AXIOMS FOR ABSOLUTE GEOMETRY

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Introduction. The axioms of Euclidean geometry may be divided into four groups: the axioms of order, the axioms of congruence, the axiom of continuity, and the Euclidean axiom of parallelism **(6)**. If we omit this last axiom, the remaining axioms give either Euclidean or hyperbolic geometry. Many important theorems can be proved if we assume only the axioms of order and congruence, and the name *absolute geometry* is given to geometry in which we assume only these axioms. In this paper we investigate what can be proved using congruence axioms that are weaker than those used previously.

Various axioms of congruence are listed in §§ 1 and 2. They will be discussed there in more detail. Axioms **C1–C7** may be called traditional axioms of congruence. They were used by Forder in 1927 **(6)** and are still used in books where not too much space can be devoted to proving basic results (e.g. **2**; **3**; **10**). In 1929 and 1930 Dorroh **(4**; **5)** showed that **C6** and **C7** are consequences of **C1–C5**. In 1965 Piesyk **(12)** gave an independent proof of one of Dorroh's results, using axioms due to Tarski **(15)** that are equivalent to **C1–C5**. In 1947 Forder **(7)** showed that **C5** can be replaced by certain special cases of **C5**, and in 1961 Szász **(14)** gave a different development of the basic results of Forder's paper of 1947.

In the present paper we shall obtain the same results using even fewer special cases of **C5** in place of **C5**. We shall also show that we can replace **C1** by the weaker existence axiom **C1*** (as used by Euclid), as long as we make suitable minor adjustments to **C2**, **C3**, **C4**. We shall also consider the construction of perpendiculars and mid-points, using weak methods of construction.

We shall assume without comment various results of ordered geometry. For the axioms of ordered geometry and for proofs of these results see **(6)**, Chapters II and III. I should like, however, to give here a proof of the *transversal theorem* that is simpler than the proofs usually given (e.g. **6**, p. 55; **3**, p. 180; **13**, p. 53). The symbol "$[ABC]$" means "A, B, and C lie in the geometrical order ABC."

THE TRANSVERSAL THEOREM. *If A, B, C are not collinear, and if $[AFB]$ and $[BCD]$, then there exists E such that $[CEA]$ and $[FED]$.*

Proof. There exists G such that $[CAG]$ (order axiom **O4**, **6**, p. 48). Apply the transversal axiom (**6**, p. 48) to triangle GDC with transversal BA. Then

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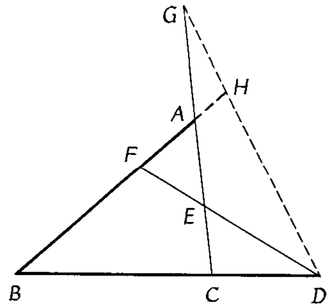


there exists H on BA such that $[GHD]$. Also $[BAH]$ **(6**, p. 49**)**. Since $[BFA]$ and $[BAH]$, we have $[FAH]$ **(6**, p. 52**)**. Apply the transversal axiom to triangle FDH with transversal GA. Then there exists E on GA such that $[FED]$. From the symmetry of the original hypotheses, we have $[CEA]$ similarly.

1. Axioms of congruence. An *absolute space* is an ordered space **(6**, Chapter II**)** together with a relation of *congruence* satisfying certain *axioms of congruence*. Congruence is a relation between ordered pairs of distinct points. (The term "ordered pair" is used in its set-theoretical sense, and has no relation to geometrical order.) If the ordered pair (A, B) is congruent to (C, D), then we shall write $AB \to CD$. The first four axioms of congruence as usually stated are:

C1. *If A, B are distinct points and if O is any point, then on any ray from O (i.e. any ray with origin O) there exists just one point C such that $AB \to OC$.*
C2. *If $AB \to CD$ and $CD \to EF$, then $AB \to EF$.*
C3. *If $[ABC]$ and $[A'B'C']$, and if $AB \to A'B'$, $BC \to B'C'$, then $AC \to A'C'$.*
C4. $AB \to BA$.

