Approximation by a composition of Chlodowsky operators and Százs–Durrmeyer operators on weighted spaces

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Abstract

In this paper we deal with the operators

$$Z_n(f;x) = \frac{n}{b_n} \sum\nolimits_{k=0}^{n} p_{n,k}\left(\frac{x}{b_n}\right) \int_0^\infty s_{n,k}\left(\frac{t}{b_n}\right) f(t)\, dt, \quad 0 \leqslant x \leqslant b_n$$

and study some basic properties of these operators where $p_{n,k}(u) = \binom{n}{k} u^k (1-u)^{n-k}$, $(0 \leqslant k \leqslant n)$, $u \in [0,1]$ and $s_{n,k}(u) = e^{-nu} (nu)^k/k!$, $u \in [0,\infty)$. Also, we establish the order of approximation by using weighted modulus of continuity.

1. Introduction

Let ρ be a polynomial or exponential function such that continuous, monotonically increasing growths to infinity on $[0, \infty)$ satisfy the condition $\rho(x) \geqslant 1$. Classes of functions which satisfy the condition $|f(x)| \leqslant M_f \times \rho(x)$ with the norm $\|f\|_\rho = \sup_{x \in [0,\infty)} |f(x)|/\rho(x)$ are said to be weighted spaces. There are many studies on approximation on weighted spaces. For instance Ditzian [6] used polynomial weight for Baskakov operators and exponential weight for Szász–Mirakjan operators. May [22], proved local equivalence theorems on shrinking intervals, using the more general 'operator of exponential type' of which the Baskakov and the Szász–Mirakjan operators are special examples. Lešniewicz and Rempulska [21] introduced some linear positive operators of the Szász–Mirakjan type in the space of continuous functions of one and two variables, having exponential growth at infinity. Gadjiev [10, 11] defined the weight spaces C_ρ and B_ρ of real functions defined on the real axis and showed that Korovkin's theorem in general does not hold on these spaces. Here

$$B_\rho := \{f : |f| \leqslant M_f \rho, \ \rho \geqslant 1 \text{ and } \rho \text{ is unbounded}\}$$

and

$$C_\rho := \{f \in B_\rho \mid f \text{ continuous}\}$$

are spaces of functions which are defined on unbounded sets. However, in [10] and [11] it has been shown that this theorem holds on a common subspaces of C_ρ and B_ρ. Coşkun [4] proved that a theorem of Korovkin type does not hold on the spaces C_{ρ_1} and B_{ρ_2} with different weights ρ_1 and ρ_2, respectively. Later, in [5], Coşkun showed that by some appropriate conditions on the weight functions it holds. Ispir [19] studied modified Baskakov operators on weighted spaces by means of Korovkin's theorems, proved by Gadjiev. We can refer the interested readers to other studies which investigated weight approximation problems for well-known linear positive operators (see, e.g., [3, 7, 12, 13, 16, 20, 24]) and for general positive linear operators (see, e.g., [4, 8, 10, 11]) on infinite intervals.

2. *Preliminaries*

İbikli and Karslı[17] defined integral modifications of Bernstein–Chlodowsky operators by means of Durrmeyer so as to approximate functions of bounded variation on the interval $[0, b_n]$ which grows to positive semi-axis where $n \to \infty$. İbikli and others [18] studied these operators for Lebesgue L_p-integrable functions on the interval $[0, b_n]$. Mazhar and Totik [23] introduced the integral modification of the Szász–Mirakjan operators to approximate functions defined on $[0, \infty)$. These operators and their modifications have been studied by many authors (see, e.g., [14, 15]).

This study combines Chlodowsky operators and Szász–Durrmeyer operators.

