_*^t \mu \ll \mathcal{L}_d$ for η-a.e. $t \in \{t' \in \Omega : \ d_\mu(t') > d\}$. Again we use a contradiction. Suppose on the contrary that this result is false. Then there exist $k \in \mathbb{N}$ and $A' \subset \Omega$ with $\eta(A') > 0$ such that

$$d_\mu(t) > d + \frac{2}{k} \quad \text{and} \quad \Pi_*^t \mu \not\ll \mathcal{L}_d \quad \text{for all } t \in A'. \tag{3.23}$$

Set

$$F_n = \{t \in B(y_n^*, \delta) : \ \Pi_*^t \mu \not\ll \mathcal{L}_d\}, \quad n \in \mathbb{N}.$$

Clearly, $A' \cap \Omega_n \subset F_n$ for each $n \in \mathbb{N}$. Since $\Omega = \bigcup_{n=1}^\infty \Omega_n$ and $\eta(A') > 0$, there exists $m \in \mathbb{N}$ so that $\eta(A' \cap \Omega_m) > 0$. Hence $A' \cap \Omega_m \neq \emptyset$. Pick $t \in A' \cap \Omega_m$. By (3.22), (3.23) and (3.18),

$$d_\mu(y_m^*) \ge d_\mu(t) - \frac{1}{k} > d + \frac{1}{k} > d + \frac{\psi(\delta)}{\log(1/\theta)}.$$

Hence $\eta(F_m) = 0$ by Proposition 3.1(ii). Since $A' \cap \Omega_m \subset F_m$, it follows that $\eta(A' \cap \Omega_m) = 0$, leading to a contradiction. $\square$

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Proof of Theorem 1.2(ii). By Lemma 2.6, for each $t \in \Omega$, we can find a σ-invariant ergodic measure μ_t on Σ such that

$$d(t) = d_{\mu_t}(t).$$

Moreover, $0 \le d(t) \le (\log \ell / \log(1/\theta))$.

Next we prove that $\dim_H K^t = \dim_B K^t = \min\{d, d(t)\}$ for η-a.e. $t \in \Omega$. By Theorem 2.7, $\overline{\dim}_B K^t \le \min\{d, d(t)\}$ for every $t \in \Omega$. Hence it is sufficient to show that

$$\dim_H K^t \ge \min\{d, d(t)\} \quad \text{for } \eta\text{-a.e. } t \in \Omega.$$

Suppose on the contrary that this statement is false. Then there exist $k \in \mathbb{N}$ and $H \subset \Omega$ with $\eta(H) > 0$ such that

$$\dim_H K^t < \min\{d, d(t)\} - 2/k \quad \text{for all } t \in H. \tag{3.24}$$

Take a number $\delta \in (0, \delta_0)$ such that (3.18) holds. Since $d(t)$ is uniformly bounded from above, similar to the construction of Y^* in the proof of part (i), we can construct a countable dense subset $Y' = \{y'_n\}_{n=1}^\infty$ of Ω such that

$$\sup_{y' \in B(t,\delta) \cap Y'} d(y') \ge d(t) - \frac{1}{k} \quad \text{for all } t \in \Omega. \tag{3.25}$$

Write

$$\Omega'_n = \{t \in B(y'_n, \delta) : \ d(y'_n) \ge d(t) - 1/k\} \quad \text{for } n \in \mathbb{N}. \tag{3.26}$$

By (3.25), $\Omega = \bigcup_{n=1}^\infty \Omega'_n$. Notice that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\Omega'_n \cap H &\subset \{t \in B(y'_n, \delta) : \ \dim_H K^t < \min\{d, d(y'_n)\} - 1/k\} \\
&\subset \{t \in B(y'_n, \delta) : \ \underline{\dim}_H \Pi^t_*(\mu_{y'_n}) < \min\{d, d_{\mu_{y'_n}}(y'_n)\} - 1/k\} \\
&\subset \left\{t \in B(y'_n, \delta) : \ \underline{\dim}_H \Pi^t_*(\mu_{y'_n}) < \min\{d, d_{\mu_{y'_n}}(y'_n)\} - \frac{\psi(\delta)}{\log(1/\theta)}\right\},
\end{aligned}$$

where we have used the facts that $\dim_H K^t \ge \underline{\dim}_H \Pi^t_*(\mu_{y'_n})$ and $d(y'_n) = d_{\mu_{y'_n}}(y'_n)$ in the second inclusion, and (3.18) in the last inclusion. Hence $\eta(\Omega'_n \cap H) = 0$ for each n by applying Proposition 3.1(i). It follows that $\eta(H) \le \sum_{n=1}^\infty \eta(\Omega'_n \cap H) = 0$, leading to a contradiction. This completes the proof of the statement that $\dim_H K^t = \min\{d, d(t)\}$ for η-a.e. $t \in \Omega$.

Finally, we prove that $\mathcal{L}_d(K^t) > 0$ for η-a.e. $t \in \{t' \in \Omega : \ d(t') > d\}$. Suppose on the contrary that this result is false. Then there exist $k \in \mathbb{N}$ and $H' \subset \Omega$ with $\eta(H') > 0$ such that

$$d(t) > d + \frac{2}{k} \quad \text{and} \quad \mathcal{L}_d(K^t) = 0 \quad \text{for all } t \in H'. \tag{3.27}$$

Set

$$F'_n = \{t \in B(y'_n, \delta) : \ \mathcal{L}_d(K^t) = 0\}, \quad n \in \mathbb{N}.$$

Clearly, $H' \cap \Omega'_n \subset F'_n$ for each $n \in \mathbb{N}$. Since $\Omega = \bigcup_{n=1}^\infty \Omega'_n$ and $\eta(H') > 0$, there exists $m \in \mathbb{N}$ so that $\eta(H' \cap \Omega'_m) > 0$. Hence $H' \cap \Omega'_m \ne \emptyset$. Taking $t \in H' \cap \Omega'_m$ and applying

(3.26), (3.27) and (3.18) gives

$$d_{\mu_{y'_m}}(y'_m) = d(y'_m) \geq d(t) - \frac{1}{k} > d + \frac{1}{k} > d + \frac{\psi(\delta)}{\log(1/\theta)}.$$

Now by Proposition 3.1(ii), $\Pi^t_*(\mu_{y'_m}) \ll \mathcal{L}_d$ for η-a.e. $t \in B(y'_m, \delta)$. This implies that $\eta(F'_m) = 0$. Since $H' \cap \Omega'_m \subset F'_m$, it follows that $\eta(H' \cap \Omega'_m) = 0$, leading to a contradiction. □

4. *Translational family of IFSs generated by a dominated lower triangular C^1 IFS*

In this section, we show that under mild assumptions, a translational family of C^1 IFSs, generated by a dominated lower triangular C^1 IFS, satisfies the GTC. To begin with, let S be a compact subset of $\mathbb{R}^d$ with non-empty interior.

Definition 4.1. Let $\ell \in \mathbb{N}$ with $\ell \geq 2$. We say that $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ is a *dominated lower triangular C^1 IFS* on S if the following conditions hold.

(i) $f_i(S) \subset \mathrm{int}(S)$, $i = 1, \ldots, \ell$.

(ii) There exists a bounded open connected set $U \supset S$ such that each f_i extends to a contracting C^1 diffeomorphism $f_i : U \to f_i(U)$ with $\overline{f_i(U)} \subset U$.

(iii) For each $z \in S$ and $i \in \{1, \ldots, \ell\}$, the Jacobian matrix $D_z f_i$ of f_i at z is a lower triangular matrix such that

$$|(D_z f_i)_{jj}| \leq |(D_z f_i)_{kk}| \quad \text{for all } 1 \leq k \leq j \leq d.$$

In the remaining part of this section, we fix a dominated lower triangular C^1 IFS $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ on S.

By continuity, there exists a small $r_0 > 0$ such that the following holds. Setting

$$f_i^{\mathbf{t}} := f_i + \mathbf{t}_i$$

for $1 \leq i \leq \ell$ and $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \mathbb{R}^{\ell d}$ with $|\mathbf{t}| < r_0$, we have $f_i^{\mathbf{t}}(S) \subset \mathrm{int}(S)$ for each i.

Write $\Delta := \{\mathbf{t} \in \mathbb{R}^{\ell d} : |\mathbf{t}| < r_0\}$ and set

$$\mathcal{F}^{\mathbf{t}} = \{f_i^{\mathbf{t}}\}_{i=1}^{\ell}, \quad \mathbf{t} \in \Delta.$$

We call $\mathcal{F}^{\mathbf{t}}$, $\mathbf{t} \in \Delta$, a *translational family of IFSs generated by $\mathcal{F}$*. For $\mathbf{i} = i_1 \ldots i_n \in \Sigma_n$, we write $f_{\mathbf{i}}^{\mathbf{t}} = f_{i_1}^{\mathbf{t}} \circ \cdots \circ f_{i_n}^{\mathbf{t}}$.

For a C^1 map $g : S \to \mathbb{R}^d$ and $z_1, \ldots, z_d \in S$, we write

$$D^*_{z_1,\ldots,z_d} g = \begin{bmatrix} \nabla^T g_1(z_1) \\ \vdots \\ \nabla^T g_d(z_d) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(z_1) & \cdots & \frac{\partial g_1}{\partial x_d}(z_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_d}{\partial x_1}(z_d) & \cdots & \frac{\partial g_d}{\partial x_d}(z_d) \end{bmatrix}, \tag{4.1}$$

where g_i is the ith component of the map g, $i = 1, \ldots, d$. Clearly,

$$(D^*_{z_1,\ldots,z_d} g)_{ij} = (D_{z_i} g)_{ij} \quad \text{for all } 1 \leq i, j \leq d. \tag{4.2}$$

The main result in this section is the following.

