UNIONS OF TWO CONVEX SETS

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1. Introduction and notation. Valentine (**1**, Theorems 2 and 3) has defined a three-point property which he called P_3 and has shown that a closed subset of the euclidean plane possessing this property is expressible as the union of at most three convex sets. He also showed that if the number of isolated points of local non-convexity of such a set is one, finite and even, or infinite, the set is the union of two convex sets. In this paper we give properties which, together with Valentine's results, characterize those subsets of a plane which may be represented as a union of two closed, convex sets.

The closed metric segment joining p and q in the euclidean plane E^2 will be denoted by $S(p, q)$. A set A is star-like with respect to $p \in A$ if $S(p, q)$ lies in A for every $q \in A$. The set of star-like points of A is called the kernel of A. The boundary of A and the kernel of A will be denoted by $B(A)$ and $K(A)$ respectively. A set A is said to have the property P_3 if for each triple of its points p, q, r at least one of the segments $S(p, q)$, $S(q, r)$, or $S(p, r)$ lies in A. By a convex polygon we mean a simple closed curve which is the union of a finite number of line segments and which bounds a convex set in E^2.

Suppose $p_1, p_2, \ldots, p_k$ are the vertices of a convex polygon ordered in the counter-clockwise manner around the polygon. Two lines containing adjacent sides of the convex polygon intersect at a vertex p_j and determine four closed sectors whose intersection is p_j. That closed sector which has only p_j in common with the polygon will be denoted by $\sum_j$. Now consider a side $S(p_j, p_{j+1})$ (where the index $k + 1$ is identified with the index 1) of the convex polygon. The line determined by p_j and p_{j+1} divides the plane into two half-planes. Let $H(p_j, p_{j+1})$ be the closed half-plane which contains $\sum_j \cup \sum_{j+1}$, and let

$$T(p_j, p_{j+1}) = \overline{H(p_j, p_{j+1})\diagdown(\sum{}_j \cup \sum{}_{j+1})}.$$

The set $T(p_j, p_{j+1})$ will be referred to as a T set. A T set may be triangular or unbounded. The intersection $\sum_j \cap T(p_j, p_{j+1})$ is either a ray with initial point p_j or is a segment with one endpoint p_j. The intersection $\sum_j \cap \sum_{j+1}$ is either null or contains interior points of both $\sum_j$ and $\sum_{j+1}$. Then the sides of the polygon determine in the plane $2k + 1$ sets: (1) the interior of the polygon, (2) the k sets $\sum_j$, and (3) the k sets $T(p_j, p_{j+1})$.

2. Preliminaries. In this section results are obtained which enable us to provide the desired characterization of certain subsets of E^2.

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LEMMA 1. *Let S, a connected subset of E^2, have property P_3, and let $p_1, p_2, \ldots, p_k$ $(k \geqslant 3)$ be all of the points of local non-convexity of S. Then $p_1, p_2, \ldots, p_k$ are the vertices of a convex polygon, and the interior of each $\sum_j (j = 1, 2, \ldots, k)$ contains no points of S.*

Proof. Since S has P_3, for each j, $p_j \in B(K(S))$ (**1**, Corollary 1). Since $K(S)$ is convex, the points p_j are the vertices of a convex polygon.

Suppose $q \in S$ and that q is an interior point of a $\sum$, say $\sum_1$. Since $S(p_2, p_k) \subset K(S)$, every point of $S(p_2, p_k)$ can be joined to q by a segment which lies in S. Then the interior and boundary of the triangle determined by q, p_2, and p_k contains only points of S. Since p_1 is an interior point of this triangle, S is locally convex at p_1. This is a contradiction.

Suppose p is a point of the boundary of a closed, convex set A in the plane. If there is a line tangent to $B(A)$ at p, then this line is the only support line for A at p. Otherwise, there exist a unique right-hand and a unique left-hand semitangent to $B(A)$ at p. Each is a support line of A.

