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NEW KINDS OF MULTIDIMENSIONAL IFR DISTRIBUTION

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Abstract

Two kinds of multidimensional IFR distribution are defined by using a partial order in $\boldsymbol{R}_+^n$, which is derived from a non-negative, strictly increasing function in $\boldsymbol{R}_+^n$. Some closure properties under operations and an application to a shock model are discussed.

1. Definitions

The one-dimensional IFR class of distributions plays an important role in reliability theory. Multidimensional IFR class of distributions should be considered when components in a system are not independent of each other. Many definitions of multidimensional IFR distributions have been presented by various authors; we present two of them below.

Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ denote an n-dimensional non-negative random vector, having the survival function

$$\bar{F}(t_1, \cdots, t_n) = P\{X_1 > t_1, \cdots, X_n > t_n\}, \quad t_i \geqq 0, \quad i = 1, \cdots, n. \tag{1.1}$$

We define a partial order $<_1$ in $\boldsymbol{R}_+^n$ as follows: $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{R}_+^n$, $\boldsymbol{s} <_1 \boldsymbol{t}$ if and only if every component of $\boldsymbol{s}$ is less than or equal to the respective component of $\boldsymbol{t}$.

Definition 1.1. (a) $\boldsymbol{X}$ belongs to the first kind of n-dimensional IFR class (denoted by $\boldsymbol{X} \in n$-IFR(I)), if $\bar{F}(\boldsymbol{s}+\boldsymbol{t})/\bar{F}(\boldsymbol{t})$ is decreasing in $\boldsymbol{t}$ about the partial order $<_1$ for all $\boldsymbol{s} \in \boldsymbol{R}_+^n$;

(b) $\boldsymbol{X}$ belongs to the second kind of n-dimensional IFR class (denoted by $\boldsymbol{X} \in n$-IFR(II)), if $\bar{F}(\boldsymbol{t}+\delta\boldsymbol{e})/\bar{F}(\boldsymbol{t})$ is decreasing in $\boldsymbol{t}$ about the partial order for all $\delta \geqq 0$, where $\boldsymbol{e} = (1, \cdots, 1)$, and n-IFR(I) $\subset n$-IFR(II) (see Marshall (1975)).

As in the one-dimensional case, they have a probabilistic meaning. Take $\boldsymbol{X} \in n$-IFR(I) as an example, since

$$P\{\boldsymbol{X} > \boldsymbol{s}+\boldsymbol{t} \mid \boldsymbol{X} > \boldsymbol{t}\} = \frac{\bar{F}(\boldsymbol{s}+\boldsymbol{t})}{\bar{F}(\boldsymbol{t})} \tag{1.2}$$

the conditional probability above is decreasing in $\boldsymbol{t}$ under the partial order $<_1$. That is, under the condition that all components survive at time $\boldsymbol{t}$, the residual life is stochastically decreasing in each component of $\boldsymbol{t}$. We give two definitions, such that the contribution of the age $\boldsymbol{X}$ to (1.2) is not $\boldsymbol{t}$, but a function of $\boldsymbol{t}: f(\boldsymbol{t})$.

Definition 1.2. Let $f(\boldsymbol{t})$ be a non-negative, strictly increasing function in $\boldsymbol{R}_+^n$. We introduce a partial order $<_f$ as follows: for $\boldsymbol{t}, \boldsymbol{t}' \in \boldsymbol{R}_+^n$, if the equality $f(\boldsymbol{t}) \leqq f(\boldsymbol{t}')$ holds, we call $\boldsymbol{t}$ less than $\boldsymbol{t}'$ about the function f, denoted by $\boldsymbol{t} <_f \boldsymbol{t}'$. We call $<_f$ the partial order induced by f.

Remark. The partial order $<_f$ lacks the anti-symmetry property. Because this property is not used in this paper, we also call it a partial order.