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THEOREM 4.2. *Let $\mathcal{F}^{\mathbf{t}}$, $\mathbf{t} \in \Delta$, be a translational family of IFSs generated by a dominated lower triangular C^1 IFS $\mathcal{F}$ defined on a compact convex subset S of $\mathbb{R}^d$. Suppose in addition that*

$$\rho := \max_{1 \le i,j \le \ell:\, i \ne j} \Big(\sup_{y \in S} \|D_y f_i\| + \sup_{z \in S} \|D_z f_j\| \Big) < 1. \tag{4.3}$$

Then $\mathcal{F}^{\mathbf{t}}$, $\mathbf{t} \in \Delta$, satisfies the GTC with respect to ℓd-dimensional Lebesgue measure $\mathcal{L}_{\ell d}$ restricted to Δ.

The proof of the above theorem is based on the following.

PROPOSITION 4.3. *Let $\mathcal{F}^{\mathbf{t}}$, $\mathbf{t} \in \Delta$, be a translational family of IFSs generated by a dominated lower triangular C^1 IFS $\mathcal{F}$ on a compact subset S of $\mathbb{R}^d$. Then there exists a function $h : (0, r_0) \to (0, \infty)$ with $\lim_{\delta \to 0} h(\delta) = 0$ such that for each $\delta \in (0, r_0)$, there is $C(\delta) \ge 1$ so that*

$$\|D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}} \cdot (D^*_{z_1,\ldots,z_d} f^{\mathbf{t}}_{\boldsymbol{\omega}})^{-1}\| \le C(\delta) e^{nh(\delta)} \tag{4.4}$$

for every $n \in \mathbb{N}$, $\boldsymbol{\omega} \in \Sigma_n$, $y, z_1, \ldots, z_d \in S$ and $\mathfrak{s}, \mathbf{t} \in \Delta$ with $|\mathfrak{s} - \mathbf{t}| < \delta$.

In the next two subsections, we prove Proposition 4.3 and Theorem 4.2, respectively.

4.1. *Proof of Proposition 4.3.* We first prove several auxiliary lemmas.

LEMMA 4.4. *Let $c \ge 1$ and $d \in \mathbb{N}$. Let A be a real $d \times d$ non-singular lower triangular matrix such that*

$$|A_{ij}| \le c|A_{jj}| \quad \text{for all } 1 \le i, j \le d. \tag{4.5}$$

Then

$$|(A^{-1})_{ij}| \le (c\sqrt{d})^{d-1} |(A^{-1})_{ii}| \quad \text{for all } 1 \le i, j \le d.$$

Proof. It is well known (see e.g. [28]) that $A^{-1} = (1/\det(A))\operatorname{adj}(A)$, where $\operatorname{adj}(A)$ is the adjugate matrix of A defined by

$$(\operatorname{adj}(A))_{ij} = (-1)^{i+j} \det(A(j,i)), \quad 1 \le i, j \le d,$$

where $A(j,i)$ is the $(d-1) \times (d-1)$ matrix that results from A by removing the jth row and ith column. By the Hadamard's inequality (see e.g. [28, Corollary 7.8.2]), $|\det(A(j,i))|$ is bounded above by the product of the Euclidean norms of the columns of $A(j,i)$. In particular, this implies that

$$|\det(A(j,i))| \le \prod_{1 \le k \le d:\, k \ne i} |v_k|,$$

where v_k denotes the kth column vector of A. By (4.5),

$$|v_k| = \sqrt{\sum_{i=1}^{d} (A_{ik})^2} \le c\sqrt{d}|A_{kk}|,$$

so $|\det(A(j,i))| \le (c\sqrt{d})^{d-1}\prod_{1\le k\le d:\, k\ne i}|A_{kk}|$. Hence for given $1\le i, j\le d$,

$$\frac{|(A^{-1})_{ij}|}{|(A^{-1})_{ii}|} = \frac{|\det(A(j,i))|}{\det(A)\cdot|(A^{-1})_{ii}|} \le \frac{(c\sqrt{d})^{d-1}\prod_{1\le k\le d:\, k\ne i}|A_{kk}|}{\det(A)\cdot|(A^{-1})_{ii}|} = (c\sqrt{d})^{d-1}. \qquad \square$$

For $c\ge 1$ and $d\in\mathbb{N}$, let $\mathcal{T}_c(d)$ denote the collection of real $d\times d$ lower triangular matrices $A=(a_{ij})$ satisfying the following two conditions:

(i) $|a_{11}|\ge|a_{22}|\ge\cdots\ge|a_{dd}|>0$;

(ii) $|a_{ij}|\le c|a_{jj}|$ for all $1\le i, j\le d$.

Then we have the following estimates.

LEMMA 4.5. *Let $n\in\mathbb{N}$ and $A_1,\ldots,A_n\in\mathcal{T}_c(d)$. Then for $1\le j\le i\le d$,*

$$|(A_1\cdots A_n)_{ij}|\le(cn)^{i-j}|(A_1\cdots A_n)_{jj}|. \tag{4.6}$$

Proof. We prove by induction on n. Since $A_1\in\mathcal{T}_c(d)$, the inequality (4.6) holds when $n=1$. Now assume that (4.6) holds when $n=k$. Below we show that it also holds when $n=k+1$.

Given $A_1,\ldots,A_{k+1}\in\mathcal{T}_c(d)$, we write $A=A_1$ and $B=A_2\cdots A_{k+1}$. Clearly B is lower triangular. By the induction assumption, $|B_{ij}|\le(ck)^{i-j}|B_{jj}|$ for each pair (i,j) with $1\le j\le i\le d$.

Now fix a pair (i,j) with $1\le j\le i\le d$. Observe that

$$\frac{(AB)_{ij}}{(AB)_{jj}} = \sum_{p=j}^{i}\frac{A_{ip}}{A_{jj}}\cdot\frac{B_{pj}}{B_{jj}} = \frac{A_{ii}}{A_{jj}}\cdot\frac{B_{ij}}{B_{jj}} + \sum_{p=j}^{i-1}\frac{A_{ip}}{A_{jj}}\cdot\frac{B_{pj}}{B_{jj}}. \tag{4.7}$$

Applying the inequalities $|A_{ii}|\le|A_{jj}|$, $|B_{ij}|\le(ck)^{i-j}|B_{jj}|$, $|A_{ip}|\le c|A_{pp}|\le c|A_{jj}|$ and $|B_{pj}|\le(ck)^{p-j}|B_{jj}|$ to (4.7) gives

$$\left|\frac{(AB)_{ij}}{(AB)_{jj}}\right| \le (ck)^{i-j} + c\sum_{p=j}^{i-1}(ck)^{p-j} \le (c(k+1))^{i-j}.$$

Hence (4.6) holds for $n=k+1$. $\square$

LEMMA 4.6. *Let $\mathcal{F}^{\mathbf{t}}=\{f_i^{\mathbf{t}}\}_{i=1}^{\ell}$, $\mathbf{t}\in\Delta$, be a translational family of IFSs on a compact subset S of $\mathbb{R}^d$ generated by a C^1 IFS $\mathcal{F}=\{f_i\}_{i=1}^{\ell}$. Let $\theta\in(0,1)$ be a common Lipschitz constant of $f_1,\ldots,f_\ell$ on S. That is,*

$$|f_i(u)-f_i(v)|\le\theta|u-v| \quad \text{for all } 1\le i\le\ell \text{ and } u,v\in S.$$

Then for $\mathfrak{s},\mathfrak{t}\in\Delta$, $u,v\in S$, $n\in\mathbb{N}$ and $\boldsymbol{\tau}\in\Sigma_n$,

$$|f_{\boldsymbol{\tau}}^{\mathbf{t}}(u)-f_{\boldsymbol{\tau}}^{\mathfrak{s}}(v)| \le \frac{|\mathbf{t}-\mathfrak{s}|}{1-\theta} + \theta^n\left(|u-v|-\frac{|\mathbf{t}-\mathfrak{s}|}{1-\theta}\right). \tag{4.8}$$

In particular,

$$|f_{\boldsymbol{\tau}}^{\mathbf{t}}(u)-f_{\boldsymbol{\tau}}^{\mathfrak{s}}(u)| \le \frac{|\mathbf{t}-\mathfrak{s}|}{1-\theta} \quad \text{and} \quad |f_{\boldsymbol{\tau}}^{\mathbf{t}}(u)-f_{\boldsymbol{\tau}}^{\mathbf{t}}(v)|\le\theta^n|u-v|. \tag{4.9}$$

Proof. To verify (4.8), we let $i \in \{1, \ldots, \ell\}$. Then

$$\begin{aligned}|f_i^{\mathfrak{t}}(u) - f_i^{\mathfrak{s}}(v)| &\le |f_i^{\mathfrak{t}}(u) - f_i^{\mathfrak{s}}(u)| + |f_i^{\mathfrak{s}}(u) - f_i^{\mathfrak{s}}(v)| \\ &= |\mathfrak{t}_i - \mathfrak{s}_i| + |f_i(u) - f_i(v)| \\ &\le |\mathfrak{t} - \mathfrak{s}| + \theta|u - v|.\end{aligned}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a contracting affine map defined by $\varphi(x) = |\mathfrak{t} - \mathfrak{s}| + \theta x$ for given $\mathfrak{s}$ and $\mathfrak{t}$. Then for every $1 \le i \le \ell$,

$$|f_i^{\mathfrak{t}}(u) - f_i^{\mathfrak{s}}(v)| \le \varphi(|u - v|). \tag{4.10}$$

Now we can prove (4.8) by using the above inequality. Indeed, using (4.10) and the fact that $\varphi(\cdot)$ is monotone increasing, we obtain that for $1 \le i, j \le \ell$,

$$|f_j^{\mathfrak{t}}(f_i^{\mathfrak{t}}(u)) - f_j^{\mathfrak{s}}(f_i^{\mathfrak{s}}(v))| \le \varphi(|f_i^{\mathfrak{t}}(u) - f_i^{\mathfrak{s}}(v)|) \le \varphi^2(|u - v|).$$

Successive application of this implies that for every $\boldsymbol{\tau} \in \Sigma_n$ and $u, v \in S$,

$$|f_{\boldsymbol{\tau}}^{\mathfrak{t}}(u) - f_{\boldsymbol{\tau}}^{\mathfrak{s}}(v)| \le \varphi^n(|u - v|) = \frac{|\mathfrak{t} - \mathfrak{s}|}{1 - \theta} + \theta^n\left(|u - v| - \frac{|\mathfrak{t} - \mathfrak{s}|}{1 - \theta}\right).$$

This proves (4.8). The assertions in (4.9) then follow directly from (4.8). □

Now we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. We divide the proof into five small steps.

