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Local geometric properties of conductive transmission eigenfunctions and applications

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Abstract

The purpose of the paper is twofold. First, we show that partial-data transmission eigenfunctions associated with a conductive boundary condition vanish locally around a polyhedral or conic corner in $\mathbb{R}^n$, $n = 2, 3$. Second, we apply the spectral property to the geometrical inverse scattering problem of determining the shape as well as its boundary impedance parameter of a conductive scatterer, independent of its medium content, by a single far-field measurement. We establish several new unique recovery results. The results extend the relevant ones in [26] in two directions: first, we consider a more general geometric setup where both polyhedral and conic corners are investigated, whereas in [26] only polyhedral corners are concerned; second, we significantly relax the regularity assumptions in [26] which is particularly useful for the geometrical inverse problem mentioned above. We develop novel technical strategies to achieve these new results.

1. Introduction

1.1. Mathematical setup and summary of major findings

The purpose of the paper is twofold. We are concerned with the spectral geometry of transmission eigenfunctions and the geometrical inverse scattering problem of recovering the shape of an anomalous scatterer, independent of its medium content, by a single far-field measurement. We first introduce the mathematical setup of our study.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, with a connected complement $\mathbb{R}^n \backslash \overline{\Omega}$. Let $V \in L^{\infty}(\Omega)$ and $\eta \in L^{\infty}(\partial\Omega)$ be complex-valued functions. Consider the following conductive transmission eigenvalue problem associated with $k \in \mathbb{R}_+$ and $(w, v) \in H^1(\Omega) \times H^1(\Omega)$:

$$\begin{cases} \Delta w + k^2(1 + V)w = 0 & \text{in } \Omega, \\ \Delta v + k^2 v = 0 & \text{in } \Omega, \\ w = v, \ \partial_\nu w = \partial_\nu v + \eta v & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where and also in what follows, $\nu \in \mathbb{S}^{n-1}$ signifies the exterior unit normal vector to $\partial\Omega$. Clearly, $(w, v) = (0, 0)$ is a trivial solution to (1.1). If there exists a non-trivial pair of solutions to (1.1), k is referred to as a conductive transmission eigenvalue and (u, v) is the corresponding pair of conductive transmission eigenfunctions. η is called the boundary impedance or conductive parameter. If $\eta \equiv 0$, then (1.1) is

reduced to the standard transmission eigenvalue problem. Hence, the conductive transmission eigenvalue problem (1.1) is a generalized formulation of the transmission eigenvalue problem. Nevertheless, it has its own physical background when $\eta \not\equiv 0$ as shall be discussed in what follows. The existence and discreteness of the conductive transmission eigenvalues can be found in [14].

One of the main purposes of this paper is to quantitatively characterize the geometric property of the conductive transmission eigenfunctions (assuming their existence). The major findings can be briefly summarized as follows. If there is a polyhedral or conic corner on $\partial\Omega$, then under certain regularity conditions the eigenfunctions must vanish at the corner. The regularity conditions are characterized by the Hölder continuity of the parameters $q := 1 + V$ and η locally around the corner as well as a certain Herglotz extension property of the eigenfunction v, which is weaker than the Hölder continuity. The results extend the relevant ones in [26] in two directions: first, we consider a more general geometric setup where both polyhedral and conic corners are investigated, whereas in [26] only polygonal and edge corners are concerned; second, we significantly relax the regularity assumptions in [26] which is particularly useful for the geometrical inverse problem discussed in what follows. We develop novel technical strategies to achieve those new results. More detailed discussion shall be given in the next subsection.

The other focus of our study is the inverse scattering problem from a conductive medium scatterer. Let V be extended by setting $V = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Throughout, we set $q = 1 + V$. Let $u^i(\mathbf{x})$ be a time-harmonic incident wave which is an entire solution to

$$\Delta u^i(\mathbf{x}) + k^2 u^i(\mathbf{x}) = 0, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1.2}$$

where $k \in \mathbb{R}_+$ signifies the wave number. Let (Ω, q, η) denote a conductive medium scatterer with Ω signifying its shape and q, η being its medium parameters. The impingement of u^i on (Ω, q, η) generates wave scattering and it is described by the following system:

$$\begin{cases} \Delta u^- + k^2 q u^- = 0, & \text{in} \quad \Omega, \\ \Delta u^+ + k^2 u^+ = 0, & \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}, \\ u^+ = u^-, \quad \partial_\nu u^+ + \eta u^+ = \partial_\nu u^-, & \text{on} \quad \partial\Omega, \\ u^+ = u^i + u^s, & \text{in} \quad \mathbb{R}^n \setminus \overline{\Omega}, \\ \lim_{r \to \infty} r^{(n-1)/2}(\partial_r u^s - \mathrm{i} k u^s) = 0, & r = |\mathbf{x}|, \end{cases} \tag{1.3}$$

where $\mathrm{i} := \sqrt{-1}$ and the last limit in (1.3) is known as the Sommerfeld radiation condition that characterizes the outward radiating of the scattered wave field u^s. The well-posedness of the direct problem (1.3) can be found in [13] for the unique existence of $u := u^- \chi_\Omega + u^+ \chi_{\mathbb{R}^n \setminus \overline{\Omega}} \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, the scattered field admits the following asymptotic expansion:

$$u^s(\mathbf{x}) = \frac{e^{\mathrm{i} k |\mathbf{x}|}}{|\mathbf{x}|^{(n-1)/2}} \left(u^\infty(\hat{\mathbf{x}}) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{(n-1)/2}} \right) \right), \quad |\mathbf{x}| \to \infty,$$

which holds uniformly in all directions $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^{n-1}$. The function u^∞ defined on the unit sphere $\mathbb{S}^{n-1}$ is known as the far-field pattern of u^s. Associated with (1.3), we are concerned with the following geometrical inverse problem:

$$u^\infty(\hat{\mathbf{x}}; u^i), \; u^i \text{ fixed} \longrightarrow \Omega \quad \text{independent of } q \text{ and } \eta. \tag{1.4}$$

That is, we intend to recover the geometrical shape of the conductive scatterer independent of its physical content by the associated far-field pattern generated by a single incident wave (which is usually referred to as a single far-field measurement in the literature).

Determining the shape of a scatterer from a single far-field measurement constitutes a longstanding problem in the inverse scattering theory [21, 22, 39]. In this paper, based on the spectral geometric results discussed earlier, we derive several new unique identifiability results for the inverse problem (1.4). In brief, we establish local unique recovery results by showing that if two conductive scatterers

possess the same far-field pattern, then their difference cannot possess a polyhedral or conic corner. If we further imposed a certain a priori global convexity on the scatterer, then one can establish the global uniqueness result. Moreover, we can show that the boundary impedance parameter η can also be uniquely recovered. It is emphasized that all of the results established in this paper hold equally for the case $\eta \equiv 0$. If $\eta \equiv 0$, (1.3) describes the scattering from a regular medium scatterer (Ω, q). In the case $\eta \neq 0$, (Ω, q, η) (effectively) characterizes a regular medium scatterer (Ω, q) by a thin layer of highly loss medium [1, 13], and in two dimensions (1.3) describes the corresponding transverse electromagnetic scattering, whereas in three dimensions (1.3) describes the corresponding acoustic scattering. In addition to its physical significance, introducing a boundary parameter η makes our study more general which includes $\eta \equiv 0$ as a special case. Hence, in what follows, we also call (v, w) to (1.1) as generalized transmission eigenfunctions.

1.2. Connection to existing studies and discussions

Before discussing the relevant existing studies, we note one intriguing connection between the scattering problem (1.3) and the spectral problem (1.1). If $u_\infty \equiv 0$, which by Rellich's theorem implies that $u^+ = u^i$ in $\mathbb{R}^n \backslash \overline{\Omega}$, one can show that $(v, w) = (u^i|_\Omega, u^-|_\Omega)$ fulfils the spectral system (1.1). In the case of $u_\infty \equiv 0$, no scattering pattern can be observed outside Ω, and hence, the scatterer (Ω, q, η) is invisible/transparent with respect to the exterior observation under the wave interrogation by u^i. On the other hand, if (w, v) is a pair of transmission eigenfunctions to (1.1), then by the Herglotz extension v can give rise to an incident wave whose impingement on (Ω, q, η) is (nearly) no scattering, i.e., (Ω, q, η) is (nearly) invisible/transparent.

Recently, there has been considerable interest in quantitatively characterizing the singularities of scattering waves induced by the geometric singularities on the shape of the underlying scatterer as well as its implications to invisibility and geometrical inverse problems. There are two perspectives in the literature. The first one is mainly concerned with occurrence or non-occurrence of non-scattering phenomenon, namely whether invisibility can occur or not. The main rationale is that if the scatterer possesses a geometric singularity (in a proper sense) on its shape, then it scatters a generic incident wave nontrivially, namely invisibility cannot occur. Here, the generic condition is usually characterized by a non-vanishing property of the incident wave at the geometrically singular place. It first started from the study in [12] for acoustic scattering with many subsequent developments in different physical contexts [4–8, 10, 18, 19, 42, 45–47].

The other one is a spectral perspective which is mainly concerned with the spectral geometry of transmission eigenfunctions. According to the connection mentioned above, the spectral geometric results characterize the patterns of the wave propagation inside a (nearly) invisible/transparent scatterer. It was first discovered in [9] that transmission eigenfunctions are generically vanishing around a corner point and such a local geometric property was further extended to conductive transmission eigenfunctions in [26], elastic transmission eigenfunctions in [5, 33] and electromagnetic transmission eigenfunctions in [10]. Though the two perspectives share some similarities, especially about the vanishing of the wave fields around the geometrically singular places, there are subtle and technical differences. In fact, it is numerically observed in [11] that there exist transmission eigenfunctions which do not vanish, instead localize, around geometrically singular places. An unobjectionable reason to account for such (locally) localizing behaviour of the transmission eigenfunctions is the regularity of the eigenfunctions at the geometrically singular places. In general, if the transmission eigenfunctions are Hölder continuous, they locally vanish around the singular places. Nevertheless, it is shown in [41] that under a certain Herglotz extension property, the locally vanishing property still holds. It is shown in [41] that the aforementioned regularity criterion in terms of the Herglotz extension is weaker than the Hölder regularity. In addition to the local geometric pattern, the spectral geometric perspective also leads to the discovery of certain global geometric patterns of the transmission eigenfunctions. Indeed, it is discovered in [20, 24, 25] that the transmission eigenfunctions tend to (globally) localize on $\partial\Omega$ with many subtle structures. Those

spectral geometric results have been proposed to produce a variety of interesting applications, including super-resolution imaging [20], wave field boundary localization [29], artificial mirage [25] and pseudo plasmon resonance [2, 3]. We also refer to [39] for more related results in different physical contexts.

In this paper, we adopt the second perspective to study the (local) geometric properties of the conductive transmission eigenfunctions as well as consider the application to address the unique identifiability issue for the geometrical inverse scattering problem. As discussed in the previous subsection, our results derived in this paper extend the relevant ones in [26] in terms of the geometric setup as well as the regularity requirements. To achieve these new results, we develop novel technical strategies. In principle, we adopt microlocal tools to quantitatively characterize the singularities of the eigenfunctions induced by the corner or conic singularities. Nevertheless, we utilize CGO (Complex Geometric Optics) solutions of the PDO (partial differential operator) $\Delta + (1 + V)$ in our quantitative analysis, whereas in [26], the analysis made use of certain CGO solutions to Δ. This induces various subtle and technical quantitative estimates and asymptotic analysis. Finally, as also discussed in the previous subsection, we apply the newly derived spectral geometric results to establish several novel unique identifiability results for the geometric inverse problem (1.4). We would also like to mention in passing some recent results on determining the shape of a scattering object by a single or at most a few far-field measurements in different physical contexts [4, 6, 7, 10, 30–32, 40, 43]. Recent developments of uniqueness and stability analysis for inverse scattering using spectral geometry can be found in [28].

The rest of the paper is organized as follows. In Section 2, we collect some preliminary results which are needed in the subsequent analysis. In Section 3, we show that the conductive transmission eigenfunctions to (1.1) near a convex sectorial corner in $\mathbb{R}^2$ must vanish. In Section 4, we study the vanishing of conductive transmission eigenfunctions to (1.1) near a convex conic or polyhedral corner in $\mathbb{R}^3$. In Section 5, we discuss the visibility of a scatterer associated with (1.3). Furthermore, the unique recovery for the shape determination Ω associated with the corresponding conductive scattering problem (1.3) is investigated.

2. Preliminaries

In this section, we present some preliminary results which shall be frequently used in our subsequent analysis.

Given $s \in \mathbb{R}$ and $p \geq 1$, the Bessel potential space is defined by

$$H^{s,p} := \{f \in L^p(\mathbb{R}^n);\ \mathcal{F}^{-1}[(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f] \in L^p(\mathbb{R}^n)\}, \tag{2.1}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively.

We introduce a complex geometrical optics (CGO) solution u_0 defined by (2.2) in Lemma 2.1.

Lemma 2.1. [19] *Given the space dimensions* $n = 2, 3$, *let* $q \in H^{1,1+\varepsilon_0}$, $\varepsilon_0 \in (0, 1)$ *and*

$$u_0(\mathbf{x}) = (1 + \psi(\mathbf{x}))e^{\rho \cdot \mathbf{x}},\ \mathbf{x} \in \mathbb{R}^n \tag{2.2}$$

where

$$\rho = -\tau(\mathbf{d} + i\mathbf{d}^\perp), \tag{2.3}$$

with $\mathbf{d}$, $\mathbf{d}^\perp \in \mathbb{S}^{n-1}$ *satisfying* $\mathbf{d} \perp \mathbf{d}^\perp$ *and* $\tau \in \mathbb{R}_+$. *For sufficient large* τ, *we have*

$$\Delta u_0 + k^2 q u_0 = 0 \quad in \quad \mathbb{R}^n, \tag{2.4}$$

and $\psi(\mathbf{x})$ *fulfils that*

$$||\psi(\mathbf{x})||_{H^{1,p}} = \mathcal{O}\left(\tau^{n(\frac{1}{\tilde{p}} - \frac{1}{p}) - 2}\right), \tag{2.5}$$

where $(\widetilde{p}, p, \varepsilon_0) = (24/19, 8, 1/2)$ *for* $n = 2$ *and* $(\widetilde{p}, p, \varepsilon_0) = (120/79, 8, 7/8)$ *for* $n = 3$.

By Laplace transform and the exponential function of negative order analysis, we can readily have the following proposition.

Proposition 2.1. *For any given $\alpha > 0$ and $0 < \epsilon < e$, we have the following estimates*

$$\left| \int_{\epsilon}^{\infty} r^{\alpha} e^{-\mu r} \mathrm{d}r \right| \le \frac{2}{\Re\mu} e^{\frac{\epsilon}{2}\Re\mu}, \tag{2.6a}$$

$$\int_{0}^{\epsilon} r^{\alpha} e^{-\mu r} \mathrm{d}r = \frac{\Gamma(\alpha+1)}{\mu^{\alpha+1}} + \mathcal{O}\left(\frac{2}{\Re\mu} e^{-\frac{\epsilon}{2}\Re\mu} \right), \tag{2.6b}$$

as $\Re(\mu) \to \infty$, where $\Gamma(s)$ stands for the Gamma function.

Lemma 2.2. [23] *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For any $f, g \in H^{1,\Delta} := \{f \in H^1(\Omega) \mid \Delta f \in L^2(\Omega)\}$, then the following Green formula holds*

$$\int_{\Omega} (g\Delta f - f\Delta g)\mathrm{d}\mathbf{x} = \int_{\partial\Omega} (g\partial_\nu f - f\partial_\nu g)\mathrm{d}\sigma, \tag{2.7}$$

where $\partial_\nu f$ is the exterior normal derivative of f to $\partial\Omega$.

3. Vanishing of transmission eigenfunctions near a convex planar corner

In this section, we consider the vanishing property of conductive transmission eigenfunctions to (1.1) near corners in $\mathbb{R}^2$. Firstly, let us introduce some notations for the subsequent use. Let (r, θ) be the polar coordinates in $\mathbb{R}^2$; that is $\mathbf{x} = (x_1, x_2) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$. For $\mathbf{x} \in \mathbb{R}^2$, $B_h(\mathbf{x})$ denotes an open ball of radius $h \in \mathbb{R}_+$ and centred at $\mathbf{x}$. For simplicity, we denote $B_h := B_h(\mathbf{0})$. Consider an open sector in $\mathbb{R}^2$ with the boundary $\Gamma^{\pm}$ as follows,

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^2 \mid \theta_m < \arg(x_1 + \mathrm{i}x_2) < \theta_M\}, \tag{3.1}$$

where $-\pi < \theta_m < \theta_M < \pi$, $\mathrm{i} := \sqrt{-1}$ and the two boundaries $\Gamma^{\pm}$ of $\mathcal{K}$ correspond to (r, θ_m) and (r, θ_M) with $r > 0$, respectively . Set

$$S_h = \mathcal{K} \cap B_h, \quad \Gamma_h^{\pm} = \Gamma^{\pm} \cap B_h, \quad \Lambda_h = \mathcal{K} \cap \partial B_h. \tag{3.2}$$

Let the Herglotz wave function be defined by

$$u(\mathbf{x}) = \int_{\mathbb{S}^1} e^{\mathrm{i}k\xi\cdot\mathbf{x}} g(\xi)\mathrm{d}\xi, \quad \xi \in \mathbb{S}^{n-1}, \quad \mathbf{x} \in \mathbb{R}^n, \quad g \in L^2(\mathbb{S}^{n-1}), \quad n = 2 \text{ or } 3, \tag{3.3}$$

which is an entire solution of

$$(\Delta + k^2)u(\mathbf{x}) = 0 \quad \text{in} \quad \mathbb{R}^n, \quad n = 2 \text{ or } 3.$$

By [48, Theorem 2 and Remark 2], we know that the set of the Herglotz wave function is dense with respect to H^1 norm in the set of the solution to

$$(\Delta + k^2)v(\mathbf{x}) = 0 \quad \text{in} \quad D, \quad D \subset \mathbb{R}^n, \quad n = 2 \text{ or } 3,$$

where D is a bounded Lipschitz domain with a connected complement.

Consider the transmission eigenvalue problem (1.1) defined in a bounded Lipschitz domain Ω with a connected complement. Since Δ is invariant under rigid motions, without loss of generality, we always assume that $\mathbf{0} \in \partial\Omega$ throughoutthe rest of this paper. In Theorem 3.1, we establish the vanishing property of the transmission eigenfunctions near a convex planar corner under H^1 regularity with certain Herglotz wave approximation assumptions in the underlying corner. We postpone the proof of Theorem 3.1 in the subsection 3.1. Compared with the assumptions in [26, Theorem 2.1], we remove the technical condition $qw \in C^{\alpha}(\overline{S}_h)$, which is critical for the analysis in [26].