Bernstein–Durrmeyer operators [9] are given by

$$M_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) \, dt \tag{1}$$

where

$$p_{n,k}(u) = \begin{cases} \binom{n}{k} u^k (1 - u)^{n-k} , \ 0 \leqslant k \leqslant n, \\ 0, k > n. \end{cases}$$

Szász–Durrmeyer operators [23] are given by

$$S(f; x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t) \, dt \tag{2}$$

where $s_{n,k}(x) = e^{-nx} (nx)^k / k!$, $x \in [0, \infty)$.

Chlodowsky [3] operators are given by

$$C_n(f; x) = \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{b_n}\right) f\left(\frac{k}{n} b_n\right), \quad 0 \leqslant x \leqslant b_n \tag{3}$$

where (b_n) is a positive and increasing sequence with properties $\lim_{n \to \infty} b_n = \infty$ and $\lim_{n \to \infty} b_n / n = 0$.

In this study we define the following operators which are combinations of (3) and (2):

$$Z_n(f; x) = \frac{n}{b_n} \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{b_n}\right) \int_0^{\infty} s_{n,k}\left(\frac{t}{b_n}\right) f(t) \, dt, \quad 0 \leqslant x \leqslant b_n \tag{4}$$

where (b_n) is given in (3) with the difference (see [16])

$$\lim_{n \to \infty} \frac{b_n^2}{n} = 0.$$

We use the weighted Korovkin's type theorems, proved by Gadjiev [10, 11], and we use the same notation as in [10].

Let $\rho(x) = 1 + x^2$, $x \in (-\infty, \infty)$ and B_ρ be the set of all functions f defined on the real axis satisfying the condition

$$|f(x)| \leqslant M_f \rho(x) \tag{5}$$

where M_f is a constant depending only on f. Here B_ρ is the normed space with the norm

$$\|f\|_\rho = \sup_{x \in (-\infty, \infty)} \frac{|f(x)|}{\rho(x)}, \quad f \in B_\rho.$$

Here C_ρ denotes the subspace of all continuous functions in B_ρ and C_ρ^k denotes the subspace of all functions $f \in C_\rho$ with

$$\lim_{|x| \to \infty} \frac{|f(x)|}{\rho(x)} = K_f < \infty$$

where K_f is a constant depending only on f.

THEOREM A [**10**, **11**]. *Let $\{T_n\}$ be a sequence of linear positive operators taking C_ρ into B_ρ and satisfying the conditions*

$$\lim_{n \to \infty} \|T_n(t^v; x) - x^v\|_\rho = 0, \quad v = 0, 1, 2.$$

Then for any $f \in C_\rho^k$,

$$\lim_{n \to \infty} \|T_n f - f\|_\rho = 0$$

and there exist a function $g \in C_\rho \setminus C_\rho^k$ such that

$$\lim_{n \to \infty} \|T_n g - g\|_\rho \geqslant 1.$$

Applying Theorem A to the operators

$$T_n(f; x) = \begin{cases} V_n(f; x) & \text{if } x \in [0, a_n] \\ f(x) & \text{if } x > a_n \end{cases}$$

then gives the following theorem.

THEOREM B [**12**]. *Let (a_n) be a sequence with $\lim_{n \to \infty} a_n = \infty$ and $\{V_n\}$ be a sequence of linear positive operators mapping $C_\rho[0, a_n]$ into $B_\rho[0, a_n]$.*

If for $v = 0, 1, 2$

$$\lim_{n \to \infty} \|V_n(t^v; x) - x^v\|_{\rho, [0, a_n]} = 0,$$

then for any $f \in C_\rho^k[0, a_n]$ we have

$$\lim_{n \to \infty} \|V_n f - f\|_{\rho, [0, a_n]} = 0,$$

where $B_\rho[0, a_n], C_\rho[0, a_n]$ and $C_\rho^k[0, a_n]$ denote the same as B_ρ, C_ρ and C_ρ^k, respectively, but the functions are taken on $[0, a_n]$ instead of $\mathbb{R}$ and the norm is taken as

$$\|f\|_{\rho, [0, a_n]} = \sup_{x \in [0, a_n]} \frac{|f(x)|}{\rho(x)}.$$

3. Auxiliary results

In this section we give some properties of $Z_n(f; x)$ which we use for the main theorems and an approximation theorem by means of Korovkin's theorem.

