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COMPATIBLE TIGHT RIESZ ORDERS ON THE GROUP OF AUTOMORPHISMS OF AN 0-2-HOMOGENEOUS SET: ADDENDUM

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The purpose of this note is to show that Theorem 8 of Davis and Fox [**1**] is sharp. That is, we show that the following result is valid.

THEOREM. *Let* Ω *be an* 0-2-*homogeneous ordered set. Then* T_ρ (*respectively*, T_λ) *is a maximal compatible tight Riesz order if and only if* Ω *has a countable cofinal* (*respectively*, *coinitial*) *subset.*

We firstly describe the candidate for a compatible tight Riesz order. For $g \in A(\Omega)$, a *zero interval* of g is a maximal convex subset Δ of $\Omega \backslash \text{supp}(g)$ with $|\Delta| > 1$. Let S be the set of all $1 \leqq g \notin T_\rho$ whose support is unbounded above and which satisfy the following:

There exists $x \in \Omega$ such that for all sequences $\Delta_1 < \Delta_2 < \ldots$ of zero intervals of g with $x \leqq \Delta_1$, there is a $\bar{z} \in \bar{\Omega}$—the Dedekind completion of Ω—such that $\bar{z} > \bar{y} = \sup \{\Delta_i : i = 1, 2, \ldots\} \in \bar{\Omega}$, and $\text{supp}(g) \cap (\bar{y}, \bar{z})$ is dense in $(\bar{y}, \bar{z}) \subseteq \bar{\Omega}$.

When Ω has no countable cofinal subset it is clear that

$$T(S) = \{g \in A^+(\Omega) : \text{supp}(g) \supseteq \text{supp}(h_1 \wedge \ldots \wedge h_n) \quad \text{for some } h_1, \ldots, h_n \in S\}$$

is strictly larger than T_ρ.

Proof of Theorem. Assume that Ω has no countable cofinal subset. Take any $x \in \Omega$ and let Γ be a cofinal subset of Ω that is well-ordered by the induced order from Ω and for which $x < \Gamma$. For each $a \in \Gamma$ choose $b \in \Omega$ such that $a < b < a + 1$ (where $a + 1$ is the successor of a in Γ). As in [**1**] we can construct $h \in A^+(\Omega)$ for which $([x, \infty] \backslash \bigcup_{a \in \Gamma} [b, a + 1]) \cap \text{supp}(h)$ is dense in $[x, \infty) \backslash \bigcup_{a \in \Gamma} [b, a + 1]$ whilst $(-\infty, x] \subseteq \Omega \backslash \text{supp}(h)$ and, for each $a \in \Gamma$, $[b, a + 1] \subseteq \Omega \backslash \text{supp}(h)$. It is not difficult to see that $h \in S$, so that $\inf S = 1$. Clearly S is a normal subset of $A^+(\Omega)$. Now suppose that $h_1, \ldots, h_n \in S$, and let $\Delta_{i,j}$ ($i = 1, \ldots, n$ and $j = 1, 2, \ldots$) be zero intervals of h_i satisfying $\Delta_{i,j} < \Delta_{i+1,j} < \Delta_{i,j+1}$ for all $i = 1, \ldots, n - 1$ and $j = 1, 2, \ldots$. Then there is a $\bar{z} \in \bar{\Omega}$ such that $\bar{z} > \bar{y} = \sup \{\Delta_{i,j} : j = 1, 2, \ldots\}$ and $(\bar{y}, \bar{z}) \cap \text{supp}(h)$ is dense in $(\bar{y}, \bar{z})$ for each $i = 1, \ldots, n$ and therefore $h_1 \wedge \ldots \wedge h_n \neq 1$. A

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straightforward calculation now shows that

$$T(S) = \{g \in A^+(\Omega) : \text{supp } (g) \supseteq \text{supp } (h_1 \wedge \ldots \wedge h_n) \text{ for some } h_1, \ldots, h_n \in S\}$$

is a compatible tight Riesz order.

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References

1. G. E. Davis and C. D. Fox, *Compatible tight Riesz orders on the automorphism group of an 0-2-homogeneous set*, Can. J. Math. *28* (1976), 1076–1081.

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