Theorem 3.1. *Consider a pair of transmission eigenfunctions $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ to (1.1) associated with $k \in \mathbb{R}_+$, where Ω is a bounded Lipschitz domain with a connected complement. Suppose*

that $\mathbf{0} \in \Gamma \subset \partial\Omega$ *such that* $\Omega \cap B_h = \mathcal{K} \cap B_h = S_h$, *where the sector* $\mathcal{K}$ *is defined by (3.1) and* $h \in \mathbb{R}_+$ *is sufficiently small such that* $q \in H^2(\overline{S}_h)$ *and* $\eta \in C^\alpha(\overline{\Gamma_h^\pm})$, *where* $\alpha \in (0, 1)$. *If the following conditions are fulfilled:*

(a) *for any given positive constants* β *and* γ *satisfying*

$$\gamma < \alpha\beta, \tag{3.4}$$

the transmission eigenfunction v *can be approximated in* $H^1(S_h)$ *by the Herglotz wave functions*

$$v_j = \int_{\mathbb{S}^1} e^{ik\xi \cdot x} g_j(\xi) \mathrm{d}\xi, j = 1, 2, \cdots,$$

with the kernels g_j *satisfying the approximation property*

$$\|v - v_j\|_{H^1} \leq j^{-\beta}, \quad \|g_j\|_{L^2(\mathbb{S}^1)} \leq j^\gamma; \tag{3.5}$$

(b) η *does not vanish at* $\mathbf{0}$, *where* $\mathbf{0}$ *is the vertex of* S_h;

(c) *the open angle of* S_h *satisfies*

$$-\pi < \theta_m < \theta_M < \pi \text{ and } 0 < \theta_M - \theta_m < \pi;$$

then one has

$$\lim_{\lambda \to +0} \frac{1}{m(B(\mathbf{0}, \lambda) \cap \Omega)} \int_{B(\mathbf{0},\lambda)\cap\Omega} |v(\mathbf{x})| \mathrm{d}\mathbf{x} = 0, \tag{3.6}$$

where $m(B(\mathbf{0}, \lambda) \cap \Omega)$ *is the area of* $B(\mathbf{0}, \lambda) \cap \Omega$.

It is remarked that the Herglotz approximation property in (3.5) characterizes a regularity lower than Hölder continuity (cf. [41]). In the following theorem, if the stronger Hölder regularity imposed on the transmission eigenfunction v near the corner is satisfied, we can prove that v vanishes near the corner point. The proof of Theorem 3.2 is a slight modification of the corresponding proof of Theorem 3.1. We only give a sketched proof of Theorem 3.2 at the end of Subsection 3.1.

Theorem 3.2. *Consider a pair of transmission eigenfunctions* $v \in H^1(\Omega)$ *and* $w \in H^1(\Omega)$ *to (1.1) associated with* $k \in \mathbb{R}_+$, *where* Ω *is a bounded Lipschitz domain with a connected complement. Suppose that* $\mathbf{0} \in \Gamma \subset \partial\Omega$ *such that* $\Omega \cap B_h = \mathcal{K} \cap B_h = S_h$, *where the sector* $\mathcal{K}$ *is defined by (3.1) and* $h \in \mathbb{R}_+$. *If the following conditions are fulfilled:*

(a) $q \in H^2(\overline{S}_h)$, $v \in C^\alpha(\overline{S}_h)$ *and* $\eta \in C^\alpha(\overline{\Gamma_h^\pm})$, *where* $0 < \alpha < 1$;

(b) *the function* η *does not vanish at the vertex* $\mathbf{0}$, *where* $\mathbf{0}$ *is the vertex of* S_h, *i.e.,*

$$\eta(\mathbf{0}) \neq 0; \tag{3.7}$$

(c) *the open angels of* S_h *satisfies*

$$-\pi < \theta_m < \theta_M < \pi, \text{ and } 0 < \theta_M - \theta_m < \pi;$$

then one has

$$v(\mathbf{0}) = 0. \tag{3.8}$$

Recall that Ω is a bounded Lipschitz domain and Γ is an open subset of $\partial\Omega$. Consider the classical transmission eigenvalue problem:

$$\begin{cases} \Delta w + k^2 q w = 0 & \text{in } \Omega, \\ \Delta v + k^2 v = 0 & \text{in } \Omega, \\ w = v, \ \partial_\nu w = \partial_\nu v \text{ on } \partial\Omega, \end{cases} \tag{3.9}$$

which can be formulated from (1.1) by setting $\eta \equiv 0$. (3.9) is referred to as *interior transmission eigenvalue problem*, which has a colourful history in inverse scattering theory (cf. [16, 17, 39] and references therein). It was revealed that in [4, Theorem 1.2], the transmission eigenfunction v and w to (3.9) must vanish near a planar corner of $\partial\Omega$ if v or w is H^2-smooth near the underlying corner and q is Hölder continuous at the corner point. In the following Corollary 3.3, we shall establish the vanishing characterization of transmission eigenfunctions to (3.9) near a convex planar corner under two regularity criteria on the underlying transmission eigenfunctions near the corner. We should emphasize that we remove the H^2-smooth near the corner assumption on v and w as stated in [4, Theorem 1.2], where we only require that v is Hölder continuous at the corner point or holds a certain regularity condition in terms of Herglotz wave approximations (which is weaker than Hölder continuity as remarked earlier). The proof of Corollary 3.3 is postponed to Subsection 3.2.

Corollary 3.3. *Consider a pair of transmission eigenfunctions $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ to (3.9) associated with $k \in \mathbb{R}_+$, where Ω is a bounded Lipschitz domain with a connected complement. Suppose that $\mathbf{0} \in \partial\Omega$ such that $\Omega \cap B_h = \mathcal{K} \cap B_h = S_h$, where the sector $\mathcal{K}$ is defined by (3.1) and $h \in \mathbb{R}_+$ is sufficient small such that $q \in H^2(\overline{S_h})$ and $q(\mathbf{0}) \neq 1$. The following two statements are valid.*

(a) *For any given positive constants β and γ satisfying $\gamma < \beta$, if the transmission eigenfunction v and Herglotz wave functions v_j with the kernel g_j satisfying the approximation property (3.5), then we have the vanishing property of v near S_h in the sense of (3.6).*

(b) *If $v \in C^\alpha(\overline{S_h})$ with $\alpha \in (0, 1)$, then it holds that $v(\mathbf{0}) = 0$.*

3.1. Proof of Theorem 3.1

Given a convex sector $\mathcal{K}$ defined by (3.1) and a positive constant ζ, we define $\mathcal{K}_\zeta$ as the open set of $\mathbb{S}^1$ which is composed of all directions $\mathbf{d} \in \mathbb{S}^1$ satisfying that

$$\mathbf{d} \cdot \hat{\mathbf{x}} > \zeta > 0, \quad \text{for all } \hat{\mathbf{x}} \in \mathcal{K} \cap \mathbb{S}^1. \tag{3.10}$$

Throughout the present section, we always assume that the unit vector $\mathbf{d}$ in the form of the CGO solution u_0 given by (2.2) fulfils (3.10).

Proposition 3.1. *Let S_h and ρ be defined in (3.2) and (2.3), respectively, where $\mathbf{d}$ satisfies (3.10). Then, we have*

$$\left| \int_{\Gamma_h^\pm} e^{\rho \cdot \mathbf{x}} \mathrm{d}\mathbf{x} \right| \geq \frac{C_{S_h}}{\tau} - \mathcal{O}\left(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{3.11}$$

for sufficiently large τ, where C_{S_h} is a positive number only depending on the opening angle $\theta_M - \theta_m$ of $\mathcal{K}$ and ζ.

Proof. Using polar coordinates transformation and Proposition 2.1, we have

$$\int_{\Gamma_h^\pm} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma = \frac{\Gamma(1)}{\tau} \frac{1}{(\mathbf{d} + i\mathbf{d}^\perp) \cdot \hat{\mathbf{x}_1}} - I_{R_1} + \frac{\Gamma(1)}{\tau} \frac{1}{(\mathbf{d} + i\mathbf{d}^\perp) \cdot \hat{\mathbf{x}_2}} - I_{R_2},$$

where $\hat{\mathbf{x}_1}$ and $\hat{\mathbf{x}_2}$ are unit vector of $\mathbf{x}$ on Γ^- and Γ^+, and

$$I_{R_1} = \int_{\Gamma^- \setminus \Gamma_h^-} e^{-\tau(\mathbf{d} + i\mathbf{d}^\perp)} \mathrm{d}\sigma, \quad I_{R_2} = \int_{\Gamma^+ \setminus \Gamma_h^+} e^{-\tau(\mathbf{d} + i\mathbf{d}^\perp)} \mathrm{d}\sigma.$$

Hence, with the help of Proposition 2.1, for sufficiently large τ, we have the following integral inequality

$$\left| \int_{\Gamma_h^{\pm}} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma \right| \geq \frac{1}{\tau} \left| \frac{1}{(\mathbf{d} + i\mathbf{d}^{\perp}) \cdot \hat{\mathbf{x}_1}} + \frac{1}{(\mathbf{d} + i\mathbf{d}^{\perp}) \cdot \hat{\mathbf{x}_2}} \right| - |I_{R_1}| - |I_{R_2}|$$

$$\geq \frac{C_h}{\tau} - O\left(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right)$$

(3.12)

by using (3.10). $\qquad\square$

The following proposition can be directly derived by using (3.10) and Proposition 2.1.

Proposition 3.2. *For any given $t > 0$, we let S_h and $\Gamma_h^{\pm}$ be defined by (3.2). Then, one has*

$$||e^{\rho \cdot \mathbf{x}}||_{L^t(S_h)} \leq C \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\frac{t}{2}\zeta h \tau} \right)^{\frac{1}{t}},$$

(3.13)

$$||e^{\rho \cdot \mathbf{x}}||_{L^t(\Gamma_h^{\pm})} \leq C \left(\frac{1}{\tau} + \frac{1}{\tau} e^{-\frac{t}{2}\zeta h \tau} \right)^{\frac{1}{t}}.$$

(3.14)

as $\tau \to \infty$, where $||e^{\rho \cdot \mathbf{x}}||_{L^t(S_h)} = \left(\int_{S_h} |e^{\rho \cdot \mathbf{x}}|^t \mathrm{d}x \right)^{1/t}$, ρ is defined in (2.3) and C is a positive constant only depending on t, ζ.

Lemma 3.1. *Under the same setup of Theorem 3.1, let the CGO solution u_0 be defined by (2.2). Denote $u = w - v$, where (v, w) is a pair of transmission eigenfunctions of (1.1) associated with k. Then, it holds that*

$$\begin{cases} \Delta u_0 + k^2 q u_0 = 0 & in \quad S_h, \\ \Delta u + k^2 q u = k^2(1-q)v & in \quad S_h, \\ u = 0, \quad \partial_{\nu} u = \eta v & on \quad \Gamma_h^{\pm}, \end{cases}$$

(3.15)

and

$$||\psi(\mathbf{x})||_{H^{1,8}} = \mathcal{O}(\tau^{-\frac{2}{3}}),$$

(3.16)

where ψ and τ are defined in (2.2).

Proof. Since $q \in H^2(\overline{S}_h)$, let $\widetilde{q}$ be the Sobolev extension of q such that $\widetilde{q} \in H^2$, then by this we have $\widetilde{q} \in H^{1,1+\varepsilon_0}$, $\varepsilon_0 \in (0, 1)$. Then by Lemma 2.1, one readily has (3.16). $\qquad\square$

Lemma 3.2. [34, 35, 44] *Let Ω be a Lipschitz bounded and connected subset of $\mathbb{R}^n$, $n = 2, 3$ whose bounded and orientable boundary is denote by Γ. Let the restriction $\gamma_0(u) = u|_{\Gamma}$, then the operator γ_0 is linear and continuous from $H^{1,p}(\Omega)$ onto $H^{1-\frac{1}{p},p}(\Gamma)$ for $1 \leq p < \infty$.*

Lemma 3.3. *Let $\Gamma_h^{\pm}$ be defined in (3.2), $e^{\rho \cdot \mathbf{x}}$ and ψ be given by (2.2) and (2.3). For sufficiently large τ, it holds that*

$$\left| \int_{\Gamma_h^{\pm}} e^{\rho \cdot \mathbf{x}} \psi(\mathbf{x}) \mathrm{d}\sigma \right| \lesssim \tau^{-\frac{17}{12}}.$$

(3.17)

Throughout the rest of this paper, $\lesssim$ means that we only give the leading asymptotic analysis by neglecting a generic positive constant C with respect to $\tau \to \infty$, where C is not a function of τ.

Proof. Taking $\mathbf{y} = \tau\mathbf{x}$, then using Hölder inequality and Lemma 3.2, one has

$$\int_{\Gamma_h^\pm} |e^{\rho\cdot\mathbf{x}}||\psi(\mathbf{x})|\mathrm{d}\sigma \lesssim \frac{1}{\tau}\|e^{-\mathbf{d}\cdot\mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma_{\tau h}^\pm)}\left\|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right\|_{L^8(\Gamma_{\tau h}^\pm)} \tag{3.18}$$

$$\lesssim \frac{1}{\tau}\|e^{-\mathbf{d}\cdot\mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma^\pm)}\left\|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right\|_{H^{1,8}(S_{\tau h})} \lesssim \frac{1}{\tau}\|e^{-\mathbf{d}\cdot\mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma^\pm)}\left\|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right\|_{H^{1,8}(\mathcal{K})},$$

for sufficiently large τ. We have $\frac{1}{\tau} < 1$, and it holds that

$$\left\|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right\|_{H^{1,8}(\mathcal{K})} \le \tau^{\frac{1}{4}}\|\psi(\mathbf{x})\|_{H^{1,8}(\mathcal{K})} = \mathcal{O}(\tau^{-\frac{5}{12}}), \quad \text{as } \tau \to \infty. \tag{3.19}$$

Furthermore,

$$\|e^{-\mathbf{d}\cdot\mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma^\pm)} \le \left(\int_{\Gamma^\pm} e^{-\frac{8}{7}\zeta|\mathbf{y}|}\mathrm{d}\sigma\right)^{\frac{7}{8}} = 2\left(\frac{7}{8}\frac{1}{\zeta}\right)^{\frac{7}{8}}, \tag{3.20}$$

where ζ is defined in (3.10). Hence, $\|e^{-\mathbf{d}\cdot\mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma^\pm)}$ is a positive constant which only depends on ζ. Combining (3.19) and (3.20) with (3.18), we can prove Lemma 3.3. $\qquad\square$

Lemma 3.4. *Let* Λ_h, S_h *be defined in (3.2) and* $u_0(\mathbf{x})$ *be given by (2.2). Then,* $u_0(\mathbf{x}) \in H^1(S_h)$ *and it holds that*

$$\|u_0(\mathbf{x})\|_{L^2(\Lambda_h)} \lesssim \left(1 + \tau^{-\frac{2}{3}}\right)e^{-\zeta h\tau}, \tag{3.21a}$$

$$\|\nabla u_0(\mathbf{x})\|_{L^2(\Lambda_h)} \lesssim (1 + \tau)\left(1 + \tau^{-\frac{2}{3}}\right)e^{-\zeta h\tau}, \tag{3.21b}$$

$$\int_{S_h} |\mathbf{x}|^\alpha |u_0(\mathbf{x})|\mathrm{d}\mathbf{x} \lesssim \tau^{-(\alpha+\frac{29}{12})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau}\right) \tag{3.21c}$$

as $\tau \to \infty$, *where* ζ *is defined in (3.10) and* $\alpha \in (0, 1)$.

Proof. Using polar coordinates transformation, (2.6a) and (3.10), we can obtain that

$$\|e^{\rho\cdot\mathbf{x}}\|_{L^t(\Lambda_h)} \lesssim e^{-\zeta h\tau}. \tag{3.22}$$

where ρ is defined in (2.3) and t is a positive constant.

According to (3.16) and Lemma 3.2, for sufficient large τ, one can show that

$$\|\psi(\mathbf{x})\|_{L^4(\Lambda_h)} \le C\|\psi(\mathbf{x})\|_{H^{1,8}(S_h)} = O(\tau^{-\frac{2}{3}}), \tag{3.23}$$

where C is a positive constant, which is not a function of τ.

By virtue of (3.23) and Hölder inequality, it can be directly verified that

$$\|u_0\|_{L^2(\Lambda_h)} \le \|e^{\rho\cdot\mathbf{x}}\|_{L^2(\Lambda_h)} + \|e^{\rho\cdot\mathbf{x}}\|_{L^4(\Lambda_h)}\|\psi(\mathbf{x})\|_{L^4(\Lambda_h)}$$

$$\lesssim \left(1 + \tau^{-\frac{2}{3}}\right)e^{-\zeta h\tau}, \quad \text{as } \tau \to \infty. \tag{3.24}$$

Similarly, using Cauchy-Schwarz inequality, (3.21a) and Proposition 2.1, we have

$$\|\nabla u_0(\mathbf{x})\|_{L^2(\Lambda_h)} \le \sqrt{2}\tau\|u_0\|_{L^2(\Lambda_h)} + \|e^{\rho\cdot\mathbf{x}}\|_{L^4(\Lambda_h)}\|\nabla\psi(\mathbf{x})\|_{L^4(\Lambda_h)}$$

$$\lesssim (1 + \tau)(1 + \tau^{-\frac{2}{3}})e^{-\zeta h\tau}, \quad \text{as } \tau \to \infty. \tag{3.25}$$

Moreover, by using Cauchy-Schwarz inequality, we know that

$$\int_{S_h} |\mathbf{x}|^\alpha |u_0|\mathrm{d}\mathbf{x} \le \int_{S_h} |\mathbf{x}|^\alpha |e^{\rho\cdot\mathbf{x}}|\mathrm{d}\mathbf{x} + \int_{S_h} |\mathbf{x}|^\alpha |e^{\rho\cdot\mathbf{x}}||\psi(\mathbf{x})|\mathrm{d}\mathbf{x}. \tag{3.26}$$

Using polar coordinates transformation and Proposition 2.1, we can deduce that

$$\int_{S_h} |\mathbf{x}|^\alpha |e^{\rho\cdot\mathbf{x}}|\mathrm{d}\mathbf{x} \lesssim \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau}\right)e^{-\frac{1}{2}\zeta h\tau}, \quad \text{as } \tau \to \infty. \tag{3.27}$$

Next, by letting $\mathbf{y} = \tau\mathbf{x}$ and Hölder inequality, it can be calculated that

$$\int_{S_h} |\mathbf{x}|^\alpha |e^{\rho\cdot\mathbf{x}}||\psi(\mathbf{x})|\mathrm{d}\mathbf{x} \le \frac{1}{\tau^{\alpha+2}} \int_{\mathcal{K}} |\mathbf{y}|^\alpha |e^{-\mathbf{d}\cdot\mathbf{y}}| \left| \psi\left(\frac{\mathbf{y}}{\tau}\right) \right| \mathrm{d}\mathbf{x}$$
$$\le \frac{1}{\tau^{\alpha+2}} \||\mathbf{y}|^\alpha |e^{-\mathbf{d}\cdot\mathbf{y}}|\|_{L^{\frac{8}{7}}(\mathcal{K})} \left\| \psi\left(\frac{\mathbf{y}}{\tau}\right) \right\|_{L^8(\mathcal{K})}. \tag{3.28}$$

With the help of variable substitution and (3.16), we can calculate that

$$\left\| \psi\left(\frac{\mathbf{y}}{\tau}\right) \right\|_{L^8(\mathcal{K})} = \tau^{\frac{1}{4}} \|\psi(\mathbf{x})\|_{L^8(\mathcal{K})} = \mathcal{O}(\tau^{-\frac{5}{12}}), \quad \text{as } \tau \to \infty. \tag{3.29}$$

Similar to (3.20), by using polar coordinates transformation, we have $\||\mathbf{y}|^\alpha |e^{-\mathbf{d}\cdot\mathbf{y}}|\|_{L^{\frac{8}{7}}(\mathcal{K})}$ is a positive constant and not a function of τ. Therefore, combining (3.29), (3.28) and (3.27) with (3.26), we have (3.21c).