If we use properties of the derivative and Leibniz formula, it will be easy to see the following equality for $Z_n(f; x)$:

$$Z_n(t^p; x) = \left(\frac{b_n}{n}\right)^p \sum_{i=0}^{p} \binom{p}{i} \frac{p!}{i!} \frac{n!}{(n-i)!} \left(\frac{x}{b_n}\right)^i. \tag{6}$$

Now we will give some special cases of (6) for some p:

$$Z_n(1; x) = 1 \tag{7}$$

$$Z_n(t; x) = x + \frac{b_n}{n} \tag{8}$$

$$Z_n(t^2; x) = x^2 + \frac{x(4b_n - x)}{n} + \frac{2b_n^2}{n^2} \tag{9}$$

$$Z_n(t^3; x) = x^3 + \frac{x[18b_n^2 + 9(n-1)b_n x - (3n-2)x^2]}{n^2} + \frac{6b_n^3}{n^3} \tag{10}$$

$$Z_n(t^4; x) = x^4 + \frac{x}{n^3}[96b_n^3 + 72(n-1)b_n^2 x$$
$$+ 16(n-1)(n-2)b_n x^2 - (6n^2 - 11n + 6)x^3] + \frac{24b_n^4}{n^4}. \tag{11}$$

LEMMA 1. *We have*

$$Z_{n,m}(x) := Z_n((t-x)^m; x) = \left(\frac{b_n}{n}\right)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j n^j$$

$$\times \sum_{i=0}^{m-j} \binom{m-j}{i} \frac{(m-j)!}{i!} \frac{n!}{(n-i)!} (\frac{nx}{b_n})^{i+j}. \tag{12}$$

Proof. It is easy to see the proof by the using the equality

$$(t-x)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j x^j t^{m-j}$$

and linearity of the operator Z_n.

By simple calculations, we get

$$Z_{n,1}((t-x); x) = \frac{b_n}{n} \tag{13}$$

$$Z_{n,2}(x) = Z_n((t-x)^2; x) = \frac{x[2b_n - x]}{n} + \frac{2b_n^2}{n^2} \tag{14}$$

$$Z_{n,2}(x) \leqslant \frac{2(nx + b_n)b_n}{n^2}$$

$$Z_{n,4}(x) = Z_n((t-x)^4; x) = \frac{x}{n^3}[3(n-2)x^3 - 4(n-8)b_n x^2$$

$$- 12(6n-1)b_n^2 x + 72b_n^3] + \frac{24b_n^4}{n^4} \tag{15}$$

$$\sup_{x \in [0,b_n]} Z_{n,2}(x) \leqslant \frac{3b_n^2}{n} \tag{16}$$

$$\sup_{x \in [0,b_n]} Z_{n,4}(x) \leqslant \frac{3(n+47)b_n^4}{n^3} \leqslant 144 \left(\frac{b_n^2}{n}\right)^2. \tag{17}$$

□

4. Approximation of $Z_n(f; x)$ in weighted spaces

Let (b_n) is be a sequence has positive terms, increasing and having the following conditions,

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n^2}{n} = 0. \tag{18}$$

LEMMA 2. *The sequence $\{Z_n\}$ defined by (4) is a sequence of linear positive operators taking $C_\rho[0, b_n]$ into $B_\rho[0, b_n]$.*