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Definition 1.3. (a) $\boldsymbol{X}$ belongs to the first kind of n-dimensional IFR class of distributions about f (denoted by $\boldsymbol{X} \in n$-IFR(I, f)), if $\bar{F}(\boldsymbol{s}+\boldsymbol{t})/\bar{F}(\boldsymbol{t})$ is decreasing in t about the partial order $<_f$ for all $\boldsymbol{s} \in \boldsymbol{R}^n_+$, that is for all $\boldsymbol{t} <_f \boldsymbol{t}'$, $\boldsymbol{s} \in \boldsymbol{R}^n_+$,

$$\frac{\bar{F}(\boldsymbol{s}+\boldsymbol{t})}{\bar{F}(\boldsymbol{t})} \geqq \frac{\bar{F}(\boldsymbol{s}+\boldsymbol{t}')}{\bar{F}(\boldsymbol{t}')}.$$

(b) $\boldsymbol{X}$ belongs to the second kind of n-dimensional IFR class about f (denoted by $\boldsymbol{X} \in n$-IFR(II, f)), if $\bar{F}(\boldsymbol{t}+\delta \boldsymbol{e})/\bar{F}(\mathbf{t})$ is decreasing in $\boldsymbol{t}$ about the partial order $<_f$ for all $\delta \geqq 0$.

Obviously we have n-IFR(I, f) $\subset n$-IFR(II, f).

For $f(\boldsymbol{t})$ as defined above, the definition has a probabilistic meaning. That is, as $f(\boldsymbol{t})$ increases, the conditional residual life of $\boldsymbol{X}$ is stochastically decreasing.

From $\boldsymbol{t} <_1 \boldsymbol{t}'$, we can get $\boldsymbol{t} <_f \boldsymbol{t}'$, so we have n-IFR(i, f) $\subset n$-IFR(i), $i =$ I, II.

2. Properties

Definition 2.1. Let h be a non-negative function on $\boldsymbol{R}^n$. If for arbitrary $\boldsymbol{t} <_f \boldsymbol{t}'$, $\boldsymbol{s} <_f \boldsymbol{s}'$, we have

$$\begin{bmatrix} h(\boldsymbol{t}-\boldsymbol{s}) & h(\boldsymbol{t}-\boldsymbol{s}') \\ h(\boldsymbol{t}'-\boldsymbol{s}) & h(\boldsymbol{t}'-\boldsymbol{s}') \end{bmatrix} \geqq 0. \tag{2.1}$$

Then we call h the first kind of Pólya function of order 2 about f, denoted by $h \in \mathrm{PF}_2(\mathrm{I}, f)$. If (2.1) holds for $\boldsymbol{s} = \delta \boldsymbol{e}$, $\boldsymbol{s}' = \delta' \boldsymbol{e}$, $\delta \leqq \delta'$, and arbitrary $\boldsymbol{t} <_f \boldsymbol{t}'$, we call h the second kind of Pólya function of order 2 about f, denoted by $h \in \mathrm{PF}_2(\mathrm{II}, f)$.

Theorem 2.1. If f is additive, then $F \in n$-IFR(i, f) if and only if $F \in \mathrm{PF}_2(i, f)$, $i =$ I, II.

Proof. We prove only the case where $i =$ I. The case where $i =$ II is similar. If $\mathbf{0} <_f \boldsymbol{v}$, we have

$$\frac{\bar{F}(\boldsymbol{u}+\boldsymbol{w})}{\bar{F}(\boldsymbol{u})} \geqq \frac{\bar{F}(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})}{\bar{F}(\boldsymbol{u}+\boldsymbol{v})}. \tag{2.2}$$

For arbitrary $\boldsymbol{t} <_f \boldsymbol{t}'$, $\boldsymbol{s} <_f \boldsymbol{s}'$, let

$$\boldsymbol{u} = \boldsymbol{t} - \boldsymbol{s}', \qquad \boldsymbol{v} = \boldsymbol{t}' - \boldsymbol{t}, \qquad \boldsymbol{w} = \boldsymbol{s}' - \boldsymbol{s}.$$

From the additivity of f, we can get $\mathbf{0} <_f \boldsymbol{v}$, then (2.2) shows that (2.1) holds.

Theorem 2.2. (a) If $\boldsymbol{X} \in n$-IFR(i, f), $i =$ I, II, then all components X_j of $\mathbf{X}$ are IFR.

(b) If $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots$ converge weakly to $\boldsymbol{X}$, and $\boldsymbol{X}_k \in n$-IFR(i, f), then $\boldsymbol{X} \in n$-IFR(i, f), $i =$ I, II.