Instead of **C1** we shall assume the weaker axiom **C1***:

C1*. *If A, B are distinct points, then on any ray from A there exists just one point C such that $AB \to AC$.*

This corresponds to Euclid's Postulate 3: *To describe a circle with any centre and distance* **(8**, p. 154**)**. Euclid proves Proposition 2: *To place at a given point (as an extremity) a straight line equal to a given straight line* **(8**, p. 244**)**, which corresponds to our **C1**, but we cannot use his method of proof as we shall not assume any axiom about the intersection of circles.

Definition. Let A, B be points. If, given any point $A' \neq A$ and any ray b from B, there exists $B' \in b$ such that $AA' \to BB'$, then A is *isometric to B*.

As an immediate consequence of **C1*** and this definition, we have

1.1. *Any point is isometric to itself.*

We shall use the symbol $AB \Rightarrow CD$ to mean that A is isometric to C and $AB \rightarrow CD$.

If we assume axioms **C1** and **C3**, then we can prove axiom **C3*** below **(6**, p. 92**)**, but the proof breaks down if we replace **C1** by **C1***. We shall assume **C3*** instead of **C3** and prove a weaker form of **C3** later. We shall also assume **C2***, **C4*** instead of **C2**, **C4**.

C2*. *If* $AB \rightarrow CD$ *and* $CD \Rightarrow EF$, *then* $AB \rightarrow EF$.
If $AB \Rightarrow CD$ *and* $CD \rightarrow EF$, *then* $AB \rightarrow EF$.

C3*. *If* B, C *lie on a ray from* A, *if* B', C' *lie on a ray from* A', *and if* $[ABC]$, $AB \Rightarrow A'B'$, $AC \Rightarrow A'C'$, *then* $[A'B'C']$ *and* $BC \rightarrow B'C'$.

C4*. *If* A *is isometric to* B, *then* $AB \Rightarrow BA$.

1.2. *If* $A \neq B$, *then* $AB \Rightarrow AB$.

Proof. We cannot give the normal proof, since this uses **C4**. On ray AB, there exist B' and B'' such that $AB \Rightarrow AB'$ and $AB' \Rightarrow AB''$ (**C1***, 1.1). Hence $AB \rightarrow AB''$ (**C2***). Since $AB \rightarrow AB'$ also, we have $B' = B''$ (**C1***). Suppose $B \neq B'$ and $[ABB']$. Since $AB \Rightarrow AB'$ and $AB' \Rightarrow AB'$, we have $[AB'B']$ (**C3***), a contradiction. Similarly, if $B \neq B'$ and $[AB'B]$. Hence $B = B'$, so $AB \Rightarrow AB$.

1.3. *Isometry is a transitive relation.*

Proof. Suppose A is isometric to B, and B to C. Given any point $A' \neq A$ and any ray c from C, let b be any ray from B. There exists $B' \in b$ such that $AA' \Rightarrow BB'$ and there exists $C' \in c$ such that $BB' \Rightarrow CC'$. Hence $AA' \rightarrow CC'$ (**C2***). Hence A is isometric to C.

1.4. *Isometry is an equivalence relation.*

Proof (Figure 1A). Because of 1.1 and 1.3, we have only to prove that isometry is a symmetric relation. Suppose A is isometric to B. Let B' be any point distinct from B, and let a be any ray from A. There exists C on ray B/A (the ray opposite to ray BA) such that $BB' \Rightarrow BC$ (**C1***, 1.1). Since A is isometric to B, there exists $D \in$ ray BA such that $AC \Rightarrow BD$. Also $AB \Rightarrow BA$ (**C4***) and $[ABC]$, so $[BAD]$ and $BC \rightarrow AD$ (**C3***). Since $BB' \Rightarrow BC$ and $BC \rightarrow AD$, we have $BB' \rightarrow AD$ (**C2***). There exists $E \in a$ such that $AD \Rightarrow AE$ (**C1***, 1.1). Hence $BB' \rightarrow AE$ (**C2***). Hence B is isometric to A.

If A is isometric to B, we can therefore say that *A and B are isometric*. In many of the subsequent figures we shall depicit isometric points by the same type of dot.

1.5. *If* A, B *are isometric, and if* $AP \Rightarrow BQ$, *then* P, Q *are isometric.*

Proof (Figure 1B). Let P' be any point distinct from P, and let q be any ray from Q. There exists A' on ray P/A such that $PP' \Rightarrow PA'$. There exists B' on ray BQ such that $AA' \Rightarrow BB'$. Since $[APA']$ and $AP \Rightarrow BQ$, we have

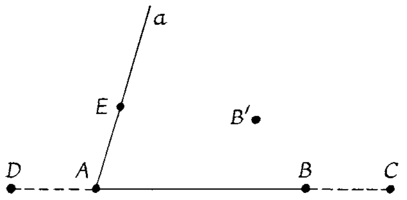


FIGURE 1A.

