Adv. Appl. Prob. **21,** 235–236 (1989)
Printed in N. Ireland

A NOTE ON COWAN'S PROCESS OF SPACE DIVISION

J. MECKE,* *University of Jena*

Abstract

Relations of a Markov chain describing the repeated splitting of polygons to processes known from queueing theory are pointed out.

R. Cowan presents an interesting interated random division procedure for polygons in his letter on pp. 233-234, which is related to the paper of Cowan and Morris (1988). His last model leads to a sequence of random polygons $P_0 \supset P_1 \supset P_2 \supset \cdots$ the number of sides of which are denoted by $X_0, X_1, X_2, \cdots$ respectively. The random variables $Y_n = X_n - 3$ $(n = 0, 1, 2, \cdots)$ form a homogeneous Markov chain with state space $\{0, 1, 2, \cdots)$ and transition probabilities $\mathrm{pr}(Y_{n+1} = s \mid Y_n = r) = 1/(r+2)$ if $0 \leq s \leq r+1$, 0 otherwise. The Poisson distribution with mean 1 is an equilibrium distribution for the Markov chain $Y_0, Y_1, Y_2, \cdots$. This fact is derived by direct computation.

The aim of this letter is to represent the process $Y_0, Y_1, Y_2, \cdots$ as an imbedded Markov chain with respect to a special birth and death process, for which it is well known that the Poisson distribution with mean 1 is an equilibrium distribution.

There is an analogue of the Cowan–Morris result in queueing theory. Let $Z(t)$ be the number of customers at time t in a service system with an infinite number of servers, Poisson input of intensity λ, exponentially distributed service times with mean $1/\lambda$ and assumptions of maximal independence (an $M/M/\infty$ system). Then the equilibrium distribution for the Z process is the Poisson distribution with mean 1, cf. for example the Erlang formulas for service systems in Khinchin (1960). Now let $T_1, T_2, \cdots$ be the instants where customers arrive. Then the sequence $Z_1 = Z(T_1 - 0)$, $Z_2 = Z(T_2 - 0)$, $\cdots$ forms a homogeneous Markov chain, which has clearly the same equilibrium distribution as the process $Z(t)$. It can be shown that the transition probabilities of the Markov chains $Z_1, Z_2, \cdots$ and $Y_1, Y_2, \cdots$ coincide.

The mathematical essence of this model is the following. We consider a homogeneous birth and death process $Z(t)$ on the time half-axis $[0, \infty)$ with state space $\{0, 1, 2, \cdots\}$, constant birth rate $\lambda_n = \lambda > 0$ and death rate $\mu_n = n\lambda$; $n = 0, 1, \cdots$. If $T_1, T_2, \cdots$ are the instants where birth occurs, then the sequence of random variables $Z_1 = Z(T_1 - 0)$, $Z_2 = Z(T_2 - 0)$, $\cdots$ forms a homogeneous Markov chain with the same transition probabilities as the chain $Y_1, Y_2, \cdots$.

Having this in mind, we may think of the polygon splitting process as generated by a birth and death process in the following way. Consider the process $C_0, C_1, C_2, \cdots$ of corners of the polygons $P_0 \supset P_1 \supset P_2 \supset \cdots$. Suppose P_n has C_n corners, numbered counterclockwise from a random reference corner as $1, 2, \cdots, C_n$. The polygon P_{n+1} may be viewed as the result of choosing a point on side $C_n \to 1$, and cutting from there to a point on side $k \to k+1$, where k is chosen at random from $\{1, 2, \cdots, C_n - 1\}$, and taking P_{n+1} as the polygon containing the corner 1. Then P_{n+1} has $k+2$ corners. The stochastic behaviour of the C process is just as if the surplus corners $C - 3$ each died at rate λ and new corners are introduced at rate λ, with this process being sampled at instants of introduction of new corners.

Acknowledgement

I thank the referee for the suggestion presented in the last paragraph of this letter.

Received 23 September 1988; revision received 24 November 1988.
* Postal address: Department of Mathematics, The University, UHH, Jena, DDR-6900, GDR.

https://doi.org/10.2307/1427209 Published online by Cambridge University Press

References

COWAN, R. AND MORRIS, V. B. (1988) Division rules for polygonal cells. *J. theor. Biol.* **131,** 33–42.
COWAN, R. (1989) The division of space and the Poisson distribution. *Adv. Appl. Prob.* **21,** 233–234.
KHINCHIN, A. Y. (1960) *Mathematical Methods in the Theory of Queueing*. Griffin, London.