Step 1. Write

$$C_n := \sup\left\{\left|\frac{(D_y f_{\boldsymbol{\omega}}^{\mathfrak{t}})_{ii}}{(D_z f_{\boldsymbol{\omega}}^{\mathfrak{t}})_{ii}}\right| : \ \mathfrak{t} \in \Delta, \ y, z \in S, \ \boldsymbol{\omega} \in \Sigma_n, \ 1 \le i \le d\right\}. \tag{4.11}$$

We claim that

$$\lim_{n\to\infty} \frac{1}{n} \log C_n = 0. \tag{4.12}$$

To prove this claim, for each $p \in \{1, \ldots, \ell\}$ and $i \in \{1, \ldots, d\}$, we define a function $a_{p,i} : S \to \mathbb{R}$ by

$$a_{p,i}(z) = \log |(D_z f_p)_{ii}|.$$

Clearly, the functions $a_{p,i}$ are continuous on S. Since the matrix $D_z f_p$ is lower triangular for each $z \in S$ and $1 \le p \le \ell$, it follows that for $\mathfrak{t} \in \Delta$, $y, z \in S$, $\boldsymbol{\omega} = \omega_1 \cdots \omega_n \in \Sigma_n$ and $1 \le i \le d$,

$$\log |(D_z f_{\boldsymbol{\omega}}^{\mathfrak{t}})_{ii}| = \sum_{k=1}^{n} a_{\omega_k,i}(f_{\sigma^k\boldsymbol{\omega}}^{\mathfrak{t}}(z)) \tag{4.13}$$

and

$$\log\left|\frac{(D_y f_{\boldsymbol{\omega}}^{\mathfrak{t}})_{ii}}{(D_z f_{\boldsymbol{\omega}}^{\mathfrak{t}})_{ii}}\right| = \sum_{k=1}^{n}\left(a_{\omega_k,i}(f_{\sigma^k\boldsymbol{\omega}}^{\mathfrak{t}}(y)) - a_{\omega_k,i}(f_{\sigma^k\boldsymbol{\omega}}^{\mathfrak{t}}(z))\right), \tag{4.14}$$

where $\sigma^k\boldsymbol{\omega} := \omega_{k+1}\cdots\omega_n$ for $1 \le k \le n-1$, $\sigma^n\boldsymbol{\omega} := \varepsilon$ (here ε stands for the empty word) and $f_\varepsilon^{\mathfrak{t}}(y) := y$. Define $\gamma : (0,\infty) \to (0,\infty)$ by

$$\gamma(u) = \max_{1\le p\le \ell,\ 1\le i\le d} \sup\{|a_{p,i}(y) - a_{p,i}(z)| :\ y, z \in S,\ |y-z| \le u\}. \tag{4.15}$$

Since S is compact and $a_{p,i}$ are continuous, it follows that $\lim_{u\to 0}\gamma(u) = 0$. To estimate the term in the left-hand side of the equality (4.14), by Lemma 4.6 we obtain that

$$|f^{\mathfrak{t}}_{\sigma^k\boldsymbol{\omega}}(y) - f^{\mathfrak{t}}_{\sigma^k\boldsymbol{\omega}}(z)| \le \theta^{n-k}|y-z| \le \theta^{n-k}\operatorname{diam}(S),$$

where $\theta \in (0,1)$ is a common Lipschitz constant of $f_1, \ldots, f_\ell$ on S. Hence by (4.14),

$$\frac{1}{n}\log\left|\frac{(D_y f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{ii}}{(D_z f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{ii}}\right| \le \frac{1}{n}\sum_{k=1}^{n}\gamma(\theta^{n-k}\operatorname{diam}(S)) \to 0 \quad \text{as } n\to\infty.$$

This proves (4.12).

Step 2. For $\mathfrak{s}, \mathfrak{t} \in \Delta$, $y \in S$, $n \in \mathbb{N}$, $\boldsymbol{\omega} \in \Sigma_n$ and $1 \le i \le d$,

$$\left|\frac{(D_y f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{ii}}{(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})_{ii}}\right| \le \exp\left(n\gamma\left(\frac{|\mathfrak{t}-\mathfrak{s}|}{1-\theta}\right)\right), \tag{4.16}$$

where $\gamma(\cdot)$ is defined as in (4.15) and $\theta \in (0,1)$ is a common Lipschitz constant for $f_1, \ldots, f_\ell$ on S.

To prove (4.16), by (4.13) we see that

$$\begin{aligned}\log\left|\frac{(D_y f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{ii}}{(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})_{ii}}\right| &= \sum_{k=1}^{n}(a_{\omega_k,i}(f^{\mathfrak{t}}_{\sigma^k\boldsymbol{\omega}}(y)) - a_{\omega_k,i}(f^{\mathfrak{s}}_{\sigma^k\boldsymbol{\omega}}(y)) \\ &\le n\gamma\left(\frac{|\mathfrak{t}-\mathfrak{s}|}{1-\theta}\right),\end{aligned}$$

where in the second inequality, we have used the fact that $|f^{\mathfrak{t}}_{\sigma^k\boldsymbol{\omega}}(y) - f^{\mathfrak{s}}_{\sigma^k\boldsymbol{\omega}}(y)| \le (|\mathfrak{t}-\mathfrak{s}|)/(1-\theta)$ (which follows from (4.9)). This proves (4.16).

Step 3. Set

$$c = \sup\left\{\left|\frac{(D_y f_p)_{ij}}{(D_y f_p)_{jj}}\right| :\ y \in S,\ 1 \le p \le \ell,\ 1 \le i, j \le d\right\}. \tag{4.17}$$

Then

$$\left|\frac{(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})_{ij}}{(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})_{jj}}\right| \le (cn)^d \quad \text{for all } \mathfrak{s} \in \Delta,\ y \in S,\ \boldsymbol{\omega} \in \Sigma_n,\ 1 \le i, j \le d. \tag{4.18}$$

To see this, we simply notice that $D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}} = \prod_{k=1}^{n} D_{f^{\mathfrak{s}}_{\sigma^k\boldsymbol{\omega}}(y)} f_{\omega_k}$ and apply Lemma 4.5.

Step 4. Let c and C_n be defined as in (4.17) and (4.11). Then for $\mathfrak{t} \in \Delta$, $y, z_1, \ldots, z_d \in S$, $\boldsymbol{\omega} \in \Sigma_n$ and $1 \le k, j \le d$,

$$|((D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1})_{kj}| \le (cn)^{d(d-1)}(\sqrt{d})^{d-1}(C_n)^d\frac{1}{|(D_y f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{kk}|}, \tag{4.19}$$

where $D^*_{z_1,\ldots,z_d} g$ is defined as in (4.1).

To prove (4.19), notice that for all $1 \le k, j \le d$,

$$\begin{aligned}|(D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{kj}| &= |(D_{z_k} f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{kj}| \quad \text{(by (4.2))}\\ &\le (cn)^d |(D_{z_k} f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{jj}| \quad \text{(by (4.18))}\\ &\le (cn)^d C_n |(D_{z_j} f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{jj}| \quad \text{(by (4.11))}\\ &= (cn)^d C_n |(D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{jj}| \quad \text{(by (4.2))}.\end{aligned}$$

Applying Lemma 4.4 (in which we replace c by $(cn)^d C_n$ and take $A = D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}}$), we obtain

$$\begin{aligned}|((D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1})_{kj}| &\le (cn)^d C_n \sqrt{d})^{d-1} |((D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1})_{kk}|\\ &= ((cn)^d C_n \sqrt{d})^{d-1} |((D_{z_k} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1})_{kk}|\\ &= ((cn)^d C_n \sqrt{d})^{d-1} \frac{1}{|(D_{z_k} f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{kk}|}\\ &\le ((cn)^d C_n \sqrt{d})^{d-1} C_n \frac{1}{|(D_y f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{kk}|},\end{aligned}$$

from which (4.19) follows.