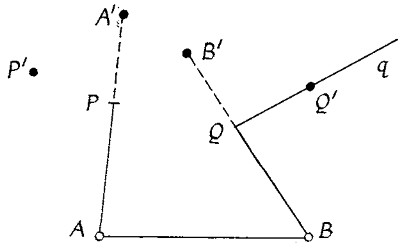


FIGURE 1B.

$[BQB']$ and $PA' \to QB'$ (**C3***). Since $PP' \Rightarrow PA'$ and $PA' \to QB'$, we have $PP' \to QB'$ (**C2***). There exists $Q' \in q$ such that $QB' \Rightarrow QQ'$. Hence $PP' \to QQ'$ (**C2***). Hence P is isometric to Q.

COROLLARY. *If $AP \Rightarrow AQ$, then P, Q are isometric.*

1.6. *If $AB \Rightarrow CD$, then $CD \Rightarrow AB$.*

Proof. Since A, C are isometric (1.4), there exists $B' \in$ ray AB such that $CD \Rightarrow AB'$. Hence $AB \to AB'$ (**C2***). But $AB \to AB$ (1.2). Hence $B' = B$ (**C1***). Hence $CD \Rightarrow AB$.

1.7. *If $[ABC]$, $[A'B'C']$, and if $AB \Rightarrow A'B'$, $BC \to B'C'$, then $AC \Rightarrow A'C'$.*

Proof. Since A, A' are isometric, there exists $D' \in$ ray $A'B'$ such that $AC \Rightarrow A'D'$. Since also $AB \Rightarrow A'B'$ and $[ABC]$, we have $[A'B'D']$ and $BC \to B'D'$. Since B, B' are isometric (1.5), we have $B'C' \Rightarrow BC$ and $BC \Rightarrow B'D'$ (1.6). Hence $B'C' \to B'D'$ (**C2***); but $B'C' \to B'C'$ and C', D' lie on the same ray from B', so $C' = D'$ (**C1***). Hence $AC \Rightarrow A'C'$.

It follows from **C2***, 1.2, 1.3, 1.6 that the relation "$\Rightarrow$" is an equivalence relation between ordered pairs of distinct points. We shall now replace the symbol "$\Rightarrow$" by the more usual symbol "$\equiv$." Thus "$AB \equiv CD$" means that A, C are isometric and $AB \to CD$.

In the statement of 1.7, B and B' are isometric (1.5). Hence the condition $BC \to B'C'$ can be written $BC \Rightarrow B'C'$ or $BC \equiv B'C'$. Using arguments similar to this, we can now deduce from **C1***, **C2***, **C3***, 1.2, 1.4, 1.5, 1.6, 1.7:

1.8 (i) *The relation denoted by $\equiv$ is an equivalence relation between ordered pairs of distinct points.*

(ii) *If A, B are isometric, if $A' \neq A$, and if b is any ray from B, then there exists $B' \in b$ such that $AA' \equiv BB'$.*

(iii) *If $[ABC]$, if B', C' lie on a ray from A', and if $AB \equiv A'B'$, $AC \equiv A'C'$, then $[A'B'C']$ and $BC \equiv B'C'$.*

(iv) *If $[ABC]$, $[A'B'C']$, and if $AB \equiv A'B'$, $BC \equiv B'C'$, then $AC \equiv A'C'$.*

The axioms **C2***, **C3***, **C4*** arise naturally when we replace **C1** by **C1***. Axiom **C2*** is rather artificial, but we do not need to assume **C2** in its entirety, whilst the axiom: *if* $AB \Rightarrow CD$ *and* $CD \Rightarrow EF$, *then* $AB \Rightarrow EF$ is too weak. By assuming **C4*** rather than **C4**, we do not postulate the existence of any congruent point-pairs such as $PQ \rightarrow XY$ *unless* P *and* X *are isometric*. The following example of a one-dimensional absolute geometry shows that this is a reasonable point of view to adopt.