The proof is complete. $\qquad\square$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. By Green's formula (2.7) and (3.15), the following integral equality holds

$$\int_{\Lambda_h} (w-v)\partial_\nu u_0 - u_0\partial_\nu(w-v)\mathrm{d}\sigma - \int_{\Gamma_h^\pm} \eta u_0 v \mathrm{d}\sigma = k^2 \int_{S_h} (q-1)v u_0 \mathrm{d}\mathbf{x}. \tag{3.30}$$

Denote

$$f_j = (q-1)v_j.$$

Since $q \in H^2(\overline{S}_h)$, by Sobolev embedding property, one has $q \in C^\alpha(\overline{S}_h)$ where $\alpha \in (0,1]$. Clearly, $v_j \in C^\alpha(\overline{S}_h)$, hence $f_j \in C^\alpha(\overline{S}_h)$. According $v_j \in C^\alpha$, $\eta \in C^\alpha$, we have the expansion

$$f_j = f_j(\mathbf{0}) + \delta f_j, \quad |\delta f_j| \le \|f_j\|_{C^\alpha(\overline{S}_h)}|\mathbf{x}|^\alpha,$$
$$v_j = v_j(\mathbf{0}) + \delta v_j, \quad |\delta v_j| \le \|v_j\|_{C^\alpha(\overline{S}_h)}|\mathbf{x}|^\alpha, \tag{3.31}$$
$$\eta = \eta(\mathbf{0}) + \delta\eta, \quad |\delta\eta| \le \|\eta\|_{C^\alpha(\overline{\Gamma_h^\pm})}|\mathbf{x}|^\alpha.$$

By virtue of (3.31) and (2.2), it yields that

$$k^2 \int_{S_h} (q-1)v u_0 \mathrm{d}\mathbf{x} = -\sum_{m=1}^3 I_m, \quad \int_{\Gamma_h^\pm} \eta u_0 v \mathrm{d}\sigma = I - \sum_{m=4}^9 I_m, \tag{3.32}$$

where

$$I_1 = -k^2 \int_{S_h} (q-1)(v-v_j)u_0 \mathrm{d}\mathbf{x}, \quad I_2 = -\int_{S_h} \delta f_j u_0 \mathrm{d}\mathbf{x},$$

$$I_3 = -f_j(\mathbf{0}) \int_{S_h} u_0 \mathrm{d}\mathbf{x}, \quad I_4 = -\eta(\mathbf{0}) \int_{\Gamma_h^\pm} (v-v_j)u_0 \mathrm{d}\sigma,$$

$$I_5 = -\int_{\Gamma_h^\pm} \delta\eta(v-v_j)u_0 \mathrm{d}\sigma, \quad I_6 = -\eta(\mathbf{0})v_j(\mathbf{0}) \int_{\Gamma_h^\pm} e^{\rho\cdot\mathbf{x}}\psi(\mathbf{x})\mathrm{d}\sigma,$$

$$I_7 = -\eta(\mathbf{0}) \int_{\Gamma_h^\pm} \delta v_j u_0 \mathrm{d}\sigma, \quad I_8 = -v_j(\mathbf{0}) \int_{\Gamma_h^\pm} \delta\eta u_0 \mathrm{d}\sigma,$$

$$I_9 = -\int_{\Gamma_h^\pm} \delta\eta \delta v_j u_0 \mathrm{d}\sigma, \quad I = \eta(\mathbf{0})v_j(\mathbf{0}) \int_{\Gamma_h^\pm} e^{\rho\cdot\mathbf{x}}\mathrm{d}\sigma.$$

Substituting (3.32) into (3.30), we have the following integral identity

$$I = \sum_{m=1}^{9} I_m + J_1 + J_2,$$

where

$$J_1 = \int_{\Lambda_h} (w - v)\partial_\nu u_0 \mathrm{d}\sigma, \quad J_2 = -\int_{\Lambda_h} u_0 \partial_\nu (w - v)\mathrm{d}\sigma. \tag{3.33}$$

Therefore, it yields that

$$|I| \leq \sum_{m=1}^{9} |I_m| + |J_1| + |J_2|. \tag{3.34}$$

In the following, we give detailed asymptotic estimates of $I_m, m = 1, \cdots, 9$ and $J_j, j = 1, 2$ as $\tau \to \infty$, separately. With the help of Proposition 3.2, Hölder inequality and (3.16), it arrives at

$$
\begin{aligned}
|I_1| &\lesssim \|v - v_j\|_{L^2(S_h)} \left(\|e^{\rho \cdot \mathbf{x}}\|_{L^2(S_h)} + \|e^{\rho \cdot \mathbf{x}}\psi(\mathbf{x})\|_{L^2(S_h)} \right) \\
&\lesssim \|v - v_j\|_{L^2(S_h)} \left(\|e^{\rho \cdot \mathbf{x}}\|_{L^2(S_h)} + \|e^{\rho \cdot \mathbf{x}}\|_{L^4(S_h)} \|\psi(\mathbf{x})\|_{L^4(S_h)} \right) \\
&\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{3}} \right]
\end{aligned}
\tag{3.35}
$$

as $\tau \to \infty$.

By virtue of (3.31), it yields that

$$|I_2| \leq \|f_j\|_{C^\alpha(\overline{S}_h)} \int_{S_h} |\mathbf{x}|^\alpha |u_0| \mathrm{d}\mathbf{x}, \tag{3.36}$$

where

$$\|f_j\|_{C^\alpha(S_h)} \leq k^2 \left(\|q\|_{C^\alpha(\overline{S}_h)} \sup_{S_h} |v_j| + \|v_j\|_{C^\alpha(\overline{S}_h)} \sup_{S_h} |q - 1| \right). \tag{3.37}$$

Using the property of compact embedding of Hölder spaces, we can derive that

$$\|v_j\|_{C^\alpha} \leq \mathrm{diam}\,(S_h)^{1-\alpha} \|v_j\|_{C^1(S_h)}, \tag{3.38}$$

where $\mathrm{diam}(S_h)$ is the diameter of S_h. By direct computations, we obtain

$$\|v_j\|_{C^1} \leq \sqrt{2\pi}(1 + k)\|g\|_{L^2(\mathbb{S}^1)}. \tag{3.39}$$

Furthermore, by Cauchy-Schwarz inequality, we also can deduce that

$$|v_j| \leq \sqrt{2\pi}\|g\|_{L^2(\mathbb{S}^1)}. \tag{3.40}$$

Due to (3.5), by using the fact that $q \in C^\alpha(\overline{S}_h)$, substituting (3.38), (3.39) and (3.40) into (3.37), we have

$$\|f_j\|_{C^\alpha(\overline{S}_h)} \lesssim j^\gamma, \quad \|v_j\|_{C^\alpha(\overline{S}_h)} \lesssim j^\gamma, \tag{3.41}$$

where γ is a given positive constant defined in (3.5). Substituting (3.21c) and (3.41) into (3.36), we can deduce that

$$|I_2| \lesssim j^\gamma \left[\tau^{-(\alpha + \frac{29}{12})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right] \tag{3.42}$$

as $\tau \to \infty$.

Using Cauchy-Schwarz inequality, it can be easily calculated that

$$|I_3| \lesssim \int_{S_h} |e^{\rho \cdot \mathbf{x}}| \mathrm{d}\mathbf{x} + \int_{S_h} |e^{\rho \cdot \mathbf{x}}\psi(\mathbf{x})| \mathrm{d}\mathbf{x} \lesssim \int_{S_h} |e^{\rho \cdot \mathbf{x}}| \mathrm{d}\mathbf{x} + \int_{\mathcal{K}} |e^{\rho \cdot \mathbf{x}}\psi(\mathbf{x})| \mathrm{d}\mathbf{x}. \tag{3.43}$$

By integral substitution and using (3.29), we obtain that

$$
\int_{\mathcal{K}} |e^{\rho \cdot \mathbf{x}} \psi(\mathbf{x})| \mathrm{d}\mathbf{x} = \frac{1}{\tau^2} \int_{\mathcal{K}} e^{-\mathbf{d} \cdot \mathbf{y}} \left| \psi \left(\frac{\mathbf{y}}{\tau} \right) \right| \mathrm{d}\mathbf{y} \le \frac{1}{\tau^2} \| e^{-\mathbf{d} \cdot \mathbf{y}} \|_{L^{\frac{8}{7}}(\mathcal{K})} \| \psi \left(\frac{\mathbf{y}}{\tau} \right) \|_{L^8(\mathcal{K})}
$$
$$
\lesssim \tau^{-\frac{29}{12}}, \quad \text{as } \tau \to \infty. \tag{3.44}
$$

With the help of Proposition 3.1, substituting (3.44) into (3.43), we can derive that

$$
|I_3| \lesssim \tau^{-\frac{29}{12}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\frac{1}{2} \zeta h \tau} \right), \text{ as } \tau \to \infty. \tag{3.45}
$$

Using Cauchy-Schwarz inequality, the trace theorem and Hölder inequality, we have

$$
|I_4| \lesssim \| v - v_j \|_{L^2(\Gamma_h^\pm)} (\| e^{\rho \cdot \mathbf{x}} \|_{L^2(\Gamma_h^\pm)} + \| e^{\rho \cdot \mathbf{x}} \|_{L^4(\Gamma_h^\pm)} \| \psi(\mathbf{x}) \|_{L^4(\Gamma_h^\pm)})
$$
$$
\lesssim j^{-\beta} \left[\left(\frac{1}{\tau} + \frac{1}{\tau} e^{-\zeta h \tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau} + \frac{1}{\tau} e^{-2\zeta h \tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{3}} \right], \tag{3.46}
$$

as $\tau \to \infty$. Similarly, by virtue of Cauchy-Schwarz inequality, the trace theorem and Hölder inequality, it can be calculated that

$$
|I_5| \lesssim \| v - v_j \|_{H^1(S_h)} \left(\| e^{\rho \cdot \mathbf{x}} |\mathbf{x}|^\alpha \|_{L^2(\Gamma_h^\pm)} + \| e^{\rho \cdot \mathbf{x}} |\mathbf{x}|^\alpha \|_{L^4(\Gamma_h^\pm)} \| \psi(\mathbf{x}) \|_{L^4(\Gamma_h^\pm)} \right)
$$
$$
\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^{(2\alpha+1)}} + \frac{1}{\tau} e^{-\zeta h \tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^{(4\alpha+1)}} + \frac{1}{\tau} e^{-2\zeta h \tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{3}} \right],
$$

as $\tau \to \infty$. By using Lemma 3.3, one can show that

$$
I_6 \lesssim \int_{\Gamma_h^\pm} |e^{\rho \cdot \mathbf{x}} \psi(\mathbf{x})| \mathrm{d}\mathbf{x} \lesssim \tau^{-\frac{17}{12}}, \quad \text{as } \tau \to \infty. \tag{3.47}
$$

Using (3.31), (3.41) and Proposition 2.1, we have the following inequality

$$
|I_7| \lesssim j^\gamma \left[\int_{\Gamma_h^\pm} |\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}| \mathrm{d}\sigma + \| |\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}| \|_{L^2(\Gamma_h^\pm)} \| \psi(\mathbf{x}) \|_{L^2(\Gamma_h^\pm)} \right]
$$
$$
\lesssim j^\gamma \left[\left(\frac{1}{\tau^{\alpha+1}} + \frac{1}{\tau} e^{-\frac{1}{2} \zeta h \tau} \right) + \left(\frac{1}{\tau^{2\alpha+1}} + \frac{1}{\tau} e^{-\zeta h \tau} \right)^{\frac{1}{2}} \tau^{-\frac{2}{3}} \right], \tag{3.48}
$$

as $\tau \to \infty$. According to (3.48), we can derive that

$$
|I_8| \lesssim \left(\frac{1}{\tau^{\alpha+1}} + \frac{1}{\tau} e^{-\frac{1}{2} \zeta h \tau} \right) + \left(\frac{1}{\tau^{2\alpha+1}} + \frac{1}{\tau} e^{-\zeta h \tau} \right)^{\frac{1}{2}} \tau^{-\frac{2}{3}}, \tag{3.49a}
$$

$$
|I_9| \lesssim j^\gamma \left[\left(\frac{1}{\tau^{2\alpha+1}} + \frac{1}{\tau} e^{-\frac{1}{2} \zeta h \tau} \right) + \left(\frac{1}{\tau^{4\alpha+1}} + \frac{1}{\tau} e^{-\zeta h \tau} \right)^{\frac{1}{2}} \tau^{-\frac{2}{3}} \right], \tag{3.49b}
$$

as $\tau \to \infty$. By the Cauchy-Schwarz inequality and the trace theorem, we deduce that

$$
|J_1| \le C \| u_0 \|_{H^1(\Lambda_h)} \| w - v \|_{H^1(\Lambda_h)} \tag{3.50}
$$
$$
\lesssim \| u_0 \|_{H^1(\Lambda_h)}
$$

as $\tau \to \infty$, where C is a positive constant arising from the trace theorem. Hence, by virtue of (3.21a) and (3.21b), from (3.50), it is readily known that

$$
|J_1| \lesssim (1 + \tau)(1 + \tau^{-\frac{2}{3}}) e^{-\zeta h \tau} \tag{3.51}
$$

as $\tau \to \infty$, where ζ is a positive constant given in (3.10).

Similarly, using Cauchy-Schwarz inequality, the trace theorem and (3.21b), we can obtain that

$$|J_2| \leq \|\partial_\nu u_0\|_{L^2(\Lambda_h)} \|w - v\|_{L^2(\Lambda_h)} \leq C\|\partial_\nu u_0\|_{L^2(\Lambda_h)} \|w - v\|_{H^1(S_h)}$$

$$\lesssim \|\nabla u_0\|_{L^2(\Lambda_h)} \lesssim (1 + \tau)(1 + \tau^{-\frac{2}{3}})e^{-\zeta h\tau}. \tag{3.52}$$

Substituting (3.35), (3.42), (3.45), (3.51) and (3.52) into (3.34), by virtue of (3.11), we derive that

$$
\begin{aligned}
\left(\frac{C_{S_h}}{\tau} - \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau}\right)|\eta(\mathbf{0})v_j(\mathbf{0})| \lesssim\ & j^{-\beta}\left[\left(\frac{1}{\tau^2} + \frac{1}{\tau}e^{-\zeta h\tau}\right)^{\frac{1}{2}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau}e^{-2\zeta h\tau}\right)^{\frac{1}{4}}\tau^{-\frac{2}{3}}\right] \\
& + j^{\gamma}\left[\tau^{-(\alpha+\frac{29}{12})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau}\right)\right] \\
& + j^{-\beta}\left[\left(\frac{1}{\tau} + \frac{1}{\tau}e^{-\zeta h\tau}\right)^{\frac{1}{2}} + \left(\frac{1}{\tau} + \frac{1}{\tau}e^{-2\zeta h\tau}\right)^{\frac{1}{4}}\tau^{-\frac{2}{3}}\right] \\
& + j^{-\beta}\left[\left(\frac{1}{\tau^{(2\alpha+1)}} + \frac{1}{\tau}e^{-\zeta h\tau}\right)^{\frac{1}{2}} + \left(\frac{1}{\tau^{(4\alpha+1)}} + \frac{1}{\tau}e^{-2\zeta h\tau}\right)^{\frac{1}{4}}\tau^{-\frac{2}{3}}\right] \\
& + (j^{\gamma} + 1)\left[\left(\frac{1}{\tau^{\alpha+1}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau}\right) + \left(\frac{1}{\tau^{2\alpha+1}} + \frac{1}{\tau}e^{-\zeta h\tau}\right)^{\frac{1}{2}}\tau^{-\frac{2}{3}}\right] \\
& + j^{\gamma}\left[\left(\frac{1}{\tau^{2\alpha+1}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau}\right) + \left(\frac{1}{\tau^{4\alpha+1}} + \frac{1}{\tau}e^{-\zeta h\tau}\right)^{\frac{1}{2}}\tau^{-\frac{2}{3}}\right] \\
& + (1 + \tau)(1 + \tau^{-\frac{2}{3}})e^{-\zeta h\tau} + \tau^{-\frac{29}{12}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau}\right) + \tau^{-\frac{17}{12}}
\end{aligned}
\tag{3.53}
$$

as $\tau \to \infty$, where C_{S_h} is a positive constant given in (3.11). Multiplying τ on both sides of (3.53) and letting $\tau = j^s$, where $s > 0$, it can be derived that

$$
\begin{aligned}
\left(C_h - e^{-\frac{1}{2}\zeta h j^s}\right)|\eta(\mathbf{0})v_j(\mathbf{0})| \lesssim\ & j^{-\beta+s} + j^{\gamma-(\alpha+1)s} + j^{-\beta+\frac{1}{2}s} + j^{-\beta+(-\alpha+\frac{1}{2})s} \\
& + j^{\gamma-\alpha s} + j^{-\frac{13}{24}s} + j^{-\frac{17}{12}s},
\end{aligned}
\tag{3.54}
$$

as $\tau \to \infty$. Under the assumption (3.4), we can choose $s \in (\gamma/\alpha, \beta)$. Hence in (3.54), let $j \to \infty$ it is readily to know that

$$\lim_{j\to\infty}\left|\eta(\mathbf{0})v_j(\mathbf{0})\right| = 0.$$

Since $\eta(\mathbf{0}) \neq 0$, one has $\lim_{j\to\infty}|v_j(\mathbf{0})| = 0$. Using (3.5) and the integral mean value theorem, we can obtain (3.6).