Step 5. Now we are ready to prove (4.4). Let $\delta \in (0, r_0)$. Write

$$u_n := (cn)^{d(d-1)} (\sqrt{d})^{d-1} (C_n)^d, \quad n \in \mathbb{N}.$$

Then, for $\mathfrak{s}, \mathfrak{t} \in \Delta$ with $|\mathfrak{t} - \mathfrak{s}| \le \delta$ and $1 \le i, j \le d$,

$$\begin{aligned}|(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}} \cdot (D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1})_{ij}| &\le \sum_{k=1}^d |(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})_{ik}| \cdot |(D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1})_{kj}|\\ &\le (cn)^d u_n \sum_{k=1}^d \left| \frac{(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})_{kk}}{(D_y f^{\mathfrak{t}}_{\boldsymbol{\omega}})_{kk}} \right| \quad \text{(by (4.18), (4.19))}\\ &\le d(cn)^d u_n \exp\left(n\gamma\left(\frac{|\mathfrak{t} - \mathfrak{s}|}{1-\theta}\right)\right) \quad \text{(by (4.16))}\\ &\le d(cn)^d u_n \exp\left(n\gamma\left(\frac{\delta}{1-\theta}\right)\right).\end{aligned}$$

This implies that

$$\|D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}} \cdot (D^*_{z_1,\ldots,z_d} f^{\mathfrak{t}}_{\boldsymbol{\omega}})^{-1}\| \le d^2 (cn)^d u_n \exp\left(n\gamma\left(\frac{\delta}{1-\theta}\right)\right), \tag{4.20}$$

where we have used an easily checked fact that

$$\|A\| \le d \max_{1 \le i,j \le d} |A_{ij}|$$

for $A = (A_{ij}) \in \mathbb{R}^{d \times d}$.

Set $h : (0, r_0) \to (0, \infty)$ by $h(x) = x + \gamma(x/(1-\theta))$. Since

$$\lim_{n\to\infty} \frac{1}{n} \log(d^2 (cn)^d u_n) = 0,$$

there exists $C(\delta) > 0$ such that $d^2(cn)^d u_n e^{-n\delta} \leq C(\delta)$ for all $n \geq 1$. According to this fact and (4.20), we obtain the desired inequality

$$\|D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}} \cdot (D^*_{z_1,\ldots,z_d} f_{\boldsymbol{\omega}}^{\mathbf{t}})^{-1}\| \leq C(\delta) \exp(nh(\delta));$$

for later convenience, we may assume that $C(\delta) \geq 1$. □

4.2. *Proof of Theorem 4.2.* The following result plays a key part in our proof.

LEMMA 4.7. *Let $\{\mathcal{F}^{\mathbf{t}}\}_{\mathbf{t}\in\Delta}$ be a translational family of IFSs on a compact set $S \subset \mathbb{R}^d$ generated by a C^1 IFS $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$. Suppose that (4.3) holds. Let $\delta > 0$. Then there exists $\widetilde{C} > 0$, which depends on $\mathcal{F}$ and δ such that the following holds. Let $\mathbf{a} = (a_n)_{n=1}^{\infty}$, $\mathbf{b} = (b_n)_{n=1}^{\infty} \in \Sigma$ with $a_1 \neq b_1$, and let A be a real invertible $d \times d$ matrix. Then for $\mathfrak{s} \in \Delta$ and $r > 0$,*

$$\begin{aligned} &\mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \ \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in A^{-1} B_{\mathbb{R}^d}(0, r)\} \\ &\quad \leq \widetilde{C} \min\left\{\frac{r^k}{\phi^k(A)} : \ k = 0, 1, \ldots, d\right\}, \end{aligned} \tag{4.21}$$

where $B_{\mathbb{R}^{\ell d}}(\cdot, \cdot)$ and $B_{\mathbb{R}^d}(\cdot, \cdot)$ stand for closed balls in $\mathbb{R}^{\ell d}$ and $\mathbb{R}^d$, respectively.

Since the proof of the above lemma is a little long, we will postpone it until we have finished the proof of Theorem 4.2.

Proof of Theorem 4.2 by assuming Lemma 4.7. Fix $\mathfrak{s} \in \Delta$ and $\delta \in (0, r_0)$. Let $\mathbf{i}, \mathbf{j} \in \Sigma$ with $\mathbf{i} \neq \mathbf{j}$. Set

$$\boldsymbol{\omega} = \mathbf{i} \wedge \mathbf{j} \quad \text{and} \quad n = |\boldsymbol{\omega}|.$$

Write $\mathbf{a} = \sigma^n \mathbf{i}$ and $\mathbf{b} = \sigma^n \mathbf{j}$. Clearly $a_1 \neq b_1$.

Fix $y \in S$. We claim that for $r > 0$,

$$\begin{aligned} &\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \ |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \\ &\quad \subset \{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \ \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in (D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}})^{-1} B_{\mathbb{R}^d}(0, C(\delta)e^{nh(\delta)} r)\}, \end{aligned} \tag{4.22}$$

where $C(\delta)$ and $h(\delta)$ are given as in Proposition 4.3.

To show (4.22), let $\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta$ so that $|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r$. Notice that

$$\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j}) = f_{\boldsymbol{\omega}}^{\mathbf{t}}(\Pi^{\mathbf{t}}(\mathbf{a})) - f_{\boldsymbol{\omega}}^{\mathbf{t}}(\Pi^{\mathbf{t}}(\mathbf{b})).$$

Since S is convex, by the mean value theorem, there exist $z_1, \ldots, z_d \in S$ such that

$$\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j}) = (D^*_{z_1,\ldots,z_d} f_{\boldsymbol{\omega}}^{\mathbf{t}})(\Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b})).$$

Hence

$$\begin{aligned} \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) &= (D^*_{z_1,\ldots,z_d} f_{\boldsymbol{\omega}}^{\mathbf{t}})^{-1}(\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})) \\ &\in (D^*_{z_1,\ldots,z_d} f_{\boldsymbol{\omega}}^{\mathbf{t}})^{-1} B_{\mathbb{R}^d}(0, r) \\ &= (D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}})^{-1} D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}} (D^*_{z_1,\ldots,z_d} f_{\boldsymbol{\omega}}^{\mathbf{t}})^{-1} B_{\mathbb{R}^d}(0, r) \\ &\subset (D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}})^{-1} B_{\mathbb{R}^d}(0, C(\delta)e^{nh(\delta)} r) \quad \text{(by Proposition 4.3)}. \end{aligned}$$

This proves (4.22).

By (4.22) and Lemma 4.7, we see that

$$\begin{aligned}
&\mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \\
&\quad \le \mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in (D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}})^{-1} B_{\mathbb{R}^d}(0, C(\delta)e^{nh(\delta)}r)\} \\
&\quad \le \widetilde{C} \cdot \min\left\{\frac{C(\delta)^k e^{nkh(\delta)} r^k}{\phi^k(D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}})} : k = 0, 1, \ldots, d\right\} \\
&\quad \le \widetilde{C} C(\delta)^d e^{ndh(\delta)} \min\left\{\frac{r^k}{\phi^k(D_y f_{\boldsymbol{\omega}}^{\mathfrak{s}})} : k = 0, 1, \ldots, d\right\}.
\end{aligned}$$

Since $y \in S$ is arbitrary and $\Pi^{\mathfrak{s}}(\Sigma) \subset S$, recalling

$$Z_{\boldsymbol{\omega}}^{\mathfrak{s}}(r) = \inf_{x \in \Sigma} \min\left\{\frac{r^k}{\phi^k(D_{\Pi^{\mathfrak{s}}x} f_{\boldsymbol{\omega}}^{\mathfrak{s}})} : k = 0, 1, \ldots, d\right\},$$

it follows that

$$\mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \le \widetilde{C} C(\delta)^d e^{ndh(\delta)} Z_{\boldsymbol{\omega}}^{\mathfrak{s}}(r).$$

This completes the proof of the theorem by letting $c_\delta = \widetilde{C} C(\delta)^d$ and $\psi(\delta) = dh(\delta)$. □

In what follows we prove Lemma 4.7. To this end, we first prove an elementary geometric lemma.

LEMMA 4.8. *Let A be a real invertible $d \times d$ matrix. Then for $r_1, r_2 > 0$,*

$$\mathcal{L}_d((A^{-1} B_{\mathbb{R}^d}(0, r_1)) \cap B_{\mathbb{R}^d}(0, r_2)) \le 2^d \min\left\{\frac{r_1^k r_2^{d-k}}{\phi^k(A)} : k = 0, 1, \ldots, d\right\},$$

where $\phi^s(\cdot)$ is the singular value function defined as in (2.5).