Proof. In order to prove this lemma, it is enough to prove $\lim_{n\to\infty} Z_n(\rho(t); x) = \rho(x)$ uniformly on $[0, b_n]$ since $\rho(x) \in C_\rho[0, b_n]$. By using (7) and (9), we have

$$Z_n(\rho(t); x) = \rho(x) + \frac{x(4b_n - x)}{n} + \frac{2b_n^2}{n^2}.$$

Therefore, $\|Z_n(f; x)\|_{\rho,[0,b_n]}$ is uniformly bounded on $[0, b_n]$ because of

$$\lim_{n\to\infty} \sup_{x \in [0,b_n]} \frac{4b_n^2}{n} + \frac{2b_n^2}{n^2} = 0$$

under condition (18). The proof is complete. □

THEOREM 1. *Let $f \in C_\rho^k[0, \infty)$. Then*

$$\lim_{n \to \infty} \|Z_n f - f\|_{\rho,[0,b_n]} = 0.$$

Proof. Using (7)–(9), we have

$$\lim_{n \to \infty} \|Z_n(1; x) - 1\|_{\rho,(0,b_n]} = 0,$$

$$\lim_{n \to \infty} \|Z_n(t; x) - x\|_{\rho,[0,b_n]} = \lim_{n \to \infty} \sup_{x \in [0,b_n]} \left| \frac{b_n}{n} \frac{1}{1 + x^2} \right|$$

$$= \lim_{n \to \infty} \frac{b_n}{n} = 0,$$

$$\lim_{n \to \infty} \|Z_n(t^2; x) - x^2\|_{\rho,[0,b_n]} = \lim_{n \to \infty} \sup_{x \in [0,b_n]} \left| \left(\frac{x(4b_n - x)}{n} + \frac{2b_n^2}{n^2} \right) \frac{1}{1 + x^2} \right|$$

$$= \lim_{n \to \infty} \frac{2(n + b_n)b_n}{n^2} = 0.$$

According to Theorem B, the proof is complete.   □

5. Rate of approximation of $Z_n(f; x)$ in weighted spaces

Now we want to find rate of approximation of the sequence of linear positive operators $\{Z_n\}$ for $f \in C_\rho^k[0, b_n]$. It is well known that the first modulus of continuity

$$\omega(f; \delta) = \sup\{|f(t) - f(x)| : t, x \in [a, b], |t - x| \leqslant \delta\}$$

does not tend to zero, as $\delta \to 0$, on any infinite interval.

Ispir [19] defined a weighted modulus of continuity $\Omega_n(f; \delta)$ which tends to zero, as $\delta \to 0$, on an infinite interval. A similar definition can be found in [1].

For each $f \in C_\rho^k[0, b_n]$ given by

$$\Delta_n(f; \delta) = \sup_{\substack{|h| \leqslant \delta \\ x \in [0,b_n]}} \frac{|f(x + h) - f(x)|}{(1 + x^2)(1 + h^2)}. \tag{19}$$

In [19] Ispir showed the following properties of $\Delta_n(f; \delta)$:
 (i) $\lim_{\delta \to 0} \Delta n(f; \delta) = 0$ for every $f \in C_\rho^k[0, b_n]$;
 (ii) for every $f \in C_\rho^k[0, b_n]$ and $t, x \in [0, b_n]$,
    $$|f(t) - f(x)| \leqslant 2(1 + \delta_n^2)(1 + x^2)\Delta_n(f; \delta_n).S_n(t, x),$$

where

$$S_n(t, x) = \left(1 + \frac{|t - x|}{\delta_n} \right)(1 + (t - x)^2).$$

It is easy to see that

$$S_n(t, x) \leqslant \begin{cases} 2(1 + \delta_n^2) & \text{if } |t - x| \leqslant \delta_n \\ 2(1 + \delta_n^2)\dfrac{(t - x)^4}{\delta_n^4} & \text{if } |t - x| \geqslant \delta_n. \end{cases} \tag{20}$$

THEOREM 2. *Let $f \in C_\rho^k[0, b_n]$. Then for all n sufficiently large,*

$$\|Z_n f - f\|_{\rho,[0,b_n]} \leqslant K.\Delta_n\left(f; \sqrt{\frac{b_n^2}{n}} \right).$$

where K is a positive constant.