Let the points of a one-dimensional geometry consist of all rational numbers of the form $p/3^r$, p an integer, r a non-negative integer, geometrical order being defined by the natural ordering of the rationals, in the obvious way. The point $p/3^r$ is called *odd* or *even* according as p is odd or even; ($p/3^r$ need not be in its lowest terms). If $a \neq b$ and $c \neq d$, where a, b, c, d are points of our geometry, we define congruence of the point-pairs ab and cd by saying that $ab \rightarrow cd$ if and only if $|a - b| = |c - d|$ and a, c are both odd or both even. Then axioms **C1*–C4*** are satisfied, and two points are isometric if and only if they are both odd or both even. However, ab and ba are not congruent unless a, b are isometric.

2. Further axioms of congruence. I have been unable to deduce the important result that $AB \equiv CD$ implies $BA \equiv DC$, assuming only **C1*–C4***, so we shall take this as an axiom. Before doing so, we shall prove this result if A, B, C, D are collinear (2.2), and shall prove also the uniqueness of mid-points.

2.1. *If* A, B *lie on the same ray from* O, *and if* $AO \equiv BO$, *then* $A = B$.

Proof (Figure 2A). Suppose $A \neq B$; then without loss of generality $[ABO]$. There exists C on ray O/B such that $OB \equiv OC$. B and C are isometric; hence there exists O' on ray CB such that $BO \equiv CO'$. Also $BC \equiv CB$ (**C4***) and $[BOC]$. Hence $[CO'B]$ and $OC \equiv O'B$ (**C3***). Also $AO \equiv CO'$, $AC \equiv CA$ (**C4***; A, B are given to be isometric, so A, C are isometric) and $[AOC]$. Hence $[CO'A]$ and $OC \equiv O'A$ (**C3***). Hence A, B lie on ray O'/C and $O'A \equiv O'B$ (1.8, i). Hence $A = B$ (**C1***, 1.2).

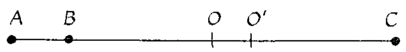


FIGURE 2A.

2.2. *If* A, B, C, D *are collinear, and if* $AB \equiv CD$, *then* $BA \equiv DC$.

Proof. We shall obtain the result in various stages.

(i) (Figure 2B, i). *If* $[POQ]$ *and* $OP \equiv OQ$, *then* $PO \equiv QO$.

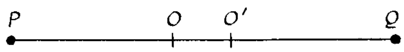


FIGURE 2B(i).

P and Q are isometric. There exists $O' \in$ ray QP such that $PO \equiv QO'$. Also $PQ \equiv QP$ and $[POQ]$. Hence $[QO'P]$ and $OQ = O'P$. Hence $OP \equiv O'P$, so $O = O'$ (2.1). Hence $PO \equiv QO$.

(ii) (Figure 2B, ii). *If* $[ABCD]$, *or if* $B = D$ *and* $[ABC]$, *and if* $AB \equiv CD$, *then* $BA \equiv DC$.

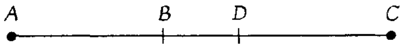


FIGURE 2B(ii).

A, C are isometric; hence so are B, D. Now $AB \equiv CD$, $AC \equiv CA$, and $[ABC]$. Hence $BC \equiv DA$ (**C3***). If $B = D$, there is nothing more to prove. If not, $BC \equiv DA$, $BD \equiv DB$, and $[BDC]$. Hence $DC \equiv BA$ (**C3***).

(iii) We can define two opposite senses on a line (**6**, p. 75 et seq.), corresponding to the intuitive notion of sense. *If* AB *and* CD *have opposite senses, and if* $AB \equiv CD$, *then* $BA \equiv DC$.

We have dealt with two possible cases in (ii). In the remaining cases, AB and DB have the same sense (Figure 2B, iii; not all cases are shown in the figure). There exist $A' \in$ ray B/A such that $BA \equiv BA'$, and $C' \in$ ray D/C such that $DC \equiv DC'$. Then $[A'BDC']$.

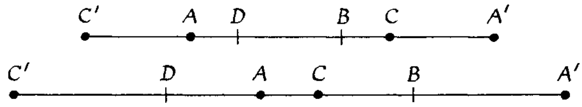


FIGURE 2B(iii).