Proof. Let $\alpha_1 \ge \cdots \ge \alpha_d$ be the singular values of A. Clearly the set

$$(A^{-1} B_{\mathbb{R}^d}(0, r_1)) \cap B_{\mathbb{R}^d}(0, r_2)$$

is contained in a rectangular parallelepiped with sides $2 \min\{r_1/\alpha_i, r_2\}$, $i = 1, \ldots, d$. It follows that

$$\begin{aligned}
\mathcal{L}_d((A^{-1} B_{\mathbb{R}^d}(0, r_1)) \cap B_{\mathbb{R}^d}(0, r_2)) &\le 2^d \prod_{i=1}^{d} \min\left\{\frac{r_1}{\alpha_i}, r_2\right\} \\
&= 2^d \min\left\{\frac{r_1^k r_2^{d-k}}{\alpha_1 \ldots \alpha_k} : k = 0, 1, \ldots, d\right\} \\
&= 2^d \min\left\{\frac{r_1^k r_2^{d-k}}{\phi^k(A)} : k = 0, 1, \ldots, d\right\}. \qquad \square
\end{aligned}$$

Proof of Lemma 4.7. Let $\mathbf{a} = (a_n)_{n=1}^{\infty}$, $\mathbf{b} = (b_n)_{n=1}^{\infty} \in \Sigma$ with $a_1 \ne b_1$. Without loss of generality, we assume that

$$a_1 = 1 \quad \text{and} \quad b_1 = 2. \tag{4.23}$$

Define $g : \Delta \to \mathbb{R}^d$ by

$$g(\mathbf{t}) = \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}).$$

Recall that we have used the notation $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \Delta \subset \mathbb{R}^{\ell d}$ with

$$\mathbf{t}_k = (t_{k,1}, \ldots, t_{k,d}) \in \mathbb{R}^d \quad \text{for all } 1 \le k \le \ell.$$

For $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \Delta$ and $k \in \{1, \ldots, \ell\}$, let $(\partial g/\partial \mathbf{t}_k)(\mathbf{t})$ denote the Jacobian matrix of the following map from $\mathbb{R}^d$ to $\mathbb{R}^d$:

$$(t_{k,1}, \ldots, t_{k,d}) \mapsto g(\mathbf{t}_1, \ldots, \mathbf{t}_{k-1}, t_{k,1}, \ldots, t_{k,d}, \mathbf{t}_{k+1}, \ldots, \mathbf{t}_\ell).$$

Write $\mathbf{I} = \mathbf{I}_d := \mathrm{diag}(\underbrace{1, \ldots, 1}_{d})$. First observe that for every $n \in \mathbb{N}$ and $\mathbf{i} = (i_k)_{k=1}^\infty \in \Sigma$,

$$\Pi^{\mathbf{t}}(\mathbf{i}) = \mathbf{t}_{i_1} + f_{i_1}(\mathbf{t}_{i_2} + f_{i_2}(\mathbf{t}_{i_3} + f_{i_3}(\ldots f_{i_{n-1}}(\mathbf{t}_{i_n} + f_{i_n}(\Pi^{\mathbf{t}}\sigma^n \mathbf{i}) \ldots)))). \tag{4.24}$$

It follows that for $k \in \{1, \ldots, \ell\}$,

$$\begin{aligned} \frac{\partial \Pi^{\mathbf{t}}(\mathbf{a})}{\partial \mathbf{t}_k} &= \delta_{k,a_1} \cdot \mathbf{I} + (D_{\Pi^{\mathbf{t}}(\sigma \mathbf{a})} f_{a_1}^{\mathbf{t}}) \times [\mathbf{I} \cdot \delta_{k,a_2} \\ &\quad + (D_{\Pi^{\mathbf{t}}(\sigma^2 \mathbf{a})} f_{a_2}^{\mathbf{t}})[\mathbf{I} \cdot \delta_{k,a_3} + (D_{\Pi^{\mathbf{t}}(\sigma^3 \mathbf{a})} f_{a_3}^{\mathbf{t}})[\mathbf{I} \cdot \delta_{k,a_4} + \cdots]]] \\ &= \delta_{k,a_1} \cdot \mathbf{I} + \sum_{\substack{n \ge 1 \\ a_{n+1} = k}} \prod_{k=1}^{n} D_{\Pi^{\mathbf{t}}(\sigma^k \mathbf{a})} f_{a_k}, \end{aligned} \tag{4.25}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

By (4.25) and the assumption (4.23), we see that for $k \in \{1, \ldots, \ell\}$,

$$\frac{\partial g}{\partial \mathbf{t}_k}(\mathbf{t}) = \frac{\partial \Pi^{\mathbf{t}}(\mathbf{a})}{\partial \mathbf{t}_k} - \frac{\partial \Pi^{\mathbf{t}}(\mathbf{b})}{\partial \mathbf{t}_k} = \delta_{k,1} \cdot \mathbf{I} - \delta_{k,2} \cdot \mathbf{I} + E_k(\mathbf{t}), \tag{4.26}$$

where

$$E_k(\mathbf{t}) = \sum_{\substack{n \ge 1 \\ a_{n+1} = k}} \prod_{i=1}^{n} D_{\Pi^{\mathbf{t}}(\sigma^i \mathbf{a})} f_{a_i} - \sum_{\substack{n \ge 1 \\ b_{n+1} = k}} \prod_{i=1}^{n} D_{\Pi^{\mathbf{t}}(\sigma^i \mathbf{b})} f_{b_i}. \tag{4.27}$$

Recall that $\rho = \max_{i \ne j}(\rho_i + \rho_j) < 1$ with $\rho_i := \max_{z \in S} \|D_z f_i\|$. □

LEMMA 4.9. *There exists $k^* = k^*(\mathbf{a}, \mathbf{b}) \in \{1, 2\}$ such that $\|E_{k^*}(\mathbf{t})\| < \rho$ for all $\mathbf{t} \in \Delta$.*

Proof. Our argument is based on an idea of Boris Solomyak, which was used to prove a corresponding statement for self-affine IFSs [3, Theorem 9.1.2].

By (4.27), for each $k \in \{1, 2\}$ and $\mathbf{t} \in \Delta$,

$$\|E_k(\mathbf{t})\| \le \sum_{\substack{n \ge 1 \\ a_{n+1} = k}} \rho_{a_1} \cdots \rho_{a_n} + \sum_{\substack{n \ge 1 \\ b_{n+1} = k}} \rho_{b_1} \cdots \rho_{b_n} =: \lambda_k. \tag{4.28}$$

Clearly, λ_k $(k = 1, 2)$ only depend on $\mathbf{a}$ and $\mathbf{b}$.

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Notice that

$$\begin{aligned}\sum_{k=1}^{2}\lambda_k(1-\rho_k)&=\sum_{n=1}^{\infty}\rho_{a_1}\cdots\rho_{a_n}(1-\rho_{a_{n+1}})+\sum_{n=1}^{\infty}\rho_{b_1}\cdots\rho_{b_n}(1-\rho_{b_{n+1}})\\&=\rho_1+\rho_2\\&\le\rho.\end{aligned}\tag{4.29}$$

This implies that one of λ_1, λ_2 is smaller than ρ; otherwise, since $\rho_1+\rho_2\le\rho<1$, it follows that

$$\sum_{k=1}^{2}\lambda_k(1-\rho_k)\ge\rho(1-\rho_1+1-\rho_2)>\rho,$$

which contradicts (4.29). Now set

$$k^*=\begin{cases}1 & \text{if } \lambda_1<\rho,\\ 2 & \text{otherwise.}\end{cases}$$

Then $\lambda_{k^*}<\rho$. Since λ_1, λ_2 only depend on $\mathbf{a}$ and $\mathbf{b}$, so does k^*. By (4.28),

$$\|E_{k^*}(\mathbf{t})\|\le\lambda_{k^*}<\rho$$

for all $\mathbf{t}\in\Delta$. □

In what follows, we always let $k^*=k^*(\mathbf{a},\mathbf{b})\in\{1,2\}$ be given as in Lemma 4.9.

LEMMA 4.10. *For all* $\mathbf{t}\in\Delta$,

$$\left\|\left(\frac{\partial g}{\partial \mathbf{t}_{k^*}}(\mathbf{t})\right)^{-1}\right\|<\frac{1}{1-\rho}.\tag{4.30}$$

Proof. Without loss of generality, we assume that $k^*=1$. The proof is similar in the case when $k^*=2$.

Let $\mathbf{t}\in\Delta$. By Lemma 4.9, $\|E_1(\mathbf{t})\|<\rho<1$. Thanks to (4.26),

$$\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t})=\mathbf{I}-(-E_1(\mathbf{t})),\tag{4.31}$$

where $\mathbf{I}=\mathrm{diag}(\underbrace{1,\ldots,1}_{d})$. Since $\|E_1(\mathbf{t})\|<\rho<1$, we see that $\partial g/\partial\mathbf{t}_1(\mathbf{t})$ is invertible with

$$\left(\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t})\right)^{-1}=\mathbf{I}+\sum_{n=1}^{\infty}(-E_1(\mathbf{t}))^n,$$

from which we obtain that

$$\left\|\left(\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t})\right)^{-1}\right\|\le1+\sum_{n=1}^{\infty}\|E_1(\mathbf{t})\|^n\le1+\sum_{n=1}^{\infty}\rho^n=\frac{1}{1-\rho}.\tag{4.32}$$

□

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Next we introduce two mappings $T_1, T_2 : \Delta \to \mathbb{R}^{\ell d}$ by

$$T_1(\mathbf{t}) = (g(\mathbf{t}), \mathbf{t}_2, \ldots, \mathbf{t}_\ell), \quad T_2(\mathbf{t}) = (\mathbf{t}_1, g(\mathbf{t}), \mathbf{t}_3, \ldots, \mathbf{t}_\ell), \tag{4.33}$$

where $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell)$. Recall that $g(\mathbf{t}) = \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b})$.