Proof. If we use (7) and the properties of linear positive operators Z_n

$$|Z_n(f; x) - f(x)| \leqslant Z_n(|f(t) - f(x)|; x)$$
$$\leqslant 2(1 + \delta_n^2)(1 + x^2)\Delta_n(f; \delta_n)Z_n(S_n(t, x); x).$$

By (20) we get

$$S_n(t, x) \leqslant 2(1 + \delta_n^2)\left[1 + \frac{(t - x)^4}{\delta_n^4}\right]$$

for all $x \in [0, b_n]$, $t \in [0, \infty)$. Thus, for $x \in [0, b_n]$ and using (17),

$$|Z_n(f; x) - f(x)| \leqslant 4(1 + \delta_n^2)^2(1 + x^2)\left[1 + \frac{1}{\delta_n^4}Z_{n,4}(x)\right]\Delta_n(f; \delta_n)$$

$$\leqslant 4(1 + \delta_n^2)^2(1 + x^2)\left[1 + \frac{144}{\delta_n^4}\left(\frac{b_n^2}{n}\right)^2\right]\Delta_n(f; \delta_n).$$

Set $\delta_n = \sqrt{b_n^2/n}$ and consider $\delta_n \leqslant 0.1$ for sufficiently large n. Since $\lim_{n \to \infty} b_n^2/n = 0$, the proof will be complete. □

REMARK 1. This kind of theorem is studied for different operators (for instance, Szász–Mirakjan and Baskakov operators: see [**19**, **20**]) in the norm $\| \cdot \|_{\rho^3}$, but our theorem is studied in the norm $\| \cdot \|_{\rho}$. For instance in [**19**] it has been given in the form of a theorem: if $f \in C_\rho^k[0, b_n]$. Then the inequality

$$\sup_{x \in [0, b_n]} \frac{|L_n(f; x) - f(x)|}{(1 + x^2)^3} \leqslant K\Delta_n(f; b_n^{-1/4})$$

holds where the sequence (b_n) is as in (3) and (L_n) is the Baskakov operators sequence $L_n(f; x) = \sum_{k=0}^{\infty} f(k/n)\varphi_n^k(x)(-x)^k/k!$, $x \in [0, \infty)$.

Thus, Theorem 2 has better order approximation that the analog theorems which were given in [**19**, **20**].

6. *A Voronovskaya-type theorem*

In this section, we prove a Voronovskaya-type theorem for the operators Z_n.

THEOREM 3. *For every* $f \in C_\rho^k[0, b_n]$ *such that* $f', f'' \in C_\rho^k[0, b_n]$, *we have*

$$\lim_{n \to \infty} \frac{n}{b_n}\{Z_n(f; x) - f(x)\} = f'(x) + xf''(x)$$

uniformly with respect to $x \in [0, b_n]$.

Proof. Let $f, f', f'' \in C_\rho^k[0, b_n]$. In order to prove the theorem, by Taylor's theorem we write

$$f(t) = \begin{cases} f(x) + (t - x)f'(x) + \frac{1}{2}(t - x)^2 f''(x) + (t - x)^2\eta(t - x) & \text{if } t \neq x \\ 0 & \text{if } t = x \end{cases}$$

where $\eta(h)$ tends to zero for all h converging to zero.

Now from (7), (13) and (14)

$$\frac{n}{b_n}\{Z_n(f; x) - f(x)\} = \frac{n}{b_n}\frac{b_n}{n}f'(x) + \frac{1}{2}\frac{n}{b_n}\left[\frac{2b_n x}{n} - \frac{x^2}{n} + \frac{2b_n^2}{n^2}\right]f''(x) + \frac{n}{b_n}Z_n((t - x)^2\eta(t - x); x).$$