LEMMA 2. *If $p_1, p_2, \ldots, p_k$ $(k \geqslant 3)$ are all the points of local non-convexity of a bounded, closed subset S of E^2 possessing property P_3, then the intersection of S and any T set is convex.*

Proof. If for any T set, say $T(p_1, p_2)$, $T \cap S$ is not convex, then there exist points $q, r \in T \cap S$ such that none of the points of $S(q, r)$ between q and r lie in S. Each of the points q and r can be joined by a segment in $T \cap S$ to $S(p_1, p_2)$. If $S(q, r)$ extended meets $S(p_1, p_2)$, there is a contradiction. Thus the segment $S(q, r)$ must be parallel to $S(p_1, p_2)$ or must be on a line which intersects $S(p_1, p_2)$ extended. In either case the points p_1, p_2, q, and r determine a convex quadrilateral. The labelling may be selected so that the diagonals of the convex quadrilateral are $S(p_1, r)$ and $S(p_2, q)$. Now there exist points $p_1' \in S(p_1, q)$ and $p_2' \in S(p_2, r)$ such that $p_1 \neq p_1' \neq r$, $p_2 \neq p_2' \neq q$ and the quadrilateral determined by p_1', p_2', q, and r is convex.

Let Q be the set of points interior to and on the boundary of the quadrilateral with vertices p_1', p_2', q, and r. Then $T \cap Q$ is convex. Since $q, r \in T \cap Q \cap S$ and $S(q, r) \not\subset T \cap Q \cap S$, then $T \cap Q \cap S$ is not convex. Hence S has a point of local non-convexity in T different from p_1 and p_2. This is a contradiction.

Let $C(j, j+1) = S \cap T(p_j, p_{j+1})$ for $j = 1, 2, \ldots, k$. In the following if we denote that the three points p_1, p_2, and p_3 are consecutive points on a simple closed curve, we mean to indicate the order in which these points are encountered in traversing the simple closed curve in such a manner that the bounded component of its complement is always on the left.

LEMMA 3. *Let p_1, p_2, p_3 be any three consecutive points of local non-convexity of a set S satisfying the hypothesis of Lemma 2. Then for $j = 1, 2$ each of the semitangents to $C(j, j+1)$ at p_2 is a support line for $K(S)$.*

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Proof. We shall show that $K(S)$ lies in only one of the closed half-planes determined by the right-hand semitangent to $C(1, 2)$ at p_2. The proof for the left-hand semitangent to $C(2, 3)$ is similar.

The right-hand semitangent is a support line for the triangle with vertices p_1, p_2, and p_3. Let H be that open half-plane which does not contain the triangle. Assume $q \in H \cap K(S)$. Clearly $q \notin C(1, 2)$; hence $q \in C(2, 3)$. Consider the line L containing $S(q, p_2)$. Since L is not a support line for $C(1, 2)$, both open half-planes determined by L contain points of $C(1, 2)$. Let r be a point of $C(1, 2)$ in that open half-plane which contains the interior of $\sum_2$. Since $q \in K(S)$, $S(q, r) \subset S$ and $S(q, r) \cap \sum_2$ is not null. This contradicts Lemma 1.

LEMMA 4. *Let p_1, p_2, p_3 be points satisfying the hypothesis of Lemma 3. If $t_1 \in C(1, 2)$ and $t_2 \in C(2, 3)$, $t_1 \neq p_2 \neq t_2$, are such that $S \cap S(t_1, t_2) = t_1 \cup t_2$, then $t_1(t_2)$ is in the closed half-plane not containing $K(S)$ determined by the left-hand (right-hand) semitangent to $C(2, 3)$ (to $C(1, 2)$) at p_2.*

Proof. Since $K(S) \cap C(1, 2)$ is not null, by Lemma 3 the right-hand semitangent at p_2 is a support line for $K(S) \cup C(1, 2)$. Note that the right-hand semitangent is also a support line for $\sum_2$ and that $\sum_2$ lies in the closed half-plane not containing $K(S)$. Since $S(t_1, t_2)$ has points in common with the interior of $\sum_2$, $S(t_1, t_2)$ intersects the right-hand semitangent to $C(1, 2)$. A similar argument holds for t_2 and the left-hand semitangent to $C(2, 3)$ at p_2.