LEMMA 4.11. *Let $k^* = k^*(\mathbf{a}, \mathbf{b}) \in \{1, 2\}$ be given as in Lemma 4.9. Then the following properties hold.*

(i) *The mapping $T_{k^*} : \Delta \to \mathbb{R}^{\ell d}$ is injective.*

(ii) *For each $\mathbf{t} \in \Delta$,*

$$|\det((D_{\mathbf{t}} T_{k^*})^{-1})| < \left(\frac{1}{1-\rho}\right)^d. \tag{4.34}$$

Proof. Without loss of generality, we may assume that $k^* = 1$. Then by Lemma 4.9 and (4.33),

$$\|E_1(\mathbf{t})\| < \rho, \quad T_1(\mathbf{t}) = (g(\mathbf{t}), \mathbf{t}_2, \ldots, \mathbf{t}_\ell) \tag{4.35}$$

for $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \Delta$. Hence to prove (i), it suffices to show that for given $\mathbf{t}_2, \ldots, \mathbf{t}_\ell \in \mathbb{R}^d$ with $\sum_{i=2}^{\ell} |\mathbf{t}_2|^2 < r_0^2$, the mapping

$$\mathbf{t}_1 \mapsto g(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_\ell)$$

is injective on $\Delta_1 := \{\mathbf{t}_1 \in \mathbb{R}^d : |\mathbf{t}_1| < \sqrt{r_0^2 - \sum_{i=2}^{\ell} |\mathbf{t}_i|^2}\}$. To this end, define $\psi : \Delta_1 \to \mathbb{R}^d$ by

$$\psi(\mathbf{t}_1) = g(\mathbf{t}_1, \ldots, \mathbf{t}_\ell) - \mathbf{t}_1.$$

Then by (4.31) and (4.35),

$$\|D_{\mathbf{t}_1} \psi\| = \left\|\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t}_1, \ldots, \mathbf{t}_\ell) - \mathbf{I}\right\| = \|E_1(\mathbf{t}_1, \ldots, \mathbf{t}_\ell)\| < \rho \quad \text{for each } \mathbf{t}_1 \in \Delta_1.$$

Since Δ_1 is a convex open subset of $\mathbb{R}^d$, by [**42**, Theorem 9.19], the above inequality implies that

$$|\psi(\mathbf{t}_1) - \psi(\mathbf{s}_1)| \le \rho|\mathbf{t}_1 - \mathbf{s_1}| < |\mathbf{t}_1 - \mathbf{s_1}|$$

for all distinct $\mathbf{t}_1, \mathbf{s_1} \in \Delta_1$. It follows that for distinct $\mathbf{t}_1, \mathbf{s_1} \in \Delta_1$,

$$\begin{aligned} |g(\mathbf{t}_1) - g(\mathbf{s}_1)| &= |\psi(\mathbf{t}_1) + \mathbf{t}_1 - \psi(\mathbf{s}_1) - \mathbf{s}_1| \\ &\ge |\mathbf{t}_1 - \mathbf{s}_1| - |\psi(\mathbf{t}_1) - \psi(\mathbf{s}_1)| \\ &> 0. \end{aligned}$$

This proves (i).

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To prove (ii), notice that

$$D_{\mathbf{t}}T_1 = \left(\begin{array}{c|cccc} \dfrac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t}) & \dfrac{\partial g}{\partial \mathbf{t}_2}(\mathbf{t}) & \dfrac{\partial g}{\partial \mathbf{t}_3}(\mathbf{t}) & \cdots & \dfrac{\partial g}{\partial \mathbf{t}_\ell}(\mathbf{t}) \\ \hline & & & & \\ \mathbf{0}_{(\ell-1)d,d} & & \mathbf{I}_{(\ell-1)d} & & \\ & & & & \end{array}\right),$$

where $\mathbf{I}_{(\ell-1)d} := \mathrm{diag}(\underbrace{1, \ldots, 1}_{(\ell-1)d})$ and $\mathbf{0}_{(\ell-1)d,d}$ is the $((\ell-1)d) \times d$ all-zero matrix. That is,

$$D_{\mathbf{t}}T_1 = \left(\begin{array}{c|c} A(\mathbf{t}) & B(\mathbf{t}) \\ \hline \mathbf{0}_{(\ell-1)d,d} & \mathbf{I}_{(\ell-1)d} \end{array}\right),$$

where $A(\mathbf{t})$ and $B(\mathbf{t})$ are given by

$$A(\mathbf{t}) = \frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t}), \quad B(\mathbf{t}) = \left(\frac{\partial g}{\partial \mathbf{t}_2}(\mathbf{t}), \ldots, \frac{\partial g}{\partial \mathbf{t}_\ell}(\mathbf{t})\right).$$

Hence, by Lemma 4.10, $A^{-1}(\mathbf{t})$ exists and

$$(D_{\mathbf{t}}T_1)^{-1} = \left(\begin{array}{c|c} A^{-1}(\mathbf{t}) & -A^{-1}(\mathbf{t}) \cdot B(\mathbf{t}) \\ \hline \mathbf{0}_{(\ell-1)d,d)} & \mathbf{I}_{(\ell-1)d} \end{array}\right).$$

It follows that

$$\det((D_{\mathbf{t}}T_1)^{-1}) = \det(A^{-1}(\mathbf{t})) = \det\left(\left(\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t})\right)^{-1}\right).$$

By the Hadamard's inequality (see e.g. [28, Corollary 7.8.2]),

$$|\det((D_{\mathbf{t}}T_1)^{-1})| = \left|\det\left(\left(\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t})\right)^{-1}\right)\right| \le \left\|\left(\frac{\partial g}{\partial \mathbf{t}_1}(\mathbf{t})\right)^{-1}\right\|^d \le \left(\frac{1}{1-\rho}\right)^d,$$

where the last inequality follows from Lemma 4.10. This completes the proof of (ii).

To shorten the notation, from now on we write

$$C_* := \left(\frac{1}{1-\rho}\right)^d. \tag{4.36}$$

Let $\mathfrak{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_\ell) \in \Delta$ and $\delta, r > 0$. Let A be a given real invertible $d \times d$ matrix.

Write

$$\begin{aligned} E : &= \{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \ \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in A^{-1}B_{\mathbb{R}^d}(0, r)\} \\ &= \{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \ g(\mathbf{t}) \in A^{-1}B_{\mathbb{R}^d}(0, r)\}. \end{aligned}$$

Below, we estimate $\mathcal{L}_{\ell d}(E)$.

Notice that $\mathcal{L}_{\ell d}(E) = \mathcal{L}_{\ell d}(T_{k^*}^{-1}(T_{k^*}(E)))$. Recall that by Lemma 4.11, the mapping $T_{k^*} : \Delta \to \mathbb{R}^{\ell d}$ is injective and $\det((D_{\mathbf{t}}T_{k^*})^{-1}) \le C_*$ for $\mathbf{t} \in \Delta$. So by the substitution rule of multiple integration (see e.g. [42, Theorem 10.9]),

$$\mathcal{L}_{\ell d}(E) \le C_* \mathcal{L}_{\ell d}(T_{k^*}(E)). \tag{4.37}$$

Next we estimate $\mathcal{L}_{\ell d}(T_{k^*}(E))$. Without loss of generality, we may assume that $k^* = 1$. Notice that for each $\mathbf{t} \in E$,

$$g(\mathbf{t}) \in A^{-1} B_{\mathbb{R}^d}(0, r);$$

in the meantime, since $\Pi^{\mathbf{t}}(\mathbf{a}), \Pi^{\mathbf{t}}(\mathbf{b}) \in S$, it follows that

$$g(\mathbf{t}) = \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in B_{\mathbb{R}^d}(0, 2\mathrm{diam}(S)).$$

Hence, for each $\mathbf{t} \in E$,

$$g(\mathbf{t}) \in (A^{-1} B_{\mathbb{R}^d}(0, r)) \cap B_{\mathbb{R}^d}(0, 2\mathrm{diam}(S)).$$

Since $T_1(\mathbf{t}) = (g(\mathbf{t}), \mathbf{t_2}, \ldots, \mathbf{t}_\ell)$, it follows that

$$T_1(E) \subset F_1 \times F_2,$$

where

$$\begin{aligned} F_1 &:= (A^{-1} B_{\mathbb{R}^d}(0, r)) \cap B_{\mathbb{R}^d}(0, 2\mathrm{diam}(S)), \\ F_2 &:= \{(\mathbf{t}_2, \ldots, \mathbf{t}_\ell) \in \mathbb{R}^{(\ell-1)d} : |\mathbf{t}_i - \mathbf{s}_i| < \delta\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathcal{L}_{\ell d}(T_1(E)) &\le \mathcal{L}_d(F_1) \cdot \mathcal{L}_{(\ell-1)d}(F_2) \\ &\le 2^d \min\left\{\frac{r^k (2\mathrm{diam}(S))^{d-k}}{\phi^k(A)} : k = 0, 1, \ldots, d\right\} \cdot (2\delta)^{(\ell-1)d} \\ &\le u \min\left\{\frac{r^k}{\phi^k(A)} : k = 0, 1, \ldots, d\right\} \end{aligned}$$

with $u := 2^{\ell d} \delta^{(\ell-1)d} \max\{1, 2^d \mathrm{diam}(S)^d\}$, where we have used Lemma 4.8 in the second inequality. Combining this with (4.37) yields that

$$\mathcal{L}_{\ell d}(E) \le C_* \mathcal{L}_{\ell d}(T_1(E)) \le u C_* \min\left\{\frac{r^k}{\phi^k(A)} : k = 0, 1, \ldots, d\right\}.$$

This completes the proof of Lemma 4.7. □

5. *Translational family of IFSs generated by a C^1 conformal IFS*

In this section, we prove the following result.

THEOREM 5.1. *Let $\mathcal{F} = \{f_i : S \to S\}_{i=1}^{\ell}$ be an IFS on a compact set $S \subset \mathbb{R}^d$. Suppose that the following properties hold.*

(i) *The set S is connected, $S = \overline{\mathrm{int}(S)}$ and $f_i(S) \subset \mathrm{int}(S)$ for all i.*

(ii) *There is a bounded connected open set $U \supset S$ such that each f_i extends to a C^1 conformal diffeomorphism $f_i : U \to f_i(U) \subset U$ with*

$$\rho_i := \sup_{x \in U} \|f_i'(x)\| < 1.$$

(iii) $\max_{i \neq j} \rho_i + \rho_j < 1$.