(c) If f is exchangeable and $(X_1, X_2, \cdots, X_n) \in n$-IFR($i, f$), then for an arbitrary permutation π of $(1, 2, \cdots, n)$, $(X_{\pi(1)}, \cdots, X_{\pi(n)}) \in n$-IFR($i, f$), $i =$ I, II.

(d) If f is additive and $f(-\boldsymbol{s}) = -f(\boldsymbol{s})$, $\boldsymbol{X}$, $\boldsymbol{Y}$ are non-negative, n-dimensional random vectors, g is the density function of $\boldsymbol{Y}$, and for arbitrary $\boldsymbol{o} <_f \boldsymbol{s}$,

$$\iint_{\{\boldsymbol{y}, \boldsymbol{z} : f(\boldsymbol{y}) = f(\boldsymbol{z})\}} g(\boldsymbol{y}) g(\boldsymbol{z}+\boldsymbol{s}) \, d\boldsymbol{y} \, d\boldsymbol{z} = 0.$$

(1) If $\boldsymbol{X} \in n$-IFR(I, f) and $g \in \mathrm{PF}_2(\mathrm{I}, f)$, then $\boldsymbol{X} + \boldsymbol{Y} \in n$-IFR(I, f);

(2) If $\boldsymbol{x} \in n$-IFR(I, f) and $g \in \mathrm{PF}_2(\mathrm{II}, f)$, then $\boldsymbol{X} + \boldsymbol{Y} \in n$-IFR(II, f).

Proof. (a)–(c) are obvious.

(d) We prove only (1); (2) is similar.

$$\bar{H}(\boldsymbol{t}) = P\{\boldsymbol{X} + \boldsymbol{Y} > \boldsymbol{t}\} = \int_{\boldsymbol{y}} \bar{F}(\boldsymbol{t} - \boldsymbol{y}) g(\boldsymbol{y}) \, d\boldsymbol{y}. \tag{2.3}$$

Let $g(\boldsymbol{y}) = 0$ if some components of $\boldsymbol{y}$ are negative. Then (2.3) can be considered as the integration on $\boldsymbol{R}^n$. If $\boldsymbol{t} <_f \boldsymbol{t}'$, $\boldsymbol{s} <_f \boldsymbol{s}'$,

$$\begin{aligned}
\bar{H}(\boldsymbol{t}+\boldsymbol{s})\bar{H}(\boldsymbol{t}) - \bar{H}(\boldsymbol{t}+\boldsymbol{s})H(\boldsymbol{t}) &= \iint_{\boldsymbol{y},\boldsymbol{z}} [\bar{F}(\boldsymbol{t}-\boldsymbol{y})\bar{F}(\boldsymbol{t}-\boldsymbol{z}) - \bar{F}(\boldsymbol{t}-\boldsymbol{y})\bar{F}(\boldsymbol{t}-\boldsymbol{z})] g(\boldsymbol{y}+\boldsymbol{s})g(\boldsymbol{z})\,d\boldsymbol{y}\,d\boldsymbol{z} \\
&= \iint_{f(\boldsymbol{y})<f(\boldsymbol{z})} + \iint_{f(\boldsymbol{y})>f(\boldsymbol{z})} \\
&= \iint_{\boldsymbol{y}<_f\boldsymbol{z}} [\bar{F}(\boldsymbol{t}-\boldsymbol{y})\bar{F}(\boldsymbol{t}-\boldsymbol{z}) - \bar{F}(\boldsymbol{t}-\boldsymbol{y})\bar{F}(\boldsymbol{t}-\boldsymbol{z})] \\
&\quad \times [g(\boldsymbol{y}+\boldsymbol{s})g(\boldsymbol{z}) - g(\boldsymbol{y})g(\boldsymbol{z}+\boldsymbol{s})]\,d\boldsymbol{y}\,d\boldsymbol{z} \\
&\geqq 0
\end{aligned}$$

then $\bar{H}(\boldsymbol{t}+\boldsymbol{s})/\bar{H}(\boldsymbol{t})$ is decreasing in $\boldsymbol{t}$ under the partial order $<_f$.