THEOREM 1. *Let S be a bounded, closed, and connected subset of E^2. Suppose S has property P_3 and that S contains at least one point of local non-convexity. If there exists a point $q \in S$ such that $q \in K(S) \cap B(S)$ and S is locally convex at q, then S can be expressed as the union of two closed and convex sets.*

Proof. If the number of points of local non-convexity of S is one, finite and even, or infinite, then the desired conclusion is immediate (**1**, p. 1232, Theorem 3). Otherwise, denote the points of local non-convexity of S by $p_1, \ldots, p_{2n+1}$, $n \geqslant 1$, ordered in a counter-clockwise manner around $B(K(S))$. Set $2n + 1 = k$. Since q lies in one of the T sets of the polygon $p_1, \ldots, p_k$, the labelling may be selected so that $q \in T(p_k, p_1)$. Since $q \in B(S)$, then by Lemma 2, q is a point of the boundary of the convex set $C(k, 1)$. Then q lies on an arc $A(p_k, p_1)$ contained in $B(S)$ with endpoints p_k and p_1. At every point of this arc except at p_k and p_1, S is locally convex. Since S is not locally convex at p_1, there is a point, say r, of the arc $A(p_k, p_1)$ such that r is not in $K(S)$ and r is in the subarc of $A(p_k, p_1)$ from q to p_1. Since $K(S)$ and $C(k, 1)$ are closed and convex, there is a subarc $A(q_1, p_1)$ of $A(p_k, p_1)$ with endpoints q_1 and p_1 such that $A(q_1, p_1) \cap K(S) = q_1 \cup p_1$. Similarly there is a subarc $A(q_k, p_k)$ of $A(p_k, p_1)$ such that $A(q_k, p_k) \cap K(S) = p_k \cup q_k$.

In $T(p_j, p_{j+1}) \cap B(S)$, denote the arc joining p_j and p_{j+1} by $A_T(p_j, p_{j+1})$ for $j = 1, 2, \ldots, k$ (where $p_{k+1} = p_1$). Also for each two distinct points a and b in $B(K(S))$, let $A_K(a, b)$ be the arc connecting a and b in $B(K(S))$ in the counter-clockwise direction. If $a = b$, then let $A_K(a, b) = a$. Let

$$D = A_K(q_1, p_1) \cup A_T(p_1, p_2) \cup A_K(p_2, p_3) \cup \ldots \cup A_T(p_{k-2}, p_{k-1}) \cup A_K(p_{k-1}, p_k) \cup A_T(p_k, q_k) \cup A_K(q_k, q_1).$$

Note that each arc of the form A_T or A_K can have only endpoints in common with another A_T or A_K. Hence D is a simple closed curve. Let M denote the closure of the bounded component of $E^2 \setminus D$. It is clear from the manner in which D was constructed that $K(S) \subset M$ and that the set M is the union of $K(S)$ and a certain collection C of the sets $C(j, j + 1)$. Hence $M \subset S$.

Assertion. *The set M is convex.*

Proof. Let $x, y \in M$. If either x or y is in $K(S)$, then $S(x, y) \subset M$. Suppose neither x nor y is in $K(S)$; then each is in a T-set. If x and y belong to the same T-set, then $S(x, y) \subset M$. If x and y are in T-sets whose intersection is null, then there exist points x_1 and y_1 in $B(K(S))$ such that $S(x, y) = S(x, x_1) \cup S(x_1, y_1) \cup S(y_1, y)$. Since each of the three segments of the decomposition of $S(x, y)$ is in M, then $S(x, y) \subset M$. Suppose then that x and y are in T-sets whose intersection is a point of local non-convexity p_j. If $j \neq 1$, then one of the points x, y is in $K(S)$. This has been considered previously. Suppose $j = 1$. If $x \in T(p_1, p_2)$, $y \in C(k, 1)$ but y is not in the closed convex set F bounded by $A_T(p_k, q_k)$ and $S(p_k, q_k)$, then $y \in K(S)$. If $y \in F$, then x and y lie on the same side of the left-hand semitangent to $C(1, 2)$ at p_1. If $S(x, y) \not\subset S$, then Lemma 4 would be contradicted. Hence $S(x, y)$ is of the form $S(x, x_1) \cup S(x_1, y_1) \cup S(y_1, y)$ where $x_1 \cup y_1 \subset B(K(S))$. That is, if x and y are in M then $S(x, y) \subset M$, and the assertion has been proved.