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Then there is a small $r_0 > 0$ such that the translational family $\mathcal{F}^{\mathbf{t}} = \{f_i^{\mathbf{t}} = f_i + \mathbf{t}_i\}_{i=1}^{\ell}$, $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \Delta := \{\mathfrak{s} \in \mathbb{R}^{\ell d} : |\mathfrak{s}| < r_0\}$, satisfies the GTC with respect to the Lebesgue measure $\mathcal{L}_{\ell d}$ on Δ.

Proof. By the assumptions (i) and (ii), we may pick two open connected sets V and W (for instance, we may let V and W be the ϵ-neighborhood and 2ϵ-neighborhood of S, respectively, for a sufficiently small $\epsilon > 0$), such that

$$S \subset V \subset \overline{V} \subset W \subset \overline{W} \subset U, \text{ and}$$

$$f_i(\overline{V}) \subset V \text{ and } f_i(\overline{W}) \subset W \quad \text{for all } i.$$

Then by continuity, we can pick a small r_0 such that for all $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \mathbb{R}^{\ell d}$ with $|\mathbf{t}| < r_0$,

$$f_i^{\mathbf{t}}(\overline{V}) \subset V \text{ and } f_i^{\mathbf{t}}(\overline{W}) \subset W \quad \text{for all } i,$$

where $f_i^{\mathbf{t}} := f_i + \mathbf{t}_i$. Fix this r_0 and set $\Delta = \{\mathfrak{s} \in \mathbb{R}^{\ell d} : |\mathfrak{s}| < r_0\}$. In what follows, we prove that the family $\mathcal{F}^{\mathbf{t}}$, $\mathbf{t} \in \Delta$, satisfies the GTC with respect to $\mathcal{L}_{\ell d}$ on Δ.

For $i = 1, \ldots, \ell$, define $g_i : \overline{W} \to \mathbb{R}$ by

$$g_i(z) = \log \|f_i'(z)\|.$$

Then g_i is continuous on $\overline{W}$ for each i. Define $\gamma : (0, \infty) \to (0, \infty)$ by

$$\gamma(u) = \max_{1 \le i \le \ell} \sup\{|g_i(x) - g_i(y)| : x, y \in \overline{W}, |x - y| \le u\}.$$

That is, γ is a common continuity modulus of $g_1, \ldots, g_\ell$. Clearly $\lim_{u \to 0} \gamma(u) = 0$. Notice that for $\mathbf{t} \in \Delta$, $y \in W$ and $\boldsymbol{\omega} \in \Sigma_n$,

$$\log \|(f_{\boldsymbol{\omega}}^{\mathbf{t}})'(y)\| = \sum_{k=1}^{n} g_{\omega_k}(f_{\sigma^k \boldsymbol{\omega}}^{\mathbf{t}}(y)).$$

Using similar arguments (with minor changes) as in Step 1 and Step 2 of the proof of Proposition 4.3, we can show that the following two properties hold.

(a) Write for $n \in \mathbb{N}$,

$$C_n := \sup \left\{ \frac{\|(f_{\boldsymbol{\omega}}^{\mathbf{t}})'(y)\|}{\|(f_{\boldsymbol{\omega}}^{\mathbf{t}})'(z)\|} : \mathbf{t} \in \Delta, \ y, z \in W, \ \boldsymbol{\omega} \in \Sigma_n \right\}. \tag{5.1}$$

Then $\lim_{n \to \infty} (1/n) \log C_n = 0$.

(b) For $y \in W$, $\mathfrak{s}$, $\mathbf{t} \in \Delta$, $n \in \mathbb{N}$ and $\boldsymbol{\omega} \in \Sigma_n$,

$$\frac{\|(f_{\boldsymbol{\omega}}^{\mathbf{t}})'(y)\|}{\|(f_{\boldsymbol{\omega}}^{\mathfrak{s}})'(y)\|} \le \exp\left(n\gamma\left(\frac{|\mathbf{t} - \mathfrak{s}|}{1 - \theta}\right)\right), \tag{5.2}$$

where $\theta := \max_{1 \le i \le \ell} \rho_i < 1$.

Let $\mathcal{H}$ denote the collection of C^1 injective conformal mappings $h : W \to W$ such that $h(\overline{V}) \subset V$. The following fact is known (for a proof, see e.g. part 3 of the proof of [39, Lemma 2.2]): there exists a constant $D \in (0, 1)$ depending on V and W, such that

$$D \cdot \Big(\inf_{z \in W} \|h'(z)\| \Big) \cdot |x - y| \le |h(x) - h(y)| \quad \text{for all } h \in \mathcal{H},\ x, y \in V. \tag{5.3}$$

Now fix $\mathfrak{s} \in \Delta$ and $\delta \in (0, r_0)$. Let $\mathbf{i}, \mathbf{j} \in \Sigma$ with $\mathbf{i} \neq \mathbf{j}$. Set

$$\boldsymbol{\omega} = \mathbf{i} \wedge \mathbf{j} \quad \text{and} \quad n = |\boldsymbol{\omega}|.$$

Write $\mathbf{a} = \sigma^n \mathbf{i}$ and $\mathbf{b} = \sigma^n \mathbf{j}$. Clearly $a_1 \neq b_1$. Fix $y \in S$. We claim that for $r > 0$,

$$\begin{aligned} &\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \\ &\quad \subset \{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in (D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})^{-1} B_{\mathbb{R}^d}(0, c(n, \delta) r)\}, \end{aligned} \tag{5.4}$$

where

$$c(n, \delta) := D^{-1} C_n \exp\bigg(n\gamma \Big(\frac{\delta}{1 - \theta} \Big) \bigg) > 1,$$

in which D is the constant from (5.3).

To show (5.4), let $\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta$ so that $|\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r$. Notice that

$$\begin{aligned} |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| &= |f^{\mathbf{t}}_{\boldsymbol{\omega}}(\Pi^{\mathbf{t}}(\mathbf{a})) - f^{\mathbf{t}}_{\boldsymbol{\omega}}(\Pi^{\mathbf{t}}(\mathbf{b}))| \\ &\ge D \cdot \Big(\inf_{z \in W} \|(f^{\mathbf{t}}_{\boldsymbol{\omega}})'(z)\| \Big) \cdot |\Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b})| \quad \text{(by (5.3))} \\ &\ge D (C_n)^{-1} \exp\bigg(-n\gamma \Big(\frac{\delta}{1 - \theta} \Big) \bigg) \cdot \|(f^{\mathfrak{s}}_{\boldsymbol{\omega}})'(y)\| \cdot |\Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b})|, \end{aligned}$$

where, in the last inequality, we have used (5.1) and (5.2). It follows that

$$|\Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b})| \le \|(f^{\mathfrak{s}}_{\boldsymbol{\omega}})'(y)\|^{-1} \cdot D^{-1} C_n \exp\bigg(n\gamma \Big(\frac{\delta}{1 - \theta} \Big) \bigg) \cdot r.$$

Since $(f^{\mathfrak{s}}_{\boldsymbol{\omega}})'(y) = D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}}$ is a scalar multiple of an orthogonal matrix, the above inequality implies that

$$\Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in (D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})^{-1} B_{\mathbb{R}^d}(0, c(n, \delta) r).$$

from which (5.4) follows.

By (5.4) and Lemma 4.7 (which is also valid in this context),

$$\begin{aligned} &\mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta :\ |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \\ &\quad \le \mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta :\ \Pi^{\mathbf{t}}(\mathbf{a}) - \Pi^{\mathbf{t}}(\mathbf{b}) \in (D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})^{-1} B_{\mathbb{R}^d}(0, c(n, \delta) r)\} \\ &\quad \le \widetilde{C} \cdot \min \bigg\{ \frac{c(n, \delta)^k r^k}{\phi^k(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})} :\ k = 0, 1 \ldots, d \bigg\} \\ &\quad \le \widetilde{C} c(n, \delta)^d \cdot \min \bigg\{ \frac{r^k}{\phi^k(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})} :\ k = 0, 1 \ldots, d \bigg\} \\ &\quad = \widetilde{C} D^{-d} (C_n)^d \exp\bigg(nd\gamma \Big(\frac{\delta}{1 - \theta} \Big) \bigg) \min \bigg\{ \frac{r^k}{\phi^k(D_y f^{\mathfrak{s}}_{\boldsymbol{\omega}})} :\ k = 0, 1 \ldots, d \bigg\}. \end{aligned}$$

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Since $y \in S$ is arbitrary and $\Pi^{\mathfrak{s}}(\Sigma) \subset S$, recalling

$$Z_{\boldsymbol{\omega}}^{\mathfrak{s}}(r) = \inf_{x\in\Sigma} \min\left\{\frac{r^k}{\phi^k(D_{\Pi^{\mathfrak{s}}x} f_{\boldsymbol{\omega}}^{\mathfrak{s}})} : k = 0, 1, \ldots, d\right\},$$

it follows that

$$\begin{aligned}
&\mathcal{L}_{\ell d}\{\mathbf{t} \in B_{\mathbb{R}^{\ell d}}(\mathfrak{s}, \delta) \cap \Delta : |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \\
&\quad \le \widetilde{C} D^{-d} (C_n)^d \exp\left(nd\gamma\left(\frac{\delta}{1-\theta}\right)\right) Z_{\boldsymbol{\omega}}^{\mathfrak{s}}(r) \\
&\quad \le c_\delta e^{n\psi(\delta)} Z_{\boldsymbol{\omega}}^{\mathfrak{s}}(r),
\end{aligned}$$

where

$$c_\delta := \sup_{n\in\mathbb{N}} \widetilde{C} D^{-d} (C_n)^d e^{-n\delta} < \infty, \quad \psi(\delta) := \delta + d\gamma\left(\frac{\delta}{1-\theta}\right).$$

Since $\lim_{u\to 0} \gamma(u) = 0$, we see that $\lim_{\delta\to 0} \psi(\delta) = 0$. Thus, $(\mathcal{F}^{\mathbf{t}})_{\mathbf{t}\in\Delta}$ satisfies the GTC. $\square$

6. *Direct product of parameterized families of C^1 IFSs*

In this section, we study the direct product of parameterized families of C^1 IFSs (cf. Definition 1.5). The main result is the following, stating that the property of the GTC is preserved under the direct product.