If we apply the Cauchy–Schwarz–Bunyakovsky inequality for $Z_n((t - x)^2\eta(t - x); x)$, we conclude that

$$\frac{n}{b_n}|Z_n((t - x)^2\eta(t - x); x)| \leqslant \sqrt{\frac{n}{b_n^2}Z_n((t - x)^4; x)}\sqrt{nZ_n((\eta(t - x))^2; x)}.$$
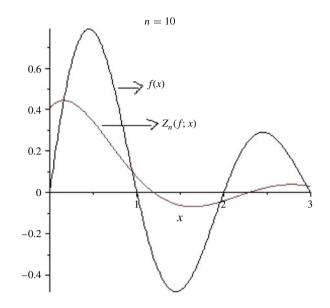
FIGURE 1. *Approximation of f by $Z_n(f; x)$, $n = 10$.*

If we consider (17) and condition (18) we obtain

$$\sqrt{\frac{n}{b_n^2} Z_n((t-x)^4; x)} \leqslant \sqrt{\frac{n}{b_n^2} \left[144 \left(\frac{b_n^2}{n} \right)^2 \right]}.$$

Here $\lim_{n \to \infty} \sqrt{(n/b_n^2) Z_n((t-x)^4; x)} = 0$ is bounded with respect to $x \in [0, b_n]$. On the other hand, by assumption $\lim_{t \to x} \eta(t - x) = 0$. So, it follows that

$$\lim_{n \to \infty} \frac{n}{b_n} |Z_n((t-x)^2 \eta(t-x); x)| = 0.$$

Then we have

$$\frac{n}{b_n} \{ Z_n(f; x) - f(x) \} = f'(x) + x f''(x) - \frac{1}{2} \frac{x^2}{b_n} f''(x) + \frac{b_n}{n} f''(x).$$

If we take lim on both sides with respect to $x \in [0, b_n]$, we will obtain the desired result. □

6.1. An example for Theorem 3

Choose $f(x) = x^3$. Then $f'(x) = 3x^2$, $f''(x) = 6x$ and $f'(x) + x f''(x) = 3x^2 + 6x^2 = 9x^2$. Now use (10)

$$\frac{n}{b_n} \{ Z_n(t^3; x) - x^3 \} = \frac{18 b_n}{n} x + \frac{9(n-1)}{n} x^2 - \frac{(3n-2)}{n b_n} x^3 + \frac{6 b_n^2}{n^2}$$

$$\lim_{n \to \infty} \frac{n}{b_n} \{ Z_n(t^3; x) - x^3 \} = 9x^2 = f'(x) + x f''(x).$$

6.2. Some plots

Figures 1–4 show plots for the function $f(x) = \sin(\pi x) \cdot \exp(-x/2)$ on the interval $[0, 3]$ for some n, generated using Maple14.
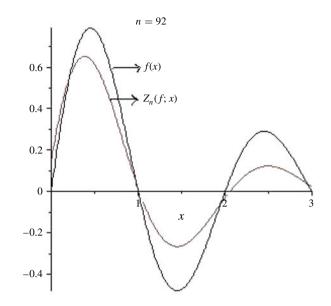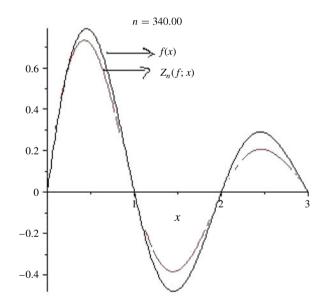
FIGURE 2. *Approximation of f by $Z_n(f;x)$, $n = 92$.*
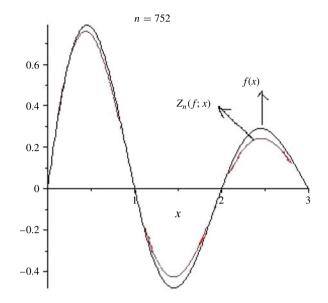

FIGURE 3. *Approximation of f by $Z_n(f;x)$, $n = 340$.*

FIGURE 4. *Approximation of f by $Z_n(f;x)$, $n = 752$.*

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