Using (i), $A'B \equiv AB \equiv CD \equiv C'D$. Hence, from (ii), $BA' \equiv DC'$. Hence $BA \equiv BA' \equiv DC' \equiv DC$.

(iv) *If* AB *and* CD *have the same sense, and if* $AB \equiv CD$, *then* $BA \equiv DC$.

There exists $E \in$ ray C/D such that $AB \equiv CD \equiv CE$. Then CE has the opposite sense to AB and CD. Hence, using (iii), $BA \equiv EC \equiv DC$.

Thus the result is proved in all cases.

Definition. A point M on the line AB, where $A \neq B$, such that $MA \equiv MB$, is a *mid-point* of the segment AB, or of the point-pair AB, and M *bisects* AB; if m is a line through M, $m \neq$ line AB, then m *bisects* AB.

2.3. *If* M *is a mid-point of* seg AB, *then* $[AMB]$.

Proof. If not, then A, B lie on the same ray from M, and $MA \equiv MB$ which contradicts **C1***.

2.4. *A segment cannot have more than one mid-point.*

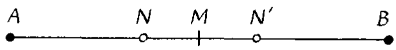


FIGURE 2C.

Proof (Figure 2C). We follow the lines of the usual proof, but we need 2.1. Suppose there exist distinct mid-points M, N of AB, and suppose without loss of generality that $[ANM]$ (using 2.3). There exists $N' \in$ ray MB such that $MN \equiv MN'$. Also $MA \equiv MB$ and $[MNA]$. Hence $[MN'B]$ and $NA \equiv N'B$. But $NA \equiv NB$, since N is a mid-point of AB. Hence $N'B \equiv NB$. Hence $N' = N$ (2.1), a contradiction since $[ANM]$ and $[MN'B]$.

We now introduce axiom **C4****.

C4.** *If $AB \equiv CD$, then $BA \equiv DC$.*

2.5. *If $[ABC]$, if A', B', C' are collinear, and if $BC \equiv B'C'$, $CA \equiv C'A'$, $AB \equiv A'B'$, then $[A'B'C']$.*

Proof. Use **C4**** and 1.8.

Definition. If ABC, $A'B'C'$ are triangles, and if $BC \equiv B'C'$, $CA \equiv C'A'$, $AB \equiv A'B'$, then we write $\triangle ABC \equiv \triangle A'B'C'$ and say that the triangles are *congruent.*

Without axiom **C4****, any definition of congruent triangles would be somewhat artificial. Our definition implies that corresponding vertices of congruent triangles are isometric. (If we assume **C1**, then all points are isometric, and the above definition gives rise to triangles that are congruent in the more usual sense. Some authors require triangles to satisfy more conditions than these before calling them congruent (e.g. **6**, p. 93, § 8, p. 97; **10**, p. 116); they then prove that two triangles are congruent if and only if corresponding sides are congruent.)

The remaining traditional axioms of congruence are listed below. These axioms presuppose axioms **C1–C4**, so that "$AB \equiv CD$" means "$AB \to CD$," and "$\triangle ABC \equiv \triangle A'B'C'$" means "$BC \to B'C'$, $CA \to C'A'$, $AB \to A'B'$."

C5. *If $\triangle ABC \equiv \triangle A'B'C'$, and if $[ABX]$ and $[A'B'X']$, and if $BX \equiv B'X'$, then $CX \equiv C'X'$.*

C6. *If the triangles ABC, ABC' lie in the same plane, if C, C' lie on the same side of line AB in this plane, and if $\triangle ABC \equiv \triangle ABC'$, then $C = C'$.*

C7. *If ABC is a triangle, if $AB \equiv A'B'$, and if π' is any plane containing $A'B'$, then there exist just two points C'_1, C'_2 in π', one on either side of $A'B'$, such that $\triangle ABC \equiv \triangle A'B'C'_1 \equiv \triangle A'B'C'_2$.*

M. *Every segment has a mid-point.*

R. *All right angles are congruent* (see **6**, p. 97 et seq.).

Forder's book *The foundations of Euclidean geometry*, based partly on the work of such authors as Veblen (**16**; **17**) and Moore (**11**), appeared in 1927. Forder used **C1–C6** and a weaker form of **R** as his axioms of congruence (**6**, Chapter IV). Axioms **C7** and **R** are equivalent (**6**, p. 111 et seq.), and **C7** clearly implies **C6**. Some authors, such as Kerékjártó (**10**, § 14) use **C7** without mentioning **C6**, but Forder used the weaker axiom **C6** to prove the existence of right angles and various properties of them, Hilbert **(9**, § 6**)** used congruence of angles as an undefined relation, but his axioms are easily seen to be equivalent to **C1–C5** and **C7**.