The proof is complete. $\qquad\square$

Proof of Theorem 3.2. Due to $q \in H^2(\overline{S_h})$, using the Sobolev embedding property, we know that $q \in C^\alpha(\overline{S_h})$ with $\alpha \in (0, 1]$. Under the assumption $v \in C^\alpha(\overline{S_h})$ ($\alpha \in (0, 1]$), it readily has $f(\mathbf{x}) := (q(\mathbf{x}) - 1)v(\mathbf{x}) \in C^\alpha(\overline{S_h})$. Hence, we have the expansion of $f(\mathbf{x})$, η and $v(\mathbf{x})$ near the origin as follows

$$
\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{0}) + \delta f, && |\delta f| \leq \|f\|_{C^\alpha}|\mathbf{x}|^\alpha \\
\eta &= \eta(\mathbf{0}) + \delta\eta, && |\delta\eta| \leq \|\eta\|_{C^\alpha}|\mathbf{x}|^\alpha \\
v(\mathbf{x}) &= v(\mathbf{0}) + \delta v, && |\delta v| \leq \|v\|_{C^\alpha}|\mathbf{x}|^\alpha
\end{aligned}
\tag{3.55}
$$

Plugging (3.55) into the integral identity (3.30), it yields that

$$\eta(\mathbf{0})v(\mathbf{0}) \int_{\Gamma_h^\pm} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma = f(\mathbf{0}) \int_{S_h} u_0 \mathrm{d}\mathbf{x} + \int_{S_h} \delta f u_0 \mathrm{d}\mathbf{x} + \eta(\mathbf{0})v(\mathbf{0}) \int_{\Gamma_h^\pm} \psi(\mathbf{x}) e^{\eta \cdot \mathbf{x}} \mathrm{d}\sigma$$

$$+ \eta(\mathbf{0}) \int_{\Gamma_h^\pm} \delta v u_0 \mathrm{d}\sigma + v(\mathbf{0}) \int_{\Gamma_h^\pm} \delta \eta u_0 \mathrm{d}\sigma + \int_{\Gamma_h^\pm} \delta v \delta \eta u_0 \mathrm{d}\sigma$$

$$- \int_{\Lambda_h} (w - v)\partial_\nu u_0 \mathrm{d}\sigma + \int_{\Lambda_h} u_0 \partial_\nu (w - v) \mathrm{d}\sigma. \tag{3.56}$$

By adopting similar asymptotic analysis for each integral in (3.56) with respect to the parameter τ as in the proof of Theorem 3.1, and letting $\tau \to \infty$, we can prove Theorem 3.2. □

3.2. Proof of Corollary 3.3

Next, we give the proof of Corollary 3.3 regarding the vanishing property of transmission eigenfunctions to (3.9) near a convex planar corner under two regularity conditions described in Corollary 3.3. Since the proof of the statement (b) in Corollary 3.3 can be obtained by using the similar asymptotic analysis for proving Corollary 3.3 (a), we omit it here. In order to prove the statement (a) in Corollary 3.3, we give the following proposition which is obtained by slightly modifying the proof of Proposition 3.1.

Proposition 3.3. *Let S_h and η be defined in (3.2) and (2.3), respectively, where $\mathbf{d}$ satisfies (3.10). Then, we have*

$$\left| \int_{S_h} e^{\eta \cdot \mathbf{x}} \mathrm{d}\mathbf{x} \right| \geq \frac{\widetilde{C_{S_h}}}{\tau^2} - \mathcal{O}\left(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{3.57}$$

for sufficiently large τ, where $\widetilde{C_{S_h}}$ is a positive number only depending on the opening angle $\theta_M - \theta_m$ of $\mathcal{K}$ and ζ.

Proof. Using polar coordinates transformation and (2.6b) in Proposition 2.1, we have

$$\int_{S_h} e^{\rho \cdot \mathbf{x}} \mathrm{d}\mathbf{x} = \int_{\theta_m}^{\theta_M} \left[\frac{\Gamma(2)}{(\tau(\mathbf{d} + \mathrm{i}\mathbf{d}^\perp) \cdot \hat{\mathbf{x}})^2} - I_\mathsf{R} \right] \mathrm{d}\theta$$

$$= \frac{\Gamma(2)}{\tau^2} \int_{\theta_m}^{\theta_M} \frac{1}{\left(\mathbf{d} \cdot \hat{\mathbf{x}} + \mathrm{i}\mathbf{d}^\perp \cdot \hat{\mathbf{x}} \right)^2} \mathrm{d}\theta - \int_{\theta_m}^{\theta_M} I_\mathsf{R} \mathrm{d}\theta,$$

where $I_\mathsf{R} = \int_h^\infty e^{-\tau(\mathbf{d} + \mathrm{i}\mathbf{d}) \cdot \hat{\mathbf{x}} r} r \mathrm{d}r$. Hence, it can be directly calculated that

$$\int_{\theta_m}^{\theta_M} \frac{1}{\left(\mathbf{d} \cdot \hat{\mathbf{x}} + \mathrm{i}\mathbf{d}^\perp \cdot \hat{\mathbf{x}} \right)^2} \mathrm{d}\theta \geq \frac{\theta_M - \theta_m}{2}$$

by using the integral mean value theorem.

With the help of Proposition 2.1, for sufficiently large τ, we have the following integral inequality

$$\left| \int_{S_h} e^{\rho \cdot \mathbf{x}} \mathrm{d}\mathbf{x} \right| \geq \frac{\Gamma(2)(\theta_M - \theta_m)}{\tau^2} \frac{1}{\left| \mathbf{d} \cdot \mathbf{x}(\theta_\xi) + \mathrm{i}\mathbf{d}^\perp \cdot \hat{\mathbf{x}}(\theta_\xi) \right|^2} - \left| \int_{\theta_m}^{\theta_M} I_\mathsf{R} \mathrm{d}\theta \right|$$

$$\geq \frac{\Gamma(2)}{(\theta_M - \theta_m)} \frac{1}{\left(\mathbf{d} \cdot \hat{\mathbf{x}}(\theta_\xi) \right)^2 + \left(\mathbf{d}^\perp \cdot \hat{\mathbf{x}}(\theta_\xi) \right)^2} - \int_{\theta_m}^{\theta_M} |I_\mathsf{R}| \mathrm{d}\theta \tag{3.58}$$

$$\geq \frac{\widetilde{C_{S_h}}}{\tau^2} - \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau},$$

by using (3.10). □

Proof of Corollary 3.3(*a*). Similar to the proof of Theorem 3.1, we have the following integral identity according to (3.30) by noting $\eta \equiv 0$ on $\Gamma_h^{\pm}$,

$$k^2 f_j(0) \int_{S_h} e^{\rho \cdot \mathbf{x}} \mathrm{d}\mathbf{x} = I_1 + I_2 + I_3 + J_1 + J_2, \tag{3.59}$$

where

$$I_1 = -k^2 \int_{S_h} (q-1)(v - v_j) u_0 \mathrm{d}\mathbf{x}, \quad I_2 = -k^2 \int_{S_h} \delta f_j u_0 \mathrm{d}\mathbf{x},$$

$$I_3 = -k^2 f_j(\mathbf{0}) \int_{S_h} e^{\rho \cdot \mathbf{x}} \psi(\mathbf{x}) \mathrm{d}\mathbf{x},$$

and J_1, J_2 are defined in (3.33), respectively.

By the Sobolev embedding theorem and $q \in H^2(\overline{S}_h)$, we have $q \in C^{\alpha}(\overline{S}_h)$, where $\alpha = 1$. Combining (3.59) with (3.35), (3.42) and (3.44), we can deduce that

$$
\begin{aligned}
k^2 \left[\frac{\widetilde{C_{S_h}}}{\tau^2} - \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right] |f_j(\mathbf{0})| &\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\zeta h \tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-2\zeta h \tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{3}} \right] \\
&\quad + j^{\gamma} \left[\tau^{-(\alpha + \frac{29}{12})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right) \right] \\
&\quad + \tau^{-\frac{29}{12}} + (1+\tau)(1+\tau^{-\frac{2}{3}}) e^{-\zeta h \tau}
\end{aligned}
\tag{3.60}
$$

as $\tau \to \infty$. Multiplying τ^2 on the both sides of (3.60), using the assumption (3.5), by letting $\tau = j^s$, it is easy to see that

$$k^2 \widetilde{C_{S_h}} |f_j(\mathbf{0})| \lesssim j^{-\beta + s} + j^{\gamma - \alpha s}. \tag{3.61}$$

And under the assumptions $\gamma/\alpha < \beta$, we choose $s \in (\gamma/\alpha, \beta)$. Letting $j \to \infty$ in (3.61), we obtain that

$$|f_j(\mathbf{0})| = 0.$$

Since $q(\mathbf{0}) \neq 1$, we finish the proof of this corollary. $\qquad \square$

4. Vanishing of transmission eigenfunctions near a convex conic corner or polyhedral corner

In this section, we study the vanishing of eigenfunctions near a corner in $\mathbb{R}^3$, respectively, where the corner in $\mathbb{R}^3$ could be a convex conic corner or polyhedral corner. Let us first introduce the corresponding geometrical setup for our study. For a given point $\mathbf{x}_0 \in \mathbb{R}^3$, let $\mathbf{v}_0 = \mathbf{y}_0 - \mathbf{x}_0$ where $\mathbf{y}_0 \in \mathbb{R}^3$ is fixed. Hence,

$$\mathcal{C} = \mathcal{C}_{\mathbf{x}_0, \mathbf{v}_0, \theta_0} := \left\{ \mathbf{y} \in \mathbb{R}^3 \mid 0 \leqslant \angle(\mathbf{y} - \mathbf{x}_0, \mathbf{v}_0) \leqslant \theta_0 \right\} \quad (\theta_0 \in (0, \pi/2)) \tag{4.1}$$

is a strictly convex conic cone with the apex $\mathbf{x}_0$ and an opening angle $2\theta_0 \in (0, \pi)$ in $\mathbb{R}^3$. Here, $\mathbf{v}_0$ is referred to be the axis of $\mathcal{C}_{\mathbf{x}_0, \theta_0}$. Specifically, when $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{v}_0 = (0, 0, 1)^{\top}$, we write $\mathcal{C}_{\mathbf{x}_0, \theta_0}$ as $\mathcal{C}_{\theta_0}$. Define the truncated conic cone $\mathcal{C}^h := \mathcal{C}_{\mathbf{0}}^h$ as

$$\mathcal{C}^h := \mathcal{C}_{\theta_0} \cap B_h, \quad \Gamma_h = \partial \mathcal{C} \cap B_h, \quad \Lambda_h = \mathcal{C} \cap \partial B_h, \tag{4.2}$$

where B_h is an open ball centred at $\mathbf{0}$ with the radius $h \in \mathbb{R}_+$.

Assume that $\mathcal{K}_{\mathbf{x}_0; \mathbf{e}_1, \ldots, \mathbf{e}_\ell}$ is a polyhedral cone with the apex $\mathbf{x}_0$ and edges $\mathbf{e}_j$ ($j = 1, \ldots, \ell$, $\ell \geq 3$). Throughout of this paper, we always suppose that $\mathcal{K}_{\mathbf{x}_0; \mathbf{e}_1, \ldots, \mathbf{e}_\ell}$ is strictly convex, which implies that it can be fitted into a conic cone $\mathcal{C}_{\mathbf{x}_0, \theta_0}$ with the opening angle $\theta_0 \in (0, \pi/2)$, where $\mathcal{C}_{\mathbf{x}_0, \theta_0}$ is defined in (4.1). Without loss of generality, we assume that the axis of $\mathcal{C}_{\mathbf{x}_0, \theta_0}$ coincides with x_3^+ and $\mathbf{x}_0 = \mathbf{0}$. Given a constant $h \in \mathbb{R}_+$,

we define the truncated polyhedral corner $\mathcal{K}^h_{\mathbf{x}_0}$ as

$$\mathcal{K}_{\mathbf{x}_0^h} = \mathcal{K}_{\mathbf{x}_0;\mathbf{e}_1\dots\mathbf{e}_\ell} \cap B_h. \tag{4.3}$$

For convenience, we have a similar geometry setup with (4.2) as

$$\mathcal{K}^h = \mathcal{K}^h_{\mathbf{0}}, \ \Gamma_h = \partial\mathcal{K} \cap B_h, \ \Lambda_h = \mathcal{K} \cap \partial B_h. \tag{4.4}$$

The following theorem states that the transmission eigenfunctions to (1.1) must vanish at a conic corner if they have H^1 regularity and v can be approximated by a sequence of Herglotz wave functions near the underlying conic corner with certain properties, where the detailed proof is postponed to Subsection 4.1.

Theorem 4.1. *Let Ω be a bounded Lipschitz domain with a connected complement and $v, w \in H^1(\Omega)$ be a pair of transmission eigenfunctions to (1.1) associated with $k \in \mathbb{R}_+$. Assume that $\mathbf{0} \in \Gamma \subset \partial\Omega$ such that $\Omega \cap B_h = \mathcal{C} \cap B_h = \mathcal{C}^h$, where $\mathcal{C}$ is defined by (4.1) and $h \in \mathbb{R}_+$ is sufficient small such that $q \in H^2(\overline{\mathcal{C}^h})$ and $\eta \in C^{\alpha_1}(\overline{\Gamma_h})$, where $\alpha_1 \in (0,1)$. If the following conditions are fulfilled:*

(a) for any given positive constants β and γ satisfying

$$\gamma < \frac{10}{11}\alpha\beta, \quad \alpha = \min\{\alpha_1, 1/2\}, \tag{4.5}$$

the transmission eigenfunction v can be approximated in $H^1(\mathcal{C}^h)$ by Herglotz functions

$$v_j = \int_{\mathbb{S}^2} e^{\mathrm{i}k\xi\cdot\mathbf{x}} g_j(\xi)\mathrm{d}\xi, \quad \xi \in \mathbb{S}^2, j = 1, 2, \cdots, \tag{4.6}$$

with the kernels g_j satisfying the approximation property

$$\|v - v_j\|_{H^1(\mathcal{C}^h)} \le j^{-\beta}, \quad \|g_j\|_{L^2(\mathcal{C}^h)} \le j^{\gamma}; \tag{4.7}$$

(b) the function η does not vanish at the apex $\mathbf{0}$ of $\mathcal{C}^h$;

then one has

$$\lim_{\lambda\to\infty} \frac{1}{m(B(\mathbf{0},\lambda)\cap\Omega)} \int_{B(\mathbf{0},\lambda)\cap\Omega} |v(\mathbf{x})|\mathrm{d}\mathbf{x} = 0, \tag{4.8}$$

where $m(B(\mathbf{0},\lambda)\cap\Omega)$ is the area of $B(\mathbf{0},\lambda)\cap\Omega$.

As remarked earlier, the Herglotz approximation property in (4.7) characterizes a regularity of v weaker than the Hölder continuity (cf. [41]). In the following theorem, if a stronger Hölder regularity condition near a conic corner on the transmission eigenfunction v to (1.1) is satisfied, we also have the vanishing characterization of the corresponding transmission eigenfunction v. Namely, when v is Hölder continuous near the underlying circular corner, we show that it must vanish at the apex of the conic corner. The proof can be obtained by modifying the corresponding proof of Theorem 4.1 directly as for the two-dimensional case, which is omitted.

Theorem 4.2. *Let $v \in H^1(\Omega)$ and $w \in H^1(\Omega)$ be a pair of transmission eigenfunctions to (1.1) associated with $k \in \mathbb{R}_+$. Assume that the Lipschitz domain $\Omega \subset \mathbb{R}^3$ with $\mathbf{0} \in \partial\Omega$ contains a conic corner $\Omega \cap B_h = \mathcal{C} \cap B_h = \mathcal{C}^h$, such that $v \in C^\alpha(\overline{\mathcal{C}^h})$, $q \in H^2(\overline{\mathcal{C}^h})$ and $\eta \in C^\alpha(\overline{\Gamma_h})$ for $0 < \alpha < 1$, where B_h, Γ_h and $\mathcal{C}^h$ are defined in (4.2). If $\eta(\mathbf{0}) \neq 0$, where $\mathbf{0}$ is the apex of $\mathcal{C}^h$, then one has*

$$v(\mathbf{0}) = 0. \tag{4.9}$$

Consider a cuboid corner $\mathcal{K}_{\mathbf{x}_0;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}$ defined by (4.3). In Theorem 4.3, we show that the transmission eigenfunctions to (1.1) vanish at the cuboid corner $\mathcal{K}_{\mathbf{x}_0;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}$ when they are Hölder continuous at the corner point. The proof of Theorem 4.3 can be found in Subsection 4.2. Since Δ is invariant under rigid

motion, we assume that the apex $\mathbf{x}_0$ of $\mathcal{K}_{\mathbf{x}_0;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}$ coincides with the origin, and the edges of $\mathcal{K}_{\mathbf{x}_0;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}$ satisfy $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$ and $\mathbf{e}_3 = (0, 0, 1)^\top$.

Theorem 4.3. *Let $v \in H^1(\Omega)$, $w \in H^1(\Omega)$ be a pair of transmission eigenfunctions of (1.1) with $k > 0$. Assume that Lipschitz domain $\Omega \subset \mathbb{R}^3$ with $\mathbf{0} \in \Gamma \subset \partial\Omega$ contains a cuboid corner $\Omega \cap B_h = \mathcal{K} \cap B_h = \mathcal{K}^h$, such that $v \in C^\alpha(\overline{\mathcal{K}^h})$, $q \in H^2(\overline{\mathcal{K}^h})$ and $\eta \in C^\alpha(\overline{\Gamma_h})$ for $0 < \alpha < 1$, where $\mathcal{K}^h$ and Γ_h are defined in (4.3). If $\eta(\mathbf{0}) \neq 0$, then*

$$v(\mathbf{0}) = 0.$$

Remark 4.1. *Consider the classical transmission eigenvalue problem (3.9) in $\mathbb{R}^3$, namely $\eta \equiv 0$ on Γ in (1.1), when the underlying domain Ω of (3.9) has a cuboid corner $\mathcal{K}_{\mathbf{x}_0;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3}$, if the corresponding potential q has α-Hölder continuity regularity for $\alpha > \frac{1}{4}$ near the cuboid corner (cf. [9, Definition 2.2 and Theorem 3.2]), then the transmission eigenfunction v must vanish near the corner. Compared with the results in [9], the vanishing property of transmission eigenfunctions to (1.1) near the underlying cuboid corner holds under a general scenario. Namely, the assumption in Theorem 4.3 only needs q fulfils H^2 regularity, v and boundary parameter η are Hölder continuous near $\mathbf{x}_0$, where $\eta(\mathbf{x}_0) \neq 0$.*

In the following two corollaries, we consider the classical transmission eigenvalue problem (3.9), namely $\eta \equiv 0$ on Γ in (1.1), where the domain Ω contains a conic or polyhedral corner. The proof of Corollary 4.4 is postponed in Subsection 4.3.