PROPOSITION 6.1. *Let $\ell \in \mathbb{N}$ with $\ell \ge 2$. Suppose that for $k = 1, \ldots, n$, $(\mathcal{F}_k^{t_k})_{t_k\in\Omega_k}$ is a parameterized family of C^1 IFSs on $Z_j \subset \mathbb{R}^{q_k}$, satisfying the GTC with respect to a locally finite Borel measure η_k on the metric space (Ω_k, d_{Ω_k}). Moreover, suppose all the individual IFSs have ℓ contractions. Set*

$$\mathcal{F}^{(t_1,\ldots,t_n)} = \mathcal{F}_1^{t_1} \times \cdots \times \mathcal{F}_n^{t_n}, \quad (t_1, \ldots, t_n) \in \Omega_1 \times \cdots \times \Omega_n.$$

Endow $\Omega := \Omega_1 \times \cdots \times \Omega_n$ with the product metric d_Ω as follows:

$$d_\Omega((t_1, \ldots, t_n), (s_1, \ldots, s_n)) = \left(\sum_{k=1}^n d_{\Omega_k}(s_k, t_k)^2\right)^{1/2}.$$

Then the family $\mathcal{F}^{(t_1,\ldots,t_n)}$, $(t_1, \ldots, t_n) \in \Omega_1 \times \cdots \times \Omega_n$, satisfies the GTC with respect to $\eta_1 \times \cdots \times \eta_n$.

To prove the above proposition, we need the following.

LEMMA 6.2.

(i) *Let A be a real non-singular $d \times d$ matrix with singular values $\alpha_1 \ge \cdots \ge \alpha_d$. Then for each $r > 0$,*

$$\min\left\{\frac{r^p}{\phi^p(A)} : p = 0, 1, \ldots, d\right\} = \prod_{i=1}^d \frac{\min\{\alpha_i, r\}}{\alpha_i},$$

where $\phi^s(\cdot)$ is the singular value function defined as in (2.5).

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(ii) *For $j = 1, \ldots, n$, let A_j be a real non-singular $d_j \times d_j$ matrix. Set*

$$M = \mathrm{diag}(A_1, \ldots, A_n) := \begin{bmatrix} A_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & A_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & A_n \end{bmatrix}.$$

Then

$$\begin{aligned}\min&\left\{\frac{r^p}{\phi^p(M)} : p = 0, 1, \ldots, d_1 + \cdots + d_n\right\} \\ &= \prod_{i=1}^n \min\left\{\frac{r^p}{\phi^p(A_i)} : p = 0, 1, \ldots, d_i\right\}.\end{aligned} \tag{6.1}$$

Proof. The proof of (i) is direct and simple. We leave it to the reader as an exercise. Part (ii) is just a consequence of (i), using the fact that the set of singular values (including the multiplicity) of M are precisely the union of those of A_i, $i = 1, \ldots, n$. □

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. Write

$$\mathcal{F}_k^{t_k} = \{f_{i,k}^{t_k}\}_{i=1}^{\ell}, \quad t_k \in \Omega_k,\ k = 1, \ldots, n.$$

For $\boldsymbol{\omega} = \omega_1 \ldots \omega_m \in \Sigma_*$, write $f_{\boldsymbol{\omega},k}^{t_k} = f_{\omega_1,k}^{t_k} \circ \cdots \circ f_{\omega_m,k}^{t_k}$. Let $\Pi_k^{t_k}$ denote the coding map associated with the IFS $\mathcal{F}_k^{t_k}$, and $\Pi^{(t_1,\ldots,t_n)}$ the coding map associated with the IFS $\mathcal{F}^{(t_1,\ldots,t_n)}$. According to the GTC assumption on the families $(\mathcal{F}_k^{t_k})_{t_k \in \Omega_k}$, $k = 1, \ldots, n$, there exist $\delta_0 > 0$ and a function $\psi : (0, \delta_0) \to [0, \infty)$ with $\lim_{\delta \to 0} \psi(\delta) = 0$ such that for every $\delta \in (0, \delta_0)$ and $(s_1, \ldots, s_n) \in \Omega_1 \times \cdots \times \Omega_n$, there is $C = C(\delta, s_1, \ldots, s_n) > 0$ satisfying the following: for each $k \in \{1, \ldots, n\}$, distinct $\mathbf{i}, \mathbf{j} \in \Sigma$ and $r > 0$,

$$\begin{aligned}&\eta_k\{t_k \in B_{\Omega_k}(s_k, \delta) : |\Pi_k^{t_k}(\mathbf{i}) - \Pi_k^{t_k}(\mathbf{j})| < r\} \\ &\quad \le C e^{|\mathbf{i} \wedge \mathbf{j}|\psi(\delta)} \inf_{x \in \Sigma} \min\left\{\frac{r^p}{\phi^p(D_{\Pi_k^{t_k} x} f_{\mathbf{i} \wedge \mathbf{j},k}^{t_k})} : p = 0, 1, \ldots, q_k\right\},\end{aligned} \tag{6.2}$$

where $B_{\Omega_k}(s_k, \delta)$ stands for the closed ball in Ω_k of radius δ centered at s_k. Writing $\mathbf{t} = (t_1, \ldots, t_n)$, $\mathbf{s} = (s_1, \ldots, s_n)$ and using (6.2),

$$\begin{aligned}&\eta_1 \times \cdots \times \eta_n\{\mathbf{t} \in B_\Omega(\mathbf{s}, \delta) : |\Pi^{\mathbf{t}}(\mathbf{i}) - \Pi^{\mathbf{t}}(\mathbf{j})| < r\} \\ &\le \prod_{k=1}^n \eta_k\{t_k \in B_{\Omega_k}(s_k, \delta) : |\Pi_k^{t_k}(\mathbf{i}) - \Pi_k^{t_k}(\mathbf{j})| < r\} \\ &\le \prod_{k=1}^n \left(C e^{|\mathbf{i} \wedge \mathbf{j}|\psi(\delta)} \inf_{x \in \Sigma} \min\left\{\frac{r^p}{\phi^p(D_{\Pi_k^{s_k} x} f_{\mathbf{i} \wedge \mathbf{j},k}^{s_k})} : p = 0, 1, \ldots, q_k\right\}\right) \\ &\le C^n e^{n|\mathbf{i} \wedge \mathbf{j}|\psi(\delta)} \inf_{x \in \Sigma} \prod_{k=1}^n \min\left\{\frac{r^p}{\phi^p(D_{\Pi_k^{s_k} x} f_{\mathbf{i} \wedge \mathbf{j},k}^{s_k})} : p = 0, 1, \ldots, q_k\right\}\end{aligned}$$

$$= C^n e^{n|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} \inf_{x\in\Sigma} \min\left\{\frac{r^p}{\phi^p(D_{\Pi^{\mathbf{t}} x} f^{\mathbf{s}}_{\mathbf{i}\wedge\mathbf{j}})} : \; p = 0, 1, \ldots, q_1 + \cdots + q_n\right\}$$
$$= C^n e^{n|\mathbf{i}\wedge\mathbf{j}|\psi(\delta)} Z^{\mathbf{s}}_{\mathbf{i}\wedge\mathbf{j}}(r),$$

where we have used (6.1) in the second last equality. Hence, the family $\mathcal{F}^{\mathbf{t}}$, $\mathbf{t} \in \Omega$, satisfies the GTC with respect to the measure $\eta_1 \times \cdots \times \eta_n$, where the involved constant and the function in the definition of the GTC are C^n and $n\psi(\cdot)$, respectively. □

7. *The proof of Theorem 1.6 and final questions*

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. This follows directly by combining Theorems 4.2, 5.1 and Proposition 6.1. □

Below we list a few 'folklore' open questions on the dimension of the attractors of C^1 IFSs. One may formulate the corresponding questions on the dimension of push-forwards of ergodic invariant measures on the attractors.

Question 7.1. Is it true that for every C^1 IFS $\mathcal{F} = \{f_i\}_{i=1}^{\ell}$ on $\mathbb{R}^d$ satisfying (1.8), there is a neighborhood Δ of $\mathbf{0}$ in $\mathbb{R}^{\ell d}$ such that for $\mathcal{L}_{\ell d}$-a.e. $\mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_\ell) \in \Delta$,

$$\dim_H K^{\mathbf{t}} = \dim_B K^{\mathbf{t}} = \min\{\dim_S \mathcal{F}^{\mathbf{t}},\ d\},$$

where $K^{\mathbf{t}}$ is the attractor of the IFS $\mathcal{F}^{\mathbf{t}} = \{f_i + \mathbf{t}_i\}_{i=1}^{\ell}$?

Question 7.2. Do we have

$$\dim_H K = \dim_B K = \min\{\dim_S \mathcal{F},\ d\}$$

for the attractor K of a 'generic' C^1 IFS $\mathcal{F}$ on $\mathbb{R}^d$ (in an appropriate sense)?

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