It can be shown that **C1–C5** and **M** imply **C6** and **R** in a plane **(6**, p. 132**)** and Dorroh showed in 1930 that the same is true in spaces of higher dimension **(5)**. In 1929 Dorroh showed that **C1–C5** imply **M** **(4)**. This result was also proved independently by Piesyk in 1965 **(12)**, using axioms due to Tarski **(15)** which are equivalent to **C1–C5**. Piesyk starts from axioms of order and congruence, and proves that every point-pair has a mid-point without defining any of the terms line, plane, and right angle.

In 1947 Forder showed that **C5** could be replaced by the following four special cases **(7)** (Figure 2D).

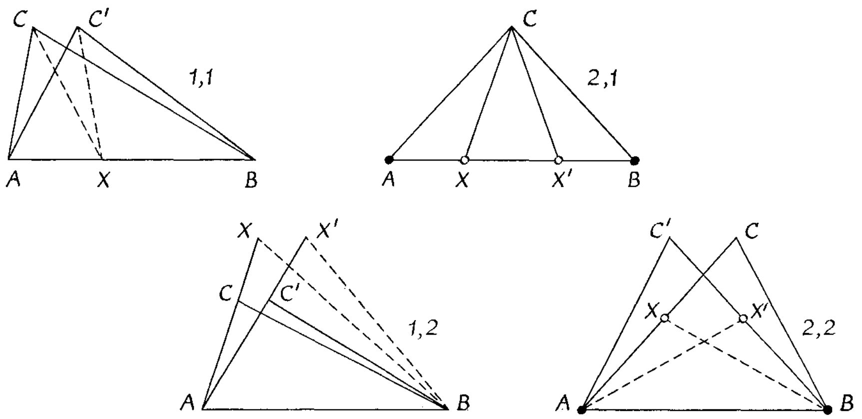


FIGURE 2D.

$C5_{1,1}$. *If $\triangle ABC \equiv \triangle ABC'$, and if X lies on line AB, then $XC \equiv XC'$.*

$C5_{1,2}$. *If $\triangle ABC \equiv \triangle ABC'$, and if X, X' lie on rays AC, AC', or on rays A/C, A/C', and if $AX \equiv AX'$, then $BX \equiv BX'$.*

$C5_{2,1}$. *If $CA \equiv CB$, and if X, X' lie on rays AB, BA, and if $AX \equiv BX'$, then $CX \equiv CX'$.*

$C5_{2,2}$. *If $\triangle ABC \equiv \triangle BAC'$, and if X, X' lie on rays AC, BC', or on rays A/C, B/C', and if $AX \equiv BX'$, then $BX \equiv AX'$.*

Forder's statement of **$C5_{2,1}$** is more general than this, but he remarks **(7**, p. 270**)** that he uses only the special case given here.

In 1961 Szász gave a different development of the basic results of Forder's paper of 1947 **(14)**. In the present paper we shall obtain the same results using fewer and weaker special cases of **C5**, as follows:

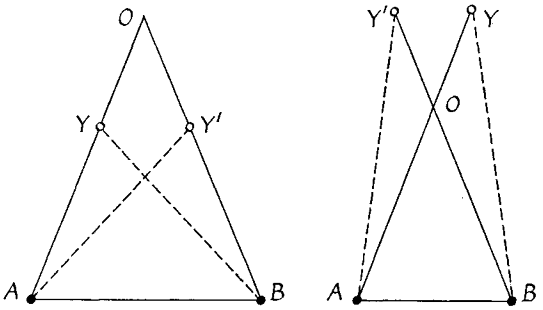
C5a. *If $\triangle ABC \equiv \triangle ABC'$, and if $[ABX]$, then $XC \equiv XC'$.*

This is a weaker form of **$C5_{1,1}$** (cf. **C5**), from which **$C5_{1,1}$** can easily be deduced (2.6).