Corollary 4.4. *Let Ω be a bounded Lipschitz domain with a connected complement and $v, w \in H^1(\Omega)$ be a pair of transmission eigenfunctions to (3.9) associated with $k \in \mathbb{R}_+$. Assume that $\mathbf{0} \in \Gamma \subset \partial\Omega$ such that $\Omega \cap B_h = \mathcal{C} \cap B_h = \mathcal{C}^h$, where $\mathcal{C}$ is defined by (4.1) and $h \in \mathbb{R}_+$ is sufficient small such that $q \in H^2(\overline{\mathcal{C}^h})$ and $q(\mathbf{0}) \neq 1$.*

   (a) *For any given positive constants β and γ satisfying $\gamma < \frac{20}{37}\alpha\beta$, if the transmission eigenfunction v can be approximated in $H^1(\mathcal{C}^h)$ by Herglotz wave functions v_j defined by (4.6) with the kernels g_j satisfying the approximation property (4.7), then we have the vanishing of the transmission eigenfunction v near $\mathcal{C}^h$ in the sense of (4.8).*
   (b) *If $v \in C^\alpha(\overline{\mathcal{C}^h})$ with $\alpha \in (0, 1)$, then one has $v(\mathbf{0}) = 0$.*

In the subsequent corollary, we consider the case that Ω contains a polyhedral corner $\mathcal{K}^h$ defined by (4.3). When the transmission eigenfunction v to (3.9) satisfies two regularity assumptions, we can establish the similar geometrical characterization of v near the polyhedral corner. The proofs are similar to the counterpart of Theorem 4.2 and Corollary 4.4, where we only need to use the asymptotic analysis [10, Lemma 2.2] with respect to the parameter in the corresponding CGO solution introduced in the following subsection. Hence, we omit its proof.

Corollary 4.5. *Let Ω be a bounded Lipschitz domain with a connected complement and $v, w \in H^1(\Omega)$ be a pair of transmission eigenfunctions to (3.9) associated with $k \in \mathbb{R}_+$. Assume that $\mathbf{0} \in \Gamma \subset \partial\Omega$ such that $\Omega \cap B_h = \mathcal{K}_{\mathbf{0};\mathbf{e}_1,\cdots,\mathbf{e}_\ell} \cap B_h = \mathcal{K}^h$, where $\mathcal{K}^h$ is defined by (4.3) and $h \in \mathbb{R}_+$ is sufficiently small such that $q \in H^2(\overline{\mathcal{K}^h})$ and $q(\mathbf{0}) \neq 1$.*

   (a) *For any given positive constants β and γ satisfying $\gamma < \frac{20}{37}\alpha\beta$, if the transmission eigenfunction v can be approximated in $H^1(\mathcal{C}^h)$ by Herglotz wave functions v_j defined by (4.6) with the kernels g_j satisfying the approximation property (4.7), then we have the vanishing property of v near $\mathcal{K}^h$ in the sense of (4.8).*
   (b) *If $v \in C^\alpha(\overline{\mathcal{K}^h})$ with $\alpha \in (0, 1)$, then one has $v(\mathbf{0}) = 0$.*

4.1. Proof of Theorem 4.1

Since the conic cone $\mathcal{C}$ defined by (4.1) is strictly convex, for any given positive constant ζ, we define $\mathcal{C}_\zeta$ as the open set of $\mathbb{S}^2$ which is composed by all unit directions $\mathbf{d} \in \mathbb{S}^2$ satisfying that

$$\mathbf{d} \cdot \hat{\mathbf{x}} > \zeta > 0, \quad \textit{for all } \hat{\mathbf{x}} \in \mathcal{C} \cap \mathbb{S}^2. \tag{4.10}$$

Throughout this subsection, we always assume that the unit vector $\mathbf{d}$ in the form of the CGO solution u_0 given by (2.2) satisfies (4.10). In order to prove Theorem 4.1, we need several key propositions and lemmas in the following.

Proposition 4.1. *Let Γ_h and ρ be defined in (4.2) and (2.3), respectively. Then, we have*

$$\left| \int_{\Gamma_h} e^{\rho \cdot \mathbf{x}} d\sigma \right| \geq \frac{C_{\mathcal{C}^h}}{\tau^2} - \mathcal{O}\left(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right), \tag{4.11}$$

for sufficiently large τ, where $C_{\mathcal{C}^h}$ is a positive number only depending on the opening angle θ_0 of $\mathcal{C}$ and ζ.

Proof. Using polar coordinates transformation and the mean value theorem for integrals, we have

$$\int_{\Gamma_h} e^{\rho \cdot \mathbf{x}} d\sigma = \sin \theta_0 \frac{2\pi \Gamma(2)}{\tau^2} \frac{1}{((\mathbf{d} + \mathrm{i} \mathbf{d}^\perp) \cdot \hat{\mathbf{x}}(\theta_0, \varphi_\xi))^2} - \sin \theta_0 \int_0^{2\pi} I_R d\varphi, \tag{4.12}$$

where $I_R = \int_h^\infty e^{-\tau(\mathbf{d}+\mathrm{i}\mathbf{d})\cdot \hat{\mathbf{x}} r} r \, dr$. Furthermore, for sufficiently large τ, it is ready to know that

$$\frac{1}{\tau^2} - \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} > 0.$$

Hence, by virtue of (4.10) and Proposition 2.1, we have the following integral inequality

$$\left| \int_{\Gamma_h} e^{\rho \cdot \mathbf{x}} d\sigma \right| \geq \sin \theta_0 \frac{2\pi \Gamma(2)}{\tau^2} \frac{1}{(\mathbf{d} \cdot \hat{\mathbf{x}}(\theta_0, \varphi_\xi))^2 + (\mathbf{d}^\perp \cdot \hat{\mathbf{x}}(\theta_0, \varphi_\xi))^2} - \sin \theta_0 \int_0^{2\pi} |I_R| d\varphi$$

$$\geq \frac{C_{\mathcal{C}^h}}{\tau^2} - \mathcal{O}(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau}),$$

which completes the proof of this proposition. $\qquad \square$

Similar to Proposition 3.2, the following proposition can be obtained by direct verifications.

Proposition 4.2. *Let $\mathcal{C}^h$ be defined by (4.2). For any given $t > 0$, it yields that*

$$\|e^{\rho \cdot \mathbf{x}}\|_{L^t(\mathcal{C}^h)} \leq C \left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-\frac{t}{2}\zeta h\tau} \right)^{\frac{1}{t}},$$

$$\tag{4.13}$$

$$\|e^{\rho \cdot \mathbf{x}}\|_{L^t(\Gamma_h)} \leq C \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\frac{t}{2}\zeta h\tau} \right)^{\frac{1}{t}},$$

as $\tau \to \infty$, where ρ is defined in (2.3) and C is a positive constant only depending on t, ζ.

The proof of the following lemma is similar to 3.1, hence we omit it here.

Lemma 4.1. *Under the same setup of Theorem 4.1, let the CGO solution u_0 be defined by (2.2). We also denote $u = w - v$, where (v, w) is a pair of transmission eigenfunctions of (1.1) associated with $k \in \mathbb{R}_+$. Then, it holds that*

$$\begin{cases} \Delta u_0 + k^2 q u_0 = 0 & \text{in} \quad \mathcal{C}^h, \\ \Delta u + k^2 q u = k^2 (1-q)v & \text{in} \quad \mathcal{C}^h, \\ u = 0, \quad \partial_\nu u = 0 & \text{on} \quad \Gamma_h, \end{cases} \tag{4.14}$$

where $\mathcal{C}^h$ and Γ_h are defined by (4.2), and

$$\|\psi(\mathbf{x})\|_{H^{1,8}} = \mathcal{O}(\tau^{-\frac{2}{5}}), \tag{4.15}$$

where ψ and τ are defined in (2.2).

Lemma 4.2. *Let Λ_h and $\mathcal{C}^h$ be defined in (4.2). Recall that $u_0(\mathbf{x})$ is given by (2.2). Then, $u_0(\mathbf{x}) \in H^1(\mathcal{C}^h)$, and it holds that*

$$\|u_0(\mathbf{x})\|_{L^2(\Lambda_h)} \lesssim \left(1 + \tau^{-\frac{2}{5}}\right) e^{-\zeta h \tau}, \tag{4.16a}$$

$$\|\nabla u_0(\mathbf{x})\|_{L^2(\Lambda_h)} \lesssim (1 + \tau)\left(1 + \tau^{-\frac{2}{5}}\right) e^{-\zeta h \tau}, \tag{4.16b}$$

$$\int_{\mathcal{C}^h} |\mathbf{x}|^\alpha |u_0(\mathbf{x})| \mathrm{d}\mathbf{x} \lesssim \tau^{-(\alpha + \frac{121}{40})} + \left(\frac{1}{\tau^{\alpha+3}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau}\right), \tag{4.16c}$$

$$\int_{\Gamma_h} |\mathbf{x}|^\alpha |u_0(\mathbf{x})| \mathrm{d}\sigma \lesssim \tau^{-(\alpha + \frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau}\right) \tag{4.16d}$$

as $\tau \to \infty$, where ζ is defined in (3.10) and $\alpha \in (0, 1)$.

Proof. Using (4.15) and Lemma 3.2 about the trace theorem, it yields that

$$\|\psi(\mathbf{x})\|_{L^4(\Lambda_h)} \le C\|\psi(\mathbf{x})\|_{H^{\frac{7}{8},8}(\Lambda_h)} \le C\|\psi(\mathbf{x})\|_{H^{1,8}(\mathcal{C}^h)} = O(\tau^{-\frac{2}{5}}),$$

as $\tau \to \infty$. By using polar coordinates transformation, (2.6a) and (4.10), one can derive that

$$\|e^{\rho \cdot \mathbf{x}}\|_{L^t(\Lambda_h)} \lesssim e^{-\zeta h \tau}, \tag{4.17}$$

where ρ is defined in (2.3) and t is a positive constant.

Due to polar coordinates transformation, (4.17), (4.15) and Hölder inequality, it can be calculated that

$$\|u_0\|_{L^2(\Lambda_h)} \le \|e^{\rho \cdot \mathbf{x}}\|_{L^2(\Lambda_h)} + \|e^{\rho \cdot \mathbf{x}}\|_{L^4(\Lambda_h)}\|\psi(\mathbf{x})\|_{L^4(\Lambda_h)}$$
$$\lesssim \left(1 + \tau^{-\frac{2}{5}}\right) e^{-\zeta h \tau}, \quad \text{as } \tau \to \infty. \tag{4.18}$$

By virtue of Cauchy-Schwarz inequality, (4.16a) and Proposition 2.1, we can deduce that

$$\|\nabla u_0\|_{L^2(\Lambda_h)} \le \sqrt{2}\tau \|u_0\|_{L^2(\Lambda_h)} + \|e^{\rho \cdot \mathbf{x}}\|_{L^4(\Lambda_h)}\|\nabla \psi(\mathbf{x})\|_{L^4(\Lambda_h)}$$
$$\lesssim (1 + \tau)(1 + \tau^{-\frac{2}{5}}) e^{-\zeta h \tau}, \quad \text{as } \tau \to \infty. \tag{4.19}$$

It is clear that we can get the following integral inequality,

$$\int_{\mathcal{C}^h} |\mathbf{x}|^\alpha |u_0| \mathrm{d}\mathbf{x} \le \int_{\mathcal{C}^h} |\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}| \mathrm{d}\mathbf{x} + \int_{\mathcal{C}^h} (|\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}|)(|\psi(\mathbf{x})|) \mathrm{d}\mathbf{x}. \tag{4.20}$$

By virtue of polar coordinates transformation and Proposition 2.1, it reveals that

$$\int_{\mathcal{C}^h} |\mathbf{x}|^\alpha e^{-\zeta \tau |\mathbf{x}|} \mathrm{d}\mathbf{x} \lesssim \frac{1}{\tau^{\alpha+3}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau}, \tag{4.21}$$

as $\tau \to \infty$. Next, letting $\mathbf{y} = \tau \mathbf{x}$, using Cauchy-Schwarz inequality and Hölder inequality, it arrives that

$$\int_{\mathcal{C}^h} (|\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}|)(|\psi(\mathbf{x})|) \mathrm{d}\mathbf{x} \le \frac{1}{\tau^{\alpha+3}} \int_{\mathcal{C}} |\mathbf{y}|^\alpha |e^{-\mathbf{d} \cdot \mathbf{y}}| \left|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right| \mathrm{d}\mathbf{y}$$
$$\le \frac{1}{\tau^{\alpha+3}} \||\mathbf{y}|^\alpha |e^{-\mathbf{d} \cdot \mathbf{y}}|\|_{L^{\frac{8}{7}}(\mathcal{C})} \left\|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right\|_{L^8(\mathcal{C})}, \tag{4.22}$$

as $\tau \to \infty$, using variable substitution and (4.15), it arrives that

$$\left\|\psi\left(\frac{\mathbf{y}}{\tau}\right)\right\|_{L^8(\mathcal{C})} = \tau^{\frac{3}{8}} \|\psi(\mathbf{x})\|_{L^8(\mathcal{C})} \le \tau^{\frac{3}{8}} \|\psi(\mathbf{x})\|_{H^{1,8}(\mathcal{C})} = \mathcal{O}(\tau^{-\frac{1}{40}}), \tag{4.23}$$

as $\tau \to \infty$. Furthermore, one has

$$\||\mathbf{y}|^\alpha |e^{\rho \cdot \mathbf{x}}|\|_{L^{\frac{8}{7}}(\mathcal{C})} = \left(\int_{\mathcal{C}} |\mathbf{y}|^{\frac{8}{7}\alpha} e^{-\frac{8}{7}\mathbf{d} \cdot \mathbf{y}} \mathrm{d}\mathbf{y}\right)^{\frac{7}{8}} \le \left(\int_{\mathcal{C}} |\mathbf{y}|^{\frac{8}{7}\alpha} e^{-\frac{8}{7}\zeta |\mathbf{y}|} \mathrm{d}\mathbf{y}\right)^{\frac{7}{8}} \le \frac{C}{\zeta^{3 + \frac{8}{7}\alpha}}, \tag{4.24}$$

where $C = 2\pi \theta_0 \Gamma(3 + \frac{8}{7}\alpha)(\frac{7}{8})^{3+\frac{8}{7}\alpha}$. Hence, $\||\mathbf{y}|^\alpha |e^{\rho \cdot \mathbf{x}}|\|_{L^{\frac{8}{7}}(\mathcal{C})}$ is a positive constant which only depends on θ_0, ζ and α. Combining (4.23), (4.22) and (4.21) with (4.20), one has (4.16c).

Furthermore, we have

$$\int_{\Gamma_h} |\mathbf{x}|^\alpha |u_0| \mathrm{d}\sigma \le \int_{\Gamma_h} |\mathbf{x}||e^{\rho \cdot \mathbf{x}}| \mathrm{d}\sigma + \int_{\Gamma_h} |\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}||\psi(\mathbf{x})| \mathrm{d}\sigma, \tag{4.25}$$

and we can easily get (4.26) by using polar coordinates transformation and Proposition 2.1,

$$\int_{\Gamma_h} |\mathbf{x}||e^{\rho \cdot \mathbf{x}}| \mathrm{d}\sigma \lesssim \frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau}, \tag{4.26}$$

as $\tau \to \infty$. Then letting $\mathbf{y} = \tau \mathbf{x}$ and utilizing Hölder inequality, it can be obtained that

$$\int_{\Gamma_h} |\mathbf{x}|^\alpha |e^{\rho \cdot \mathbf{x}}||\psi(\mathbf{x})| \mathrm{d}\sigma \le \frac{1}{\tau^{\alpha+2}} \int_{\partial \mathcal{C}} |\mathbf{y}|^\alpha |e^{-\mathbf{d} \cdot \mathbf{y}}| \left| \psi\left(\frac{\mathbf{y}}{\tau}\right) \right| \mathrm{d}\sigma$$

$$\le \frac{1}{\tau^{\alpha+2}} \||\mathbf{y}|^\alpha |e^{-\mathbf{d} \cdot \mathbf{y}}|\|_{L^{\frac{8}{7}}(\partial \mathcal{C})} \left\| \psi\left(\frac{\mathbf{y}}{\tau}\right) \right\|_{L^8(\partial \mathcal{C})}. \tag{4.27}$$

Similar to (4.24), we know that $\||\mathbf{y}|^\alpha |e^{-\mathbf{d} \cdot \mathbf{y}}|\|_{L^{\frac{8}{7}}(\partial \mathcal{C})}$ is a positive constant. By virtue of variable substitution, trace theorem and (4.15), it arrives that

$$\left\| \psi\left(\frac{\mathbf{y}}{\tau}\right) \right\|_{L^8(\partial \mathcal{C})} \lesssim \tau^{\frac{1}{4}} \|\psi(\mathbf{x})\|_{H^{\frac{7}{8},8}(\partial \mathcal{C})} \lesssim \tau^{\frac{1}{4}} \|\psi(\mathbf{x})\|_{H^{1,8}(\mathcal{C})} \lesssim \tau^{-\frac{3}{20}}, \tag{4.28}$$

as $\tau \to \infty$. Combining (4.26), (4.27) and (4.28) with (4.25), one has (4.16d). □

Now, we are in the position to prove Theorem 4.1.