Let E denote the simple closed curve which is the set-theoretic union of the sequence of arcs $A_T(q_1, p_1), A_K(p_1, p_2), \ldots, A_T(p_{k-1}, p_k), A_K(p_k, q_k), A_T(q_k, q_1)$ and let N be the closure of the set of points interior to and on E. By an argument similar to that used for M, the set N is convex and $K(S) \subset N \subset S$.

If $x \in S$, then either $x \in K(S)$ or $x \in C(j, j + 1)$ for some j. In either case x is in M or N. By construction $M \cup N \subset S$, and hence $S = M \cup N$ where M and N are convex. Thus the theorem has been proved.

Theorem 2. *If a bounded, closed, connected subset S of the plane E^2 is the union of two closed, convex sets, each of which contains the kernel of S, and if S has an odd number $\geqslant 3$ of points of local non-convexity, then S contains a point of local convexity which lies in $K(S) \cap B(S)$.*

Proof. Let the points of local non-convexity be denoted $p_1, \ldots, p_k$, k odd and $k \geqslant 3$, as $K(S)$ is traversed in the counter-clockwise manner. Let $S = M \cup N$ where $K(S) \subset M \cap N$ with M and N each closed and convex.

Consider $A_T(p_1, p_2)$. Since S has P_3, points p_1 and p_2 each belong to $K(S) = M \cap N$. If any other point of $A_T(p_1, p_2)$ belongs to both M and N, the proof is complete. So let it be assumed that $A_T(p_1, p_2) \subset M$, and $A_T(p_1, p_2) \cap N = p_1 \cup p_2$.

Then consider $A_T(p_2, p_3)$. There exists a point of $A_T(p_2, p_3)$ and a point of $A_T(p_1, p_2)$ which cannot be joined by a segment in S. Such a point of $A_T(p_2, p_3)$ is in N. Again if a point of local convexity (with respect to S) of $A_T(p_2, p_3)$ belongs to $M \cap N$, the proof is complete. Thus it may be assumed that every point of $A_T(p_2, p_3)$ belongs to N, and that the only points of $A_T(p_2, p_3)$ belonging to M are p_2 and p_3.

Continuing around the polygon $p_1, \ldots, p_k$ in this manner gives

$$\begin{aligned} &A_T(p_{2j-1}, p_{2j}) \subset M \\ &A_T(p_{2j}, p_{2j+1}) \subset N \end{aligned} \qquad j = 1, \ldots, \frac{k-1}{2},$$

and

$$A_T(p_k, p_1) \subset M.$$

But $A_T(p_k, p_1)$ and $A_T(p_1, p_2)$ are contained in M. Then any point of either of these arcs may be joined to any point of the other by a segment lying in $M \subset S$. This violates the local non-convexity of S at p_1 and nullifies the assumption that no point of any of the A_T arcs except endpoints belongs to both M and N.

The following example shows that the hypothesis of Theorem 2 cannot be weakened by deleting the condition that S be bounded. Let M be the set of points $(x, y) \in E^2$ in the closed first quadrant; and let N be the set of points $(x, y) \in E^2$ satisfying both $y \geqslant 1 - x$ and $y \geqslant x - 3$. The points of local non-convexity of $S = M \cup N$ are $(0, 1)$, $(1, 0)$, and $(3, 0)$. This set S has no point q of local convexity such that $q \in B(S) \cap K(S)$.

Reference

1. F. A. Valentine, *A three point convexity property*, Pac. J. Math., 7 (1957), 1227–1235.

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