We do not need to assume either $\mathbf{C5}_{1,2}$ or $\mathbf{C5}_{2,1}$.

C5b (Figure 2E). *If* $OA \equiv OB$, *and if* Y, Y' *lie on rays* OA, OB, *or on rays* O/A, O/B, *and if* $OY \equiv OY'$, *then* $BY \equiv AY'$.

This is a special case of $\mathbf{C5}_{2,2}$, with $C = C' = O$. In Figure 2E(i), we have $\triangle ABY' \equiv \triangle BAY$. As our final axiom we shall assume $\mathbf{C5}_{2,2}$ for triangles such as these.



(i) (ii)

FIGURE 2E.

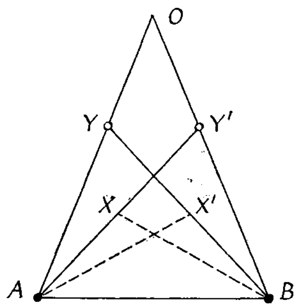


FIGURE 2F.

C5c (Figure 2F). *If* $OA \equiv OB$, *if* Y, Y' *lie on rays* OA, OB *and if* $OY \equiv OY'$, *if* $[AXY']$, $[BX'Y]$ *and if* $AX \equiv BX'$, *then* $BX \equiv AX'$.

Apart from making stronger restrictions on the order of A, X, Y', etc., what we are essentially doing here is restricting $\mathbf{C5}_{2,2}$ to the case when AC', BC meet at O, say, and $OA \equiv OB$. We shall see later (at the end of § 5) that this is a significant restriction.

Because of the further weakening of **C5** and the assumption of **C1*** rather than **C1**, some of the theorems in (**7**; **14**) must be proved by different methods. Other proofs in (**7**; **14**) can be adapted. Some proofs I have simplified. For the sake of completeness I have included certain proofs that are essentially the same as those given in (**7**) or (**14**).

We can still prove the existence of perpendicular lines, but the proof of the existence of mid-points breaks down. However, using more powerful methods, we shall prove the existence of mid-points in § 7; this means that, using axioms **C1*–C4***, **C4****, **C5a**, **b**, **c**, we can prove **C1**.

Many results depend only on axioms **C5a** and **C5b**, together with the previous axioms. These will be proved in §§ 3, 4.

Axiom **C5c** is not an existence axiom, but we shall use it in § 5 to establish the existence of perpendicular lines. It is only once used directly, in the proof of 5.1.

2.6 (an extension of **C5a**). *If* $\triangle ABC \equiv \triangle ABC'$, *and if* $[AYB]$, *then* $YC \equiv YC'$.

Proof (**6**, p. 97). There exists X such that $[ABX]$. Then $XC \equiv XC'$ (**C5a**). Hence $\triangle XBC \equiv \triangle XBC'$. Also $[XBY]$, so $YC \equiv YC'$ (**C5a**).

For convenience we shall refer to both **C5a** and 2.6 as **C5a**.

3. Basic results.

3.1 (**C5**$_{1,2}$). *If $\triangle ABC \equiv \triangle ABC'$, and if X, X' lie on rays AC, AC', or on rays A/C, A/C', and if $AX \equiv AX'$, then $BX \equiv BX'$.*

Proof (Figure 3A). Suppose X, X' lie on rays AC, AC'. There exist C'', X'' on ray AB such that $AC'' \equiv AC \equiv AC'$, $AX'' \equiv AX \equiv AX'$. Then $XC'' \equiv X''C$, $X'C'' \equiv X''C'$ (**C5b**). But $X''C \equiv X''C'$ (**C5a**), so $XC'' \equiv X'C''$. Hence $\triangle AC''X \equiv \triangle AC''X'$. Hence $BX \equiv BX'$ (**C5a**).
A similar proof holds if X, X' lie on rays A/C, A/C'.

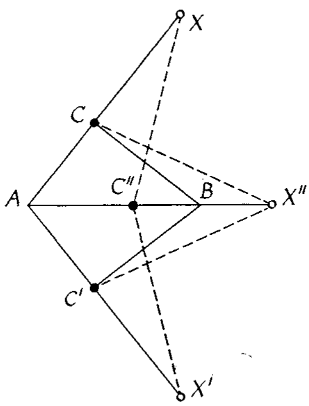


FIGURE 3A.