3. Multidimensional shock model

Suppose $\boldsymbol{N}(t) = (N_1(t), \cdots, N_n(t))$ is an n-dimensional shock process, where $N_1(t), \cdots, N_n(t)$ are independent Poisson processes with parameters $\lambda_1, \cdots, \lambda_n$. Under the condition $\boldsymbol{N}(t) = (k_1, \cdots, k_n)$, the probability that the system survives till time t is $\bar{P}_{k_1,\cdots,k_n}$; then the probability that the system survives till time t without failure is

$$\bar{H}(t) = \sum_{k_1,\cdots,k_n=0}^{\infty} \bar{P}_{k_1,\cdots,k_n} \frac{(\lambda_1 t)^{k_1}}{k_1!} \cdots \frac{(\lambda_n t)^{k_n}}{k_n!} \exp(-(\lambda_1 + \cdots + \lambda_n)t). \tag{3.1}$$

Theorem 3.1. If $P_{k_1,\cdots,k_n} \in n\text{-IFR}(\mathrm{I}, f)$, where $f(k_1, \cdots, k_n) = \sum_{i=1}^n k_i$, then $H \in \mathrm{IFR}$.

Proof. We need only prove that for $t < t'$, $x \geqq 0$,

$$\Delta = \bar{H}(x+t)\bar{H}(t') - \bar{H}(x+t')\bar{H}(t) \geqq 0.$$

Let $\Lambda = \sum_{i=0}^n \lambda_i$, then

$$\begin{aligned}
\bar{H}(x+t) &= \sum_{k_1,\cdots,k_n=0}^{\infty} \bar{P}_{k_1,\cdots,k_n} \frac{\lambda_1^{k_1}(x+t)^{k_1}}{k_1!} \cdots \frac{\lambda_n^{k_n}(x+t)^{k_n}}{k_n!} \exp(-\Lambda(x+t)) \\
&= \sum_{l_1,\cdots,l_n=0}^{\infty} \sum_{k_1,\cdots,k_n=0}^{\infty} \bar{P}_{k_1+l_1,\cdots,k_n+l_n} \frac{(\lambda_1 x)^{l_1} \cdots (\lambda_n x)^{l_n} (\lambda_1 t)^{k_1} \cdots (\lambda_n t)^{k_n}}{l_1! \cdots l_n!\, k_1! \cdots k_n!} \exp(-\Lambda(x+t)).
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta \exp \Lambda(x+t+t') &= \sum_{l_1,\cdots,l_n=0}^{\infty} \frac{(\lambda_1 x)^{l_1}}{l_1!} \cdots \frac{(\lambda_n x)^{l_n}}{l_n!} \sum_{m_1,\cdots,m_n=0}^{\infty} \sum_{k_1,\cdots,k_n=0}^{\infty} \bar{P}_{m_1,\cdots,m_n} \\
&\quad \times \bar{P}_{k_1+l_1,\cdots,k_n+l_n} \frac{\lambda_1^{k_1} \cdots \lambda_n^{k_n} \lambda_1^{m_1} \cdots \lambda_n^{m_n}}{k_1! \cdots k_n!\, m_1! \cdots m_n!} [t^K t'^M - t^M t'^K] \\
&= \sum_{l_1,\cdots l_n=0}^{\infty} \frac{(\lambda_1 x)^{l_1}}{l_n!} \cdots \frac{(\lambda_n x)^{l_n}}{l_n!} \sum_{K<M} [t^K t'^M - t^M t'^K] \frac{\lambda_1^{k_1} \cdots \lambda_n^{k_n} \lambda_1^{m_1} \cdots \lambda_n^{m_n}}{k_1! \cdots k_n!\, m_1! \cdots m_n!} \\
&\quad \times [\bar{P}_{m_1,\cdots,m_n} \bar{P}_{k_1+l_1,\cdots,k_n+l_n} - \bar{P}_{m_1+l_1,\cdots,m_n+l_n} \bar{P}_{k_1,\cdots,k_n}] \\
&\geqq 0
\end{aligned}$$

where $K = \sum_{i=1}^n k_i$, $M = \sum_{i=1}^n m_i$, so that $\Delta \geqq 0$..

Reference

Marshall, A. W. (1975) Multivariate distributions with monotone hazard rate. *Reliability and Fault Tree Analysis,* SIAM, Philadelphia, 259–284.