Proof of Theorem 4.1. The proof of this theorem is similar to the counterpart of Theorem 3.1. Recall that (v, w) is a pair of transmission eigenfunctions to (1.1). Using Green formula (2.7) and boundary conditions in (4.14), the following integral identity holds

$$\int_{\mathcal{C}^h} k^2(q-1)vu_0 \mathrm{d}\mathbf{x} = \int_{\Lambda_h} (w-v)\partial_\nu u_0 - u_0 \partial(w-v) \mathrm{d}\sigma - \int_{\Gamma_h} \eta u_0 v \mathrm{d}\sigma \tag{4.29}$$

where $\mathcal{C}^h$, Λ_h and Γ_h are defined by (4.2). Let

$$f_j = (q-1)v_j.$$

Due to $q \in H^2(\overline{\mathcal{C}^h})$, we know that $q \in C^{1/2}(\overline{\mathcal{C}^h})$ by using the property of embedding of Sobolev space. Recall that $\eta \in C^{\alpha_1}(\overline{\Gamma_h})$. Let $\alpha = \{\alpha_1, 1/2\}$. Furthermore, since the Herglotz wave function $v_j \in C^\alpha(\overline{\mathcal{C}^h})$, it yields that $f_j \in C^\alpha(\overline{\mathcal{C}}^h)$. Hence, one has the expansion

$$\begin{aligned} f_j &= f_j(\mathbf{0}) + \delta f_j, \ |\delta f_j| \le \|f_j\|_{C^\alpha(\overline{\mathcal{C}^h})}|\mathbf{x}|^\alpha, \\ v_j &= v_j(\mathbf{0}) + \delta v_j, \ |\delta v_j| \le \|v_j\|_{C^\alpha(\overline{\mathcal{C}^h})}|\mathbf{x}|^\alpha, \\ \eta &= \eta(\mathbf{0}) + \delta \eta, \ |\delta \eta| \le \|\eta\|_{C^\alpha(\overline{\Gamma_h})}|\mathbf{x}|^\alpha. \end{aligned} \tag{4.30}$$

By virtue of (4.30), we have the following integral identities

$$k^2 \int_{\mathcal{C}_h} (q-1)vu_0 \mathrm{d}\mathbf{x} = -\sum_{m=1}^{3} I_m, \quad \int_{\Gamma_h} \eta u_0 v \mathrm{d}\sigma = I - \sum_{m=4}^{9} I_m, \tag{4.31}$$

where

$$I_1 = -k^2 \int_{\mathcal{C}^h} (q-1)(v-v_j)u_0 \mathrm{d}\mathbf{x}, \quad I_2 = -\int_{\mathcal{C}^h} \delta f_j u_0 \mathrm{d}\mathbf{x},$$

$$I_3 = -f_j(\mathbf{0}) \int_{\mathcal{C}^h} u_0 \mathrm{d}\mathbf{x}, \quad I_4 = -\eta(\mathbf{0}) \int_{\Gamma_h} (v-v_j)u_0 \mathrm{d}\sigma$$

$$I_5 = -\int_{\Gamma_h} \delta\eta(v-v_j)u_0 \mathrm{d}\sigma, \quad I_6 = -\eta(\mathbf{0})v_j(\mathbf{0}) \int_{\Gamma_h} e^{\rho\cdot\mathbf{x}}\psi(\mathbf{x})\mathrm{d}\sigma, \tag{4.32}$$

$$I_7 = \eta(\mathbf{0}) \int_{\Gamma_h} \delta v_j u_0 \mathrm{d}\sigma, \quad I_8 = -v_j(\mathbf{0}) \int_{\Gamma_h} \delta\eta u_0 \mathrm{d}\sigma,$$

$$I_9 = \int_{\Gamma_h} \delta\eta \delta v_j u_0 \mathrm{d}\sigma, \quad I = \eta(\mathbf{0})v_j(\mathbf{0}) \int_{\Gamma_h} e^{\rho\cdot\mathbf{x}} \mathrm{d}\sigma.$$

Substituting (4.31) into (4.29), it yields that

$$I = \sum_{m=1}^{9} I_m + J_1 + J_2,$$

where

$$J_1 = \int_{\Lambda_h} u_0 \partial_\nu(w-v) \mathrm{d}\sigma, \quad J_2 = -\int_{\Lambda_h} (w-v)\partial_\nu u_0 \mathrm{d}\sigma. \tag{4.33}$$

Hence, it readily yields that

$$|I| \leq \sum_{m=1}^{9} |I_m| + |J_1| + |J_2|. \tag{4.34}$$

In the sequel, we derive the asymptotic estimates of I_j ($j = 1, \ldots, 9$) and J_j, $j = 1, 2$ with respect to the parameter τ in the CGO solution u_0 when $\tau \to \infty$, separately. Using Hölder inequality, Proposition 4.2 and (4.15), it is clear that

$$|I_1| \leq \|v - v_j\|_{L^2(\mathcal{C}^h)} \|e^{\rho\cdot\mathbf{x}}\|_{L^2(\mathcal{C}^h)} + \|v - v_j\|_{L^2(\mathcal{C}^h)} \|e^{\rho\cdot\mathbf{x}}\psi(\mathbf{x})\|_{L^2(\mathcal{C}^h)}$$

$$\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right], \tag{4.35}$$

as $\tau \to \infty$.

With the help of (4.30), we have

$$|I_2| \leq k^2 \|f_j\|_{C^\alpha} \int_{\mathcal{C}^h} |\mathbf{x}|^\alpha |u_0| \mathrm{d}\mathbf{x}, \tag{4.36}$$

and

$$\|f_j\|_{C^\alpha(\mathcal{C}^h)} \leq \|q\|_{C^\alpha(\mathcal{C}^h)} \sup_{\mathcal{C}^h} |v_j| + \|v_j\|_{C^\alpha(\mathcal{C}^h)} \sup_{\mathcal{C}^h} |q-1|. \tag{4.37}$$

Moreover, due to the property of compact embedding of Hölder spaces, one has

$$\|v_j\|_{C^\alpha(\mathcal{C}^h)} \leq \mathrm{diam}\left(\mathcal{C}^h\right)^{1-\alpha} \|v_j\|_{C^1(\mathcal{C}^h)}, \tag{4.38}$$

where $\mathrm{diam}(\mathcal{C}^h)$ is the diameter of $\mathcal{C}^h$. It can be directly shown that

$$\|v_j\|_{C^1(\mathcal{C}^h)} \leq 4\sqrt{\pi}(1+k)\|g\|_{L^2(\mathbb{S}^2)}. \tag{4.39}$$

On the other hand, we can obtain the following estimate by using the Cauchy-Schwarz inequality,

$$|v_j| \leq 4\sqrt{\pi} \|g\|_{L^2(\mathbb{S}^2)}. \tag{4.40}$$

Using (4.7) and $q \in C^\alpha(\overline{C^h})$, plugging (4.7), (4.38), (4.39) and (4.40) into (4.37), one can arrive at

$$\|f_j\|_{C^\alpha(C^h)} \lesssim j^\gamma, \tag{4.41}$$

where γ is a given positive constant defined in (4.7). Substituting (4.16c) and (4.41) into (4.36), we obtain

$$|I_2| \lesssim j^\gamma \left[\tau^{-(\alpha+\frac{121}{40})} + \left(\frac{1}{\tau^{\alpha+3}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right] \tag{4.42}$$

as $\tau \to \infty$.

With the help of Cauchy-Schwarz inequality and (4.7), it yields that

$$|I_3| \leq \int_{C^h} |e^{\rho \cdot \mathbf{x}}| d\mathbf{x} + \int_{C} |e^{\rho \cdot \mathbf{x}}| |\psi(\mathbf{x})| d\mathbf{x}, \tag{4.43}$$

Similar to (4.24), we have that $\|e^{-\mathbf{d} \cdot \mathbf{y}}\|_{L^{\frac{8}{7}}(C)}$ is a positive constant depending only on ζ and θ_0. Letting $\mathbf{y} = \tau \mathbf{x}$ and using (4.23), it can be calculated that

$$\int_{C} |e^{\rho \cdot \mathbf{x}}| |\psi(\mathbf{x})| d\mathbf{x} \leq \frac{1}{\tau^3} \|e^{-\mathbf{d} \cdot \mathbf{y}}\|_{L^{\frac{8}{7}}(C)} \left\| \psi \left(\frac{\mathbf{y}}{\tau} \right) \right\|_{L^8(C)} \lesssim \tau^{-\frac{121}{40}}, \tag{4.44}$$

as $\tau \to \infty$. Therefore, with the help of Proposition 4.2, and plugging (4.44) into (4.43), one has

$$|I_3| \lesssim \tau^{-\frac{121}{40}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right), \quad \text{as } \tau \to \infty. \tag{4.45}$$

By virtue of Cauchy-Schwarz inequality and Lemma 3.2, we can obtain that

$$|I_4| \lesssim \|v - v_j\|_{L^2(\Gamma_h)} \|e^{\rho \cdot \mathbf{x}}\|_{L^2(\Gamma_h)} + \|v - v_j\|_{L^2(\Gamma_h)} \|e^{\rho \cdot \mathbf{x}}\|_{L^4(\Gamma_h)} \|\psi(\mathbf{x})\|_{L^4(\Gamma_h)}$$

$$\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right] \tag{4.46}$$

as $\tau \to \infty$.

With the help of Cauchy-Schwarz inequality, Lemma 3.2 and Hölder inequality, one has

$$|I_5| \lesssim \|v - v_j\|_{L^2(\Gamma_h)} (\||e^{\rho \cdot \mathbf{x}}||\mathbf{x}|^\alpha\|_{L^2(\Gamma_h)} + \||e^{\rho \cdot \mathbf{x}}||\mathbf{x}|^\alpha\|_{L^4(\Gamma_h)} \|\psi(\mathbf{x})\|_{L^4(\Gamma_h)})$$

$$\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^{(2\alpha+2)}} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^{(4\alpha+2)}} + \frac{1}{\tau} e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right], \tag{4.47}$$

as $\tau \to \infty$.

Similar to (3.19), it can be directly obtained that

$$\left\| \psi \left(\frac{\mathbf{y}}{\tau} \right) \right\|_{H^{1,8}(C)} \leq \tau^{\frac{3}{8}} \|\psi(\mathbf{x})\|_{H^{1,8}(C)} = \mathcal{O}(\tau^{-\frac{1}{40}}), \tag{4.48}$$

as $\tau \to \infty$. Therefore, following the proof of Lemma 3.3 and using Hölder inequality and Lemma 3.2, we have

$$|I_6| \lesssim \frac{1}{\tau^2} \|e^{-\mathbf{d} \cdot \mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma)} \left\| \psi \left(\frac{\mathbf{y}}{\tau} \right) \right\|_{L^8(\Gamma)}$$

$$\lesssim \frac{1}{\tau^2} \|e^{-\mathbf{d} \cdot \mathbf{y}}\|_{L^{\frac{8}{7}}(\Gamma)} \tau^{\frac{3}{8}} \|\psi(\mathbf{x})\|_{H^{1,8}(C)} \lesssim \tau^{-\frac{81}{40}}, \quad \text{as } \tau \to \infty. \tag{4.49}$$

Moreover, we have the following estimates for I_7, I_8 and I_9 by virtue of (4.16d) directly,

$$|I_7| \lesssim \|v_j\|_{C^\alpha(\Gamma_h)} \int_{\Gamma_h} |\mathbf{x}|^\alpha |u_0| d\sigma \lesssim j^\gamma \left[\tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right],$$

$$|I_8| \lesssim \tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right),$$

$$|I_9| \lesssim j^\gamma \left[\tau^{-(2\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{2\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right], \quad \text{as } \tau \to \infty. \tag{4.50}$$

Using Cauchy-Schwarz inequality and Lemma 3.2, we obtain that

$$|J_1| \leq \|u_0\|_{H^{\frac{1}{2}}(\Lambda_h)} \|\partial_\nu(w-v)\|_{H^{-\frac{1}{2}}(\Lambda_h)} \leq C \|u_0\|_{H^1(\Lambda_h)} \|\partial_\nu(w-v)\|_{H^1(\Lambda_h)}$$

$$\lesssim \|u_0\|_{H^1(\Lambda_h)} \tag{4.51}$$

as $\tau \to \infty$, where C is a positive constant arising from the trace theorem. By virtue of (4.16a) and (4.16b), it can be calculated that

$$|J_1| \lesssim (1+\tau)(1+\tau^{-\frac{2}{5}}) e^{-\zeta h\tau} \tag{4.52}$$

as $\tau \to \infty$, where ζ is a positive constant given in (4.10). Finally, using Cauchy-Schwarz inequality, the trace theorem and (4.16b), we can obtain that

$$|J_2| \leq \|\partial_\nu u_0\|_{L^2(\Lambda_h)} \|w-v\|_{L^2(\Lambda_h)} \leq C \|\partial_\nu u_0\|_{L^2(\Lambda_h)} \|w-v\|_{H^1(\mathcal{C}^h)}$$

$$\lesssim (1+\tau)(1+\tau^{-\frac{2}{5}}) e^{-\zeta h\tau}, \tag{4.53}$$

as $\tau \to \infty$.

Substituting (4.35), (4.42), (4.45)−(4.50), (4.52) and (4.53) into (4.34), we have

$$|\eta(\mathbf{0})v_j(\mathbf{0})| \left(\frac{C_{\mathcal{C}^h}}{\tau^2} - \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \lesssim j^{-\beta} \left[\left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-4\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right]$$

$$+ j^\gamma \left[\tau^{-(\alpha+\frac{121}{40})} + \left(\frac{1}{\tau^{\alpha+3}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right]$$

$$+ j^{-\beta} \left[\left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^2} + \frac{1}{\tau} e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right]$$

$$+ j^{-\beta} \left[\left(\frac{1}{\tau^{(2\alpha+2)}} + \frac{1}{\tau} e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^{(4\alpha+2)}} + \frac{1}{\tau} e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right]$$

$$+ (j^\gamma + 1) \left[\tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right]$$

$$+ j^\gamma \left[\tau^{-(2\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{2\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) \right]$$

$$+ \tau^{-\frac{81}{40}} + \tau^{-\frac{121}{40}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} \right) + (1+\tau)(1+\tau^{-\frac{2}{5}}) e^{-\zeta h\tau} \tag{4.54}$$

as $\tau \to \infty$, where $C_{\mathcal{C}^h}$ is a positive constant given in (4.11). Moreover, for sufficiently large τ, we know that

$$\frac{C_{\mathcal{C}^h}}{\tau^2} - \frac{1}{\tau} e^{-\frac{1}{2}\zeta h\tau} > 0.$$

Hence, multiplying τ^2 on both sides of (4.54) and taking $\tau = j^s$ and $s > 0$, we derive that

$$
\left(C_{\mathcal{C}^h} - j^s e^{-\frac{1}{2}\zeta h j^s} \right) |\eta(\mathbf{0}) v_j(\mathbf{0})| \lesssim j^{-\beta + \frac{17}{20}s} + j^{\gamma - (\alpha+1)s} + j^{-\beta + \frac{11}{10}s}
$$
$$
+ j^{-\beta + (-\alpha + \frac{11}{10})s} + j^{\gamma - \alpha s} + j^{\gamma - 2\alpha s}
\tag{4.55}
$$

as $\tau \to \infty$. Recalling that $\gamma/\alpha < \frac{10}{11}\beta$, we can choose $s \in (\gamma/\alpha, \frac{10}{11}\beta)$. Hence in (4.55), by letting $j \to \infty$, we prove that

$$
\lim_{j \to \infty} |\eta(\mathbf{0}) v_j(\mathbf{0})| = 0.
$$

Since $\eta(\mathbf{0}) \neq 0$, we have $\lim_{j \to \infty} |v_j(\mathbf{0})| = 0$. Using (4.6) and integral mean value theorem, we can obtain (4.8).

  The proof is complete. $\square$

4.2. Proof of Theorem 4.3

In order to prove Theorem 4.3, we first give a crucial estimate in the following proposition. It is pointed out that $\mathcal{K}$ is a cuboid cone in this subsection, where $\mathbf{0}$ is the apex of $\mathcal{K}$. Denote $\mathsf{cone}(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = c_1 \mathbf{a} + c_2 \mathbf{b}, \ \forall c_i \geq 0, \ i = 1, 2\}$, where $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors. Let $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$ and $\mathbf{e}_3 = (0, 0, 1)^\top$. Suppose that the faces $\partial \mathcal{K} = \cup_{i=1}^3 \partial \mathcal{K}_i$, where $\mathcal{K}_1 = \mathsf{cone}(\mathbf{e}_1, \mathbf{e}_3)$, $\mathcal{K}_1 = \mathsf{cone}(\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{K}_1 = \mathsf{cone}(\mathbf{e}_2, \mathbf{e}_3)$.

Proposition 4.3. *Let $\mathbf{d} = (1, 1, 1)^\top$ and $\mathbf{d}^\perp = (1, -1, 0)^\top$. Denote $z_j = \rho_1 \cdot \hat{\mathbf{x}}_j(\theta_\xi)$, where*

$$
\hat{\mathbf{x}}_1(\theta_\xi) = \begin{bmatrix} 0 \\ \sin \theta_\xi \\ \cos \theta_\xi \end{bmatrix}, \quad \hat{\mathbf{x}}_2(\theta_\xi) = \begin{bmatrix} \sin \theta_\xi \\ 0 \\ \cos \theta_\xi \end{bmatrix}, \quad \hat{\mathbf{x}}_3(\theta_\xi) = \begin{bmatrix} \cos \theta_\xi \\ \sin \theta_\xi \\ 0 \end{bmatrix}
\tag{4.56}
$$

with a fixed $\theta_\xi \in (0, \pi/2)$, and $\rho_1 = \mathbf{d} + \mathrm{i}\mathbf{d}^\perp$. It holds that

$$
\left| \sum_{j=1}^3 \frac{1}{z_j^2} \right| \geq \frac{\sin^3 \theta_\xi}{30} > 0.
\tag{4.57}
$$

Proof. By direct calculations, we have

$$
\sum_{j=1}^3 \frac{1}{z_j^2} = \frac{S(\theta_\xi)}{z_1^2}, \quad S(\theta_\xi) = 1 + \left(\frac{z_1}{\bar{z}_1} \right)^2 + c_1 z_4,
\tag{4.58}
$$

where

$$
c_1 = \frac{|z_1|^4}{||z_1|^2 - \sin \theta_\xi \cos \theta_\xi + \mathrm{i} \cos \theta_\xi (\cos \theta_\xi + \sin \theta_\xi)|^4},
$$
$$
z_4 = \left(|z_1|^2 - 1/2 \sin 2\theta_\xi - \mathrm{i} \cos \theta_\xi (\cos \theta_\xi + \sin \theta_\xi) \right)^2.
$$

By noting $\theta_\xi \in (0, \pi/2)$, it yields that $c_1 \geq 0.05 \sin \theta_\xi$ and $\Re(z_4) \geq 2 \sin^2 \theta_\xi$. Hence according to (4.58), we obtain (4.57). $\square$

Proposition 4.4. *Assume that $\mathcal{K}^h$ is a truncated cuboid. Let $\Gamma_h = \partial \mathcal{K}^h \cap B_h$ and ρ be defined in (2.3), where $\mathbf{d} = (1, 1, 1)^\top$ and $\mathbf{d}^\perp = (1, -1, 0)^\top$. Then, one has*

$$
\left| \int_{\Gamma_h} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma \right| \geq \frac{C'_{\mathcal{K}^h}}{\tau^2} - \mathcal{O}\left(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right),
\tag{4.59}
$$

for sufficiently large τ, where $C'_{\mathcal{K}^h}$ is a positive number not depending on τ.

Proof. Since $\mathcal{K}$ is a cuboid, by the geometrical setup and notations in this subsection, we have $\Gamma_h = \Gamma_{h1} \cup \Gamma_{h2} \cup \Gamma_{h3}$, where $\Gamma_{h1} := \partial \mathcal{K}_1 \cap B_h$, $\Gamma_{h2} := \partial \mathcal{K}_2 \cap B_h$, $\Gamma_{h3} := \partial \mathcal{K}_3 \cap B_h$.

According to Proposition 2.1, it can be derived that

$$\int_{\Gamma_h} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma = \int_{\Gamma_{h1}} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma + \int_{\Gamma_{h2}} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma + \int_{\Gamma_{h3}} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma$$

$$= \frac{1}{2\pi \tau^2} \sum_{j=1}^{3} \frac{1}{(\rho_1 \cdot \hat{\mathbf{x}}_j(\theta_\xi))^2} - \mathcal{O}(\frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau}),$$

where $\theta_\xi \in (0, \pi/2)$ is fixed. By virtue of Proposition 4.3, we complete the proof. $\qquad \square$

The proof of Theorem 4.3. Using the fact that $f = (q-1)v \in C^\alpha(\overline{\mathcal{K}^h})$, $v \in C^\alpha(\overline{\mathcal{K}^h})$, $\eta \in C^\alpha(\overline{\Gamma_h})$, we have the following expansion

$$\begin{aligned} f &= f(\mathbf{0}) + \delta f, & |\delta f| &\leq \|f\|_{C^\alpha} |\mathbf{x}|^\alpha, \\ v &= v(\mathbf{0}) + \delta v, & |\delta v| &\leq \|v\|_{C^\alpha} |\mathbf{x}|^\alpha, \\ \eta &= \eta(\mathbf{0}) + \delta\eta, & |\delta\eta| &\leq \|\eta\|_{C^\alpha} |\mathbf{x}|^\alpha. \end{aligned} \tag{4.60}$$

Combining the integral identity (4.29) with (4.60), it arrives that

$$\eta(\mathbf{0})v(\mathbf{0}) \int_{\Gamma_h} e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma = \sum_{i=1}^{6} I_i + J_1 + J_2, \tag{4.61}$$

where

$$\begin{aligned} I_1 &= f(\mathbf{0}) \int_{\mathcal{K}^h} u_0 \mathrm{d}\mathbf{x}, \quad I_2 = \int_{\mathcal{K}^h} \delta f u_0 \mathrm{d}\mathbf{x}, \quad I_3 = \eta(\mathbf{0})v(\mathbf{0}) \int_{\Gamma_h} \psi(\mathbf{x}) e^{\rho \cdot \mathbf{x}} \mathrm{d}\sigma, \\ I_4 &= \eta(\mathbf{0}) \int_{\Gamma_h} \delta v u_0 \mathrm{d}\sigma, \quad I_5 = v(\mathbf{0}) \int_{\Gamma_h} \delta\eta u_0 \mathrm{d}\sigma, \quad I_6 = \int_{\Gamma_h} \delta\eta \delta v u_0 \mathrm{d}\sigma, \\ J_1 &= -\int_{\Lambda_h} (w-v)\partial_\nu u_0 \mathrm{d}\sigma, \quad J_2 = \int_{\Lambda_h} u_0 \partial_\nu(w-v) \mathrm{d}\sigma. \end{aligned} \tag{4.62}$$

There must exist a convex conic cone $\mathcal{C}$ contains the cuboid cone $\mathcal{K}$, namely $\mathcal{K} \subset \mathcal{C}$. Hence, by virtue of (4.45) and (4.16d), we have

$$|I_1| \leq |f(\mathbf{0})| \int_{\mathcal{K}^h} |u_0| \mathrm{d}\mathbf{x} \leq |f(\mathbf{0})| \int_{\mathcal{C}^h} |u_0| \mathrm{d}\mathbf{x} \lesssim \tau^{-\frac{121}{40}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{4.63}$$

and

$$|I_2| \leq \int_{\mathcal{K}^h} |\delta f u_0| \mathrm{d}\mathbf{x} \leq \int_{\mathcal{C}^h} |\delta f u_0| \mathrm{d}\mathbf{x} \lesssim \tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{4.64}$$

as $\tau \to \infty$.

In view of (4.49), we have

$$|I_3| \lesssim \tau^{-\frac{81}{40}}. \tag{4.65}$$

In addition, by using (4.16d) in Lemma 4.2, we have the following inequalities:

$$|I_4| \lesssim \tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{4.66}$$

$$|I_5| \lesssim \tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{4.67}$$

$$|I_6| \lesssim \tau^{-(2\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{2\alpha+2}} + \frac{1}{\tau} e^{-\frac{1}{2}\zeta h \tau} \right), \tag{4.68}$$

as $\tau \to \infty$. Moreover, by using (4.52) and (4.53), we have

$$|J_1| \lesssim (1+\tau)(1+\tau^{-\frac{2}{5}})e^{-\zeta h\tau} \tag{4.69}$$

and

$$|J_2| \lesssim (1+\tau)(1+\tau^{-\frac{2}{5}})e^{-\zeta h\tau}. \tag{4.70}$$

as $\tau \to \infty$. Let ρ be defined in (2.3) with $\mathbf{d} = (1,1,1)^\top$ and $\mathbf{d}^\perp = (1,-1,0)^\top$. By Proposition 4.4, one has (4.59). Plugging (4.63)-(4.70) and (4.59) into (4.61), it arrives that

$$\begin{aligned}
|\eta(\mathbf{0})v(\mathbf{0})| \left(\frac{C'_{\mathcal{K}^h}}{\tau^2} - \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right) &\lesssim \tau^{-\frac{121}{40}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right) + \tau^{-(\alpha+\frac{43}{20})} \\
&+ \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right) + \tau^{-\frac{81}{40}} + \tau^{-(\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{\alpha+2}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right) \\
&+ \tau^{-(2\alpha+\frac{43}{20})} + \left(\frac{1}{\tau^{2\alpha+2}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right) \\
&+ (1+\tau)(1+\tau^{-\frac{2}{5}})e^{-\zeta h\tau},
\end{aligned} \tag{4.71}$$

where the positive constant $C'_{\mathcal{K}^h}$ not depending on τ is defined in (4.59). Multiplying τ^2 on both sides of (4.71) and letting $\tau \to \infty$, one has

$$|\eta(\mathbf{0})v(\mathbf{0})| = 0.$$

Due to $\eta(\mathbf{0}) \neq 0$, we complete the proof of Theorem 4.3.   □

4.3. Proof of Corollary 4.4

Due to the proof of Corollary 4.4, (b) can be obtained by adopting the similar process as one of Corollary 4.4 (a); hence, we only give the proof of Corollary 4.4 (a). Firstly, we give the following proposition.

Proposition 4.5 (27, Lemma 2.4). *Let $\mathcal{C}^h$ and ρ be defined in (4.2) and (2.3), respectively. Then, we have*

$$\left| \int_{\mathcal{C}^h} e^{\rho \cdot \mathbf{x}} d\mathbf{x} \right| \geq \frac{\widetilde{C_{\mathcal{C}^h}}}{\tau^3} - \mathcal{O}\left(\frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right), \tag{4.72}$$

for sufficiently large τ, where $\widetilde{C_{\mathcal{C}^h}}$ is a positive number only depending on the opening angle θ_0 of $\mathcal{C}$ and ζ.

Proof of Corollary 4.4(a). The following integral identity can be obtained according to (4.29):

$$k^2 f_j(\mathbf{0}) \int_{\mathcal{C}^h} e^{\rho \cdot \mathbf{x}} = I_1 + I_2 + I_3 + J_1 + J_2, \tag{4.73}$$

where I_m, $m = 1,2,3$, J_1 and J_2 defined in (4.32).

With the help of (4.35), (4.42), (4.45), (4.52) and Proposition 4.5, we have the following integral inequality

$$\begin{aligned}
k^2 \left[\frac{\widetilde{C_{\mathcal{C}^h}}}{\tau^3} - \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right] |f_j(\mathbf{0})| &\lesssim j^{-\beta} \left[\left(\frac{1}{\tau^3} + \frac{1}{\tau}e^{-\zeta h\tau} \right)^{\frac{1}{2}} + \left(\frac{1}{\tau^3} + \frac{1}{\tau}e^{-2\zeta h\tau} \right)^{\frac{1}{4}} \tau^{-\frac{2}{5}} \right] \\
&+ j^\gamma \left[\tau^{-(\alpha+\frac{121}{40})} + \left(\frac{1}{\tau^{\alpha+3}} + \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} \right) \right] \\
&+ \tau^{-\frac{121}{40}} + (1+\tau)(1+\tau^{-\frac{2}{5}})e^{-\zeta h\tau}
\end{aligned} \tag{4.74}$$

as $\tau \to \infty$. For sufficiently large τ, we know that

$$\frac{\widetilde{C_{\mathcal{C}^h}}}{\tau^3} - \frac{1}{\tau}e^{-\frac{1}{2}\zeta h\tau} > 0.$$

Then, multiplying τ^3 in the both sides of (4.74) and letting $\tau \to \infty$ and $\tau = j^s$, one has

$$k^2 \widetilde{C_{\mathcal{C}^h}}|f_j(\mathbf{0})| \lesssim j^{-\beta+\frac{37}{20}s} + j^{\gamma-\alpha s}. \tag{4.75}$$

Due to the assumption that $\gamma < \frac{20}{37}\alpha\beta$, we choose $s \in (\gamma/\alpha, \frac{20}{37}\beta)$. By letting $j \to \infty$, we have

$$|f_j(\mathbf{0})| = 0.$$

Since $q(\mathbf{0}) \neq 1$, the proof of this corollary is complete. $\qquad\square$

5. Visibility and unique recovery results for the inverse scattering problem

In this section, we show that when a medium scatter with a conductive transmission boundary condition possesses either one of a convex planar corner, a convex polyhedral corner, or a convex conic corner, it radiates a non-trivial far-field pattern, namely the visibility of this scatterer occurs. Furthermore, when the medium scatter is visible, it can be uniquely determined by a single far-field measurement under generic physical scenarios.

In the following theorem, it indicates that a conductive medium possesses an aforementioned corner under generic physical conditions always scatters.

Theorem 5.1. *Consider the conductive medium scattering problems (1.3). Let $(\Omega; q, \eta)$ be the medium scatterer associated with (1.3), where Ω is a bounded Lipschitz domain with a connected complement in $\mathbb{R}^n$, $n = 2, 3$. If either of the following conditions is fulfilled, namely,*

(a) *when $\Omega \Subset \mathbb{R}^2$, there exists a sufficient small $h \in \mathbb{R}_+$ such that $\Omega \cap B_h = S_h$, where S_h is defined by (3.1), $q \in H^2(\overline{S_h})$, $\eta \in C^\alpha(\overline{\Gamma_h^\pm})$ satisfying $\alpha \in (0, 1)$ and $\eta(\mathbf{0}) \neq 0$, and $\Gamma_h^\pm = \partial S_h \setminus \partial B_h$;*

(b) *when $\Omega \Subset \mathbb{R}^3$, there exists a sufficient small $h \in \mathbb{R}_+$ such that $\Omega \cap B_h = \mathcal{K}^h$, where $\mathcal{K}^h$ is a cuboid defined by (4.4), $q \in H^2(\overline{\mathcal{K}^h})$, $\eta \in C^\alpha(\overline{\Gamma_h})$ satisfying $\alpha \in (0, 1)$ and $\eta(\mathbf{0}) \neq 0$, and $\Gamma_h = \partial \mathcal{K}^h \setminus \partial B_h$;*

(c) *when $\Omega \Subset \mathbb{R}^3$, there exists a sufficient small $h \in \mathbb{R}_+$ such that $\Omega \cap B_h = \mathcal{K}^h$, where $\mathcal{K}^h$ is a polyhedral corner but not a cuboid, then $q \in H^2(\overline{\mathcal{K}^h})$ satisfying $q(\mathbf{0}) \neq 1$ and $\eta \equiv 0$ on $\partial \mathcal{K}^h \setminus \partial B_h$;*

(d) *when $\Omega \Subset \mathbb{R}^3$, there exists a sufficient small $h \in \mathbb{R}_+$ such that $\Omega \cap B_h = \mathcal{C}^h$, where $\mathcal{C}^h$ is defined by (4.1), $q \in H^2(\overline{\mathcal{C}^h})$, $\eta \in C^\alpha(\overline{\Gamma_h})$ satisfying $\alpha \in (0, 1)$, $\eta(\mathbf{0}) \neq 0$ and $\Gamma_h = \partial \mathcal{C}^h \setminus \partial B_h$;*

then Ω always scatters for any incident wave satisfying (1.2).

Proof. By contradiction, suppose that the mediums scatterer Ω possesses either one of a convex planar corner, a convex polyhedral corner and a convex conic corner, where the assumptions (a)-(d) are fulfilled. Assume that Ω is non-radiating, namely the far-field pattern $u^\infty \equiv 0$. By virtue of Rellich lemma, the total wave field u and incident wave u^i satisfies (1.1) associated with the incident wave number k. It is clear that the incident u^i is α-Hölder continuous and non-vanishing near the underlying corner. According to Corollaries 3.3 and 4.5, Theorems 4.2 and 4.3, one has u^i must vanish at the corresponding corner point, where we get the contradiction.

The proof is complete. $\qquad\square$

In the following, we shall study the unique recovery for the inverse problem (1.4) associated with the conductive scattering problem (1.3) in $\mathbb{R}^3$. In the field of inverse scattering problems, it is concerned with the shape determination of Ω by a minimum far-field measurement (cf. [22]). We utilize the local geometrical characterization of transmission eigenfunctions near a corner in Section 4 to establish the uniqueness regarding the shape determination of (1.4) by a single measurement under generic physical scenario, where a single far-field measurement means that the underlying far-field pattern is generated

only by a single incident wave u^i. The unique determination results of (1.4) for recovering the material parameters associated with (1.3) by infinitely many far-field measurements with a fixed frequency can be found in [14, 15, 36, 37]. We obtain local unique recovery results for the determination of Ω without a-prior knowledge on the material parameters q and η in this section. When Ω is a cuboid or a corona shape scatterer with a conductive transmission boundary condition, the corresponding global uniqueness results on the shape determination can be drawn under generic physical scenarios. It is pointed out that when $\eta \equiv 0$ on $\partial\Omega$, namely consider the inverse problem (1.4) associated with the corresponding scattering problem

$$
\begin{cases}
\Delta u^- + k^2 q u^- = 0 & \text{in} \quad \Omega, \\
\Delta u^+ + k^2 u^+ = 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega, \\
u^+ = u^-, \quad \partial_\nu u^+ = \partial_\nu u^-, & \text{on} \quad \partial\Omega, \\
u^+ = u^i + u^s, & \text{in} \quad \mathbb{R}^n, \\
\lim_{r \to \infty} r^{(n-1)/2}(\partial_r u^s - iku^s) = 0, & r = |\mathbf{x}|,
\end{cases}
\tag{5.1}
$$

we can establish global unique recovery results for the shape of Ω within convex polyhedral or corona geometries by a single far-field measurement, whereas the corresponding single measurement uniqueness result regarding the shape determination of a convex polygonal or cuboid shape associated with (5.1) was studied in [38].

In Theorem 5.2, we show the local uniqueness results for (1.4), which aims to recover a scatterer $(\Omega; q, \eta)$ by knowledge of the far-field pattern $u^\infty(\hat{\mathbf{x}}; u^i)$ with a single measurement. First, let us introduce the admissible class of the conductive scatterer and the related notations in our study.

Definition 5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with a connected complement and $(\Omega; k, \mathbf{d}, q, \eta)$ be a conductive scatterer with the incident plane wave $u^i = e^{ik\mathbf{x}\cdot\mathbf{d}}$, where $\mathbf{d} \in \mathbb{S}^2$ and $k \in \mathbb{R}_+$. Consider the scattering problem (1.3). Denote u by the total wave field, which is associated with (1.3). The scatterer Ω is said to be admissible if the following conditions are fulfilled:*

(a) *$q \in L^\infty(\Omega)$ and $\eta \in L^\infty(\partial\Omega)$.*

(b) *After rigid motions, we assume that $\mathbf{0} \in \partial\Omega$. Recall that $\mathcal{C}^h$ and $\mathcal{K}^h$ are defined in (4.2) and (4.3) respectively, where $\mathbf{0}$ is the apex of the conic corner $\mathcal{C}^h$ or the convex polyhedral corner $\mathcal{K}^h$. If Ω possesses a convex conic corner $\mathcal{C}^h$ (or a cuboid corner $\mathcal{K}^h$), then $q \in H^2(\overline{\mathcal{C}^h})$ (or $q \in H^2(\overline{\mathcal{K}^h})$) and $\eta \in C^\alpha(\overline{\Gamma_h})$ satisfying $\eta(\mathbf{0}) \neq 0$ and $\alpha \in (0,1)$, where $\Gamma_h = \mathcal{C}^h \cap \partial\Omega$ (or $\Gamma_h = \mathcal{K}^h \cap \partial\Omega$). If Ω possesses a convex polyhedral corner $\mathcal{K}^h = B_h \cap \Omega$, then $q \in H^2(\overline{\mathcal{K}^h})$ satisfying $q(\mathbf{0}) \neq 1$ and $\eta \equiv 0$ on $\overline{\mathcal{K}^h} \cap \partial\Omega$.*

(c) *The total wave field u is non-vanishing everywhere in the sense that for any $\mathbf{x} \in \mathbb{R}^3$,*

$$
\lim_{\lambda \to +0} \frac{1}{m(B(\mathbf{x}, \rho))} \int_{B(\mathbf{x},\lambda)} |u(\mathbf{x})| d\mathbf{x} \neq 0,
\tag{5.2}
$$

where $m(B(\mathbf{x}, \lambda))$ is the measure of $B(\mathbf{x}, \lambda)$.

Remark 5.1. *The assumption (5.2) is a technical condition for deriving the unique results, which can be fulfilled under generic physical scenarios. For example, when $k \cdot \text{diam}(\Omega) \ll 1$, by the well-posedness of the direct scattering problem (1.3) (cf. [15, Theorem 2.4]), the condition (5.2) can be satisfied. The detailed discussion on this point can be found in [26, Page 44]. We believe that (5.2) can be fulfilled under other physical settings, where we choose not to explore this aspect in this paper and shall investigate it in the future.*

Theorem 5.2. *Consider the conductive scattering problem (1.3) with two conductive scatterers $(\Omega_j; k, \mathbf{d}, q_j, \eta_j), j = 1, 2$, in $\mathbb{R}^3$. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the scatterers $(\Omega_j; k, \mathbf{d}, q_j, \eta_j), j = 1, 2$ and the incident field u^i. If $(\Omega_j; k, \mathbf{d}, q_j, \eta_j)$ are admissible and*

$$
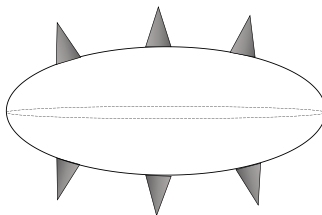u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i)
\tag{5.3}
$$

Figure 1 *Schematic illustration of corona shape scatterers.*

for all $\hat{\mathbf{x}} \in \mathbb{S}^2$ with a fixed incident u^i. Then,

$$\Omega_1 \Delta \Omega_2 := (\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1) \tag{5.4}$$

cannot contain a convex conic corner or a cuboid corner. Furthermore, if Ω_1 and Ω_2 are two cuboids, then $\Omega_1 = \Omega_2$.

Proof. We prove this theorem by contradiction. Suppose that $\Omega_1 \Delta \Omega_2$ contains a convex conic corner. Without loss of generality, we assume that the underlying convex conic corner $\mathcal{C}^h \subset \Omega_2 \setminus \Omega_1$, where $\mathbf{0} \in \partial \Omega_2$ and $\Omega_2 \cap B_h = \mathcal{C}^h$ with a sufficient small $h \in \mathbb{R}_+$ such that $B_h \subset \mathbb{R}^3 \setminus \overline{\Omega_1}$.

Due to (5.3), with the help of Rellich's Theorem (cf. [21]), it holds that $u_1^s = u_2^s$ in $\mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, we have

$$u_1(\mathbf{x}) = u_2(\mathbf{x}), \ \ \forall \mathbf{x} \in \mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2). \tag{5.5}$$

Since $\Gamma_h = \partial \mathcal{C}^h \cap \partial \Omega_2$, by virtue of transmission conditions on $\partial \Omega_2$ of (1.3) and (5.5), it yields that

$$u_2^+ = u_2^- = u_1^+, \ \ \partial u_2^- = \partial u_2^+ + \eta_2 u_2^+ = \partial u_1^+ + \eta_2 u_1^+ \text{ on } \Gamma_h. \tag{5.6}$$

According to (5.6) and direct scattering problems (1.3) associated with $(\Omega_j; k, \mathbf{d}, q_j, \eta_j)$, one has

$$\begin{cases} \Delta u_2^- + k^2 q_2 u_2^- = 0 & \text{in } \mathcal{C}^h, \\ \Delta u_1^+ + k^2 u_1^+ = 0 & \text{in } \mathcal{C}^h, \\ u_2^- = u_1^+, \ \partial_\nu u_2^- = \partial_\nu u_1^+ + \eta_2 u_1^+ & \text{on } \Gamma_h. \end{cases}$$

By the well-posedness of the direct scattering problem (1.3), it yields that $u_2^- \in H^1(\mathcal{C}^h)$ and u_1^+ is real analytic in B_h. By virtue of the condition (b) in Definition 5.1, using Theorem 4.2, we know that $u_1(\mathbf{0}) = 0$, which is contradicted to the admissibility condition (c) in Definition 5.1.

The first conclusion of this theorem concerning a cuboid corner can be proved similarly by using Theorem 4.3. We omit the proof.

By the convexity of two cuboids Ω_1 and Ω_2 and the first conclusion of this theorem, it is ready to know that $\Omega_1 = \Omega_2$.

The proof is complete. $\square$

In the following, we introduce an admissible class $\mathcal{T}$ of corona shape, which shall be used in Theorem 5.3. The schematic illustration of corona shape scatterers can be found in Figure 1.

Definition 5.2. *Let D be a convex open bounded Lipschitz domain with a connected complement $\mathbb{R}^3 \setminus \overline{D}$. If there exist finite many strictly convex conic cones $\mathcal{C}_{\mathbf{x}_j, \mathbf{v}_j, \theta_j} (j = 1, 2, \ldots, \ell, \ell \in \mathbb{N})$ defined in (4.1) such that*

   (a) the apex $\mathbf{x}_j \in \mathbb{R}^3 \setminus \overline{D}$, $\mathcal{C}_{\mathbf{x}_j, \mathbf{v}_j, \theta_j} \cap D \neq \emptyset$ and $\mathcal{C}_{\mathbf{x}_j, \mathbf{v}_j, \theta_j} \setminus D$ has two disconnected components, where $\mathcal{C}^*_{\mathbf{x}_j, \mathbf{v}_j, \theta_j}$ is the bounded component of $\mathcal{C}_{\mathbf{x}_j, \mathbf{v}_j, \theta_j} \setminus D$;

   (b) $\partial \overline{\mathcal{C}^*_{\mathbf{x}_j, \mathbf{v}_j, \theta_j}} \setminus \partial \overline{\mathcal{C}_{\mathbf{x}_j, \mathbf{v}_j, \theta_j}} \subset \partial \overline{D}$ and $\cap_{j=1}^{\ell} \left(\partial \overline{\mathcal{C}^*_{\mathbf{x}_j, \mathbf{v}_j, \theta_j}} \setminus \partial \overline{\mathcal{C}_{\mathbf{x}_j, \mathbf{v}_j, \theta_j}} \right) = \emptyset$;

   (c) $\Omega := \cup_{j=1}^{\ell} \mathcal{C}^*_{\mathbf{x}_j, \mathbf{v}_j, \theta_j} \cup D$ is admissible described by Definition 5.1;

then Ω is said to belong to an admissible class $\mathcal{T}$ of corona shape.

A global unique recovery for the admissible scatter belonging to $\mathcal{T}$ of corona shape is shown in Theorem 5.3, which can be proved by using Theorem 5.2 and the assumptions in Theorem 5.3. Indeed, the assumptions (5.7a) and (5.7b) imply that the set difference of two scatters Ω_1 and Ω_2 cannot contain a convex conic corner if $\Omega_j \in \mathcal{T}, j = 1, 2$.

Theorem 5.3. *Suppose that $\Omega_m, m = 1, 2$ belong to the admissible class $\mathcal{T}$ of corona shape, where*

$$\Omega_m = \cup_{j^{(m)}=1}^{\ell^{(m)}} \mathcal{C}^*_{\mathbf{x}_{j^{(m)}}, \mathbf{v}_{j^{(m)}}, \theta_{j^{(m)}}} \cup D_m, \quad m = 1, 2.$$

Consider the conductive scattering problem (1.3) associated with the admissible conductive scatterers $\Omega_m, m = 1, 2$. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the scatterers $(\Omega_m; \mathcal{C}^*_{\mathbf{x}_{j^{(m)}}, \mathbf{v}_{j^{(m)}}, \theta_{j^{(m)}}}), m = 1, 2$ and the incident field u^i. If the following conditions:

$$D_1 = D_2, \tag{5.7a}$$

$$\theta_{i^{(1)}} = \theta_{j^{(2)}}, \mathbf{v}_{i^{(1)}} = \mathbf{v}_{j^{(2)}} \text{ for } i^{(1)} \in \{1, \ldots, \ell^{(1)}\} \text{ and } j^{(2)} \in \{1, \ldots, \ell^{(2)}\} \text{ when } \mathbf{x}_{i^{(1)}} = \mathbf{x}_{j^{(2)}}, \tag{5.7b}$$

and (5.3) are satisfied, then $\ell^{(1)} = \ell^{(2)}$, $\mathbf{x}_{j^{(1)}} = \mathbf{x}_{j^{(2)}}$ and $\theta_{j^{(1)}} = \theta_{j^{(2)}}$, where $j^{(m)} = 1, \ldots \ell^{(m)}, m = 1, 2$. Namely, one has $\Omega_1 = \Omega_2$.

In Theorem 5.4, we first show a local uniqueness result regarding a polyhedral corner by a single measurement, where we can prove this theorem in a similar manner as for Theorem 5.2 by utilizing Corollary 4.5. Hence, the detailed proof of Theorem 5.4 is omitted. We emphasize that an admissible convex polyhedral scatterer Ω can be uniquely determined by a single far-field measurement, which a global uniqueness result for (1.4) associated with (1.3) is established.

Theorem 5.4. *Consider the conductive scattering problem (1.3) with conductive scatterers $(\Omega_j; k, \mathbf{d}, q_j, \eta_j), j = 1, 2,$ in $\mathbb{R}^3$. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the scatterers $(\Omega_j; k, \mathbf{d}, q_j, \eta_j), j = 1, 2$ and the incident field u^i. If $(\Omega_j; k, \mathbf{d}, q_j, \eta_j)$ are admissible and (5.3) is fulfilled, then $\Omega_1 \Delta \Omega_2$ defined by (5.4) cannot contain a convex polyhedral corner. Furthermore, if Ω_1 and Ω_2 are two admissible convex polyhedrons, then*

$$\Omega_1 = \Omega_2.$$

Consider the direct scattering problem (5.1) associated with a convex polyhedron medium $(\Omega; k, \mathbf{d}, q)$, which is a special case of (1.3) by letting $\eta \equiv 0$ on $\partial\Omega$. In Corollary 5.5, we give a global unique determination for a convex polyhedron Ω associated with the direct scattering problem (5.1) by a single far-field measurement under generic physical settings. Corollary 5.5 can be proved directly by using Theorem 5.4 and the detailed proof is omitted. Compared with the corresponding uniqueness result in [38] for the shape determination of a cuboid scatterer by a single measurement, we relax the geometrical restriction on the uniqueness determination regarding medium shapes by a single measurement from a cuboid to a general convex polyhedron.

Corollary 5.5. *Consider the scattering problem (5.1) with scatterers $(\Omega_j; k, \mathbf{d}, q_j), j = 1, 2,$ in $\mathbb{R}^3$. Let $u_j^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern associated with the scatterers $(\Omega_j; k, \mathbf{d}, q_j), j = 1, 2$ and the incident field u^i. Assume that the total wave field u_j corresponding to (5.1) associated with $(\Omega_j; k, \mathbf{d}, q_j)$ $(j = 1, 2)$ satisfies (5.2). Suppose that Ω_j is a convex polyhedron, $j = 1, 2$. Denote $\mathcal{V}(\Omega_j)$ by a set composed of all vertexes of Ω_j with $j = 1, 2$. For any $\mathbf{x}_{c,j} \in \mathcal{V}(\Omega_j)$, if there exists sufficient small $h \in \mathbb{R}_+$ such that $q_j \in H^2(\overline{\mathcal{K}^h_{\mathbf{x}_{c,j}}})$ with $q_j(\mathbf{x}_{c,j}) \neq 1$ for $j = 1, 2$, where $\mathcal{K}^h_{\mathbf{x}_{c,j}} = \Omega \cap B_h(\mathbf{x}_{c,j}) \Subset \Omega_j$, then the condition (5.3) implies that $\Omega_1 = \Omega_2$.*

When the shape of an admissible scatter Ω is uniquely determined by a single measurement, under a-prior knowledge the potential q associated with Ω we can recover the surface parameter η by a single measurement provided that η is a non-zero constant. We can use a similar argument for proving [26,

Theorem 4.2] to establish Theorem 5.6. The detailed proof is omitted. The technical condition (5.8) can be easily fulfilled under generic physical scenarios; see the detailed discussion in [26, Remark 4.2].

Theorem 5.6. *Consider the conductive scattering problem (1.3) with the admissible conductive scatterers $(\Omega_m; k, \mathbf{d}, q, \eta_m)$ in $\mathbb{R}^3$, where $\eta_m \neq 0$, $m = 1, 2$, are two constants. Let $u_m^\infty(\hat{\mathbf{x}}; u^i)$ be the far-field pattern with the scatterers $(\Omega_m; k, \mathbf{d}, q, \eta_m)$, $m = 1, 2$ and the incident field u^i. Suppose that*

$$u_1^\infty(\hat{\mathbf{x}}; u^i) = u_2^\infty(\hat{\mathbf{x}}; u^i), \text{ for all } \hat{\mathbf{x}} \in \mathbb{S}^2$$

with a fixed incident wave u^i. If

$$k \text{ is not an eigenvalue of the partial differential operator } \Delta + k^2 q, \tag{5.8}$$

and Ω_m is a cuboid ($m = 1, 2$), we have $\eta_1 = \eta_2$. Similarly, when

$$\Omega_m = \cup_{j^{(m)}=1}^{\ell^{(m)}} \mathcal{C}^*_{\mathbf{x}_{j^{(m)}}, \mathbf{v}_{j^{(m)}}, \theta_{j^{(m)}}} \cup D_m \in \mathcal{T}, \quad m = 1, 2,$$

if the conditions (5.8), (5.7a) and (5.7b) are fulfilled, one has $\eta_1 = \eta_2$.

References

[1] Angell, T. S. & Kirsch, A. (1992) The conductive boundary condition for Maxwell's equations. *SIAM J. Appl. Math* **52**(6), 1597–1610.

[2] Ammari, H., Chow, Y. T. & Liu, H. (2023) Quantum ergodicity and localization of plasmon resonances. *J Funct Anal* **285**(4), 109976.

[3] Ammari, H., Chow, Y. T., Liu, H. & Sunkula, M. (2024) Quantum integrable systems and concentration of plasmon resonance. *J. Eur. Math. Soc.,* Doi:10.4171/JEMS/1437.

[4] Blåsten, E. (2018) Nonradiating sources and transmission eigenfunctions vanish at corner and edges. *SIAM J. Math. Anal* **50**(6), 6255–7270.

[5] Blåsten, E. & Lin, Y.-H. (2019) Radiating and non-radiating sources in elasticity. *Inverse Probl* **35**(1), 015005.

[6] Blåsten, E. & Liu, H. (2020) Recovering piecewise-constant refractive indices by a single far-field pattern. *Inverse Probl* **36**, 085005.

[7] Blåsten, E. & Liu, H. (2021) Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems. *SIAM J Math Anal* **53**(4), 3801–3837.

[8] Blåsten, E. & Liu, H. (2021) On corners scattering stably and stable shape determination by a single far-field pattern. *Indiana Univ. Math. J* **70**(3), 907–947.

[9] Blåsten, E. & Liu, H. (2017) On vanishing near corners of transmission eigenfunctions. *J. Funct. Anal* **273**(11), 3616–3632.

[10] Blåsten, E., Liu, H. & Xiao, J. (2021) On an electromagnetic problem in a corner and its applications. *Anal. PDE* **14**(7), 2207–2224.

[11] Blåsten, E., Li, X., Liu, H. & Wang, Y. (2017) On vanishing and localizing of transmission eigenfunctions near singular points: Numberical study. *Inverse Probl* **37**, 105001.

[12] Blåsten, E., Päivärinta, L. & Sylvester, J. (2014) Corners always scatter. *Comm. Math. Phys* **331**, 725–753.

[13] Bondarenko, O. (2016). The factorization method for conducting transmission conditions. Karlsruher Institut für Technologie (KIT).

[14] Bondarenko, O., Harris, I. & Kleefeld, A. (2017) The interior transmission eigenvalue problem for an inhomogeneous media with a conductive boundary. *Appl. Anal* **96**(1), 2–22.

[15] Bondarenko, O. & Liu, X. (2013) The factorization method for inverse obstacle scattering with conductive boundary condition. *Inverse Probl* **29**(9), 095021.

[16] Cakoni, F., Colton, D. & Haddar, H. (2016). *Inverse Scattering Theory and Transmission eigenvalues, CBMS-NSF*, SIAM Publications, Philadelphia.

[17] Cakoni, F. & Haddar, H. (2013) Transmission eigenvalues in inverse scattering theory, in "Inverse problems and applications: Inside out II. *Math. Sci. Res. Inst. Publ* **60**, 529–580.

[18] Cakoni, F. & Vogelius, M. (2023) Singularities almost always scatter: Regularity results for non-scattering inhomogeneities. *Comm. Pure Appl. Math* **76**(12), 4022–4047.

[19] Cakoni, F. & Xiao, J. (2021) On corner scattering for operators of divergence and applications to inverse scattering. *Commun. Part. Diff. Eq* **46**(3), 413–441.

[20] Chow, Y.-T., Deng, Y., He, Y., Liu, H. & Wang, X. (2021) Surface-localized transmission eigenstates, super-resolution imaging and pseudo surface plasmon modes. *SIAM J Imaging Sci* **14**(3), 946–975.

[21] Colton, D. & Kress, R. (2019). *Inverse Acoustic and Electromagnetic Scattering Theory*. 4th ed. Springer Nature, Switzerland.

[22] Colton, D. & Kress, R. (2018) Looking back on inverse scattering theory. *SIAM Rev* **60**(4), 779–807.

[23] Costabel, M. (1988) Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal* **19**(3), 613–626.

[24] Deng, Y., Jiang, Y., Liu, H. & Zhang, K. (2022) On new surface-localized transmission eigenmodes. *Inverse Probl Imag* **16**(3), 595–611.

[25] Deng, Y., Liu, H., Wang, X. & Wu, W. (2022) On geometrical properties of electromagnetic transmission eigenfunctions and artificial mirage. *SIAM J. Appl. Math* **82**(1), 1–24.

[26] Diao, H., Cao, X. & Liu, H. (2021) On the geometric structures of transmission eigenfunctions with a conductive boundary condition and application. *Commun. Part. Diff. Eq* **46**(4), 630–679.

[27] Diao, H., Fei, X., Liu, H. & Yang, K. (2024) Visibility, invisibility and unique recovery of inverse electromagnetic problems with conical singularities. *Inverse Probl. Imaging* **18**(3), 541–570.

[28] Diao, H. & Liu, H. (2023). *Spectral Geometry and Inverse Scattering Theory*, Springer, Cham.

[29] Diao, H., Li, H., Liu, H. & Tang, J. (2023) Spectral properties of an acoustic-elastic transmission eigenvalue problem with applications. *J. Diff. Eq* **371**, 629–659.

[30] Diao, H., Liu, H. & Tao, L. (2024) Stable determination of an impedance obstacle by a single far-field measurement. *Inverse Probl* **40**, 055005.

[31] Diao, H., Tang, R., Liu, H. & Tang, J. (2024) Unique determination by a single far-field measurement for an inverse elastic problem. *Inverse Probl. Imaging*, Doi:10.3934/ipi.2024020.

[32] Diao, H., Liu, H. & Wang, L. (2022) Further results on generalized Holmgren's principle to the Lamé operator and applications. *J. Diff. Eq* **309**, 841–882.

[33] Diao, H., Liu, H. & Sun, B. (2021) On a local geometric property of the generalized elastic transmission eigenfunctions and application. *Inverse Probl* **37**(10), 105015.

[34] Gagliardo, E. (1957) Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in *n* variabili. *Rend Semin Ma Univ. P* **27**, 284–305.

[35] Geymonat, G. (2007) Trace theorem for Sobolev spaces on Lipschitz domain: Necessary conditions. *Annal. Mathéma. Blaise Pas* **14**, 187–197.

[36] Harris, I. & Kleefeld, A. (2022) Analysis and computation of the transmission eigenvalues with a conductive boundary condition. *Appl Anal* **101**(6), 1880–1895.

[37] Harris, I. & Kleefeld, A. (2020) The inverse scattering problem for a conductive boundary condition and transmission eigenvalues. *Appl Anal* **99**(3), 508–529.

[38] Hu, G., Salo, M. & Vesalainen, E. V. (2016) Shape identification in inverse medium scattering problems with a single far-field pattern. *SIAM J Math Anal* **48**(1), 152–165.

[39] Liu, H. (2022) On local and global structures of transmission eigenfunctions and beyond. *J. Inverse Ill-Posed Probl* **30**(2), 287–305.

[40] Liu, H., Petrini, M., Rondi, L. & Xiao, J. (2017) Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements. *J Diff. Eq.* **262**(3), 1631–1670.

[41] Liu, H. & Tsou, C.-H. (2022) Stable determination by a single measurement, scattering bound and regularity of transmission eigenfunction. *Calcul. Vari Part Diff. Eq* **61**(3), Article 91.

[42] Liu, H. & Xiao, J. (2017) On electromagnetic scattering from a penetrable corner. *SIAM J. Math. Anal* **49**(6), 5207–5241.

[43] Liu, H. & Zou, J. (2007) On unique determination of partially coated polyhedral scatterers with far field measurements. *Inverse Probl* **23**, 297–308.

[44] Mikhailov, S. E. (2011) Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J Math Anal Appl* **378**(1), 324–342.

[45] Päivärinta, L., Salo, M. & Vesalainen, E. V. (2017) Strictly convex corners scatter. *Rev. Mat Iberoam* **33**(4), 1369–1396.

[46] Salo, M. & Shahgholian, H. (2021) Free boundary methods and non-scattering phenomena. *Res. Math. Sci.* **8**(4), 19.

[47] Vogelius, M. & Xiao, J. (2021) Finiteness results concerning non-scattering wave numbers for incident plane and Herglotz waves. *SIAM J. Math. Anal* **53**(5), 5436–5464.

[48] Weck, N. (2004) Approximation by herglotz wave functions. *Math. Methods Appl. Sci* **27**(2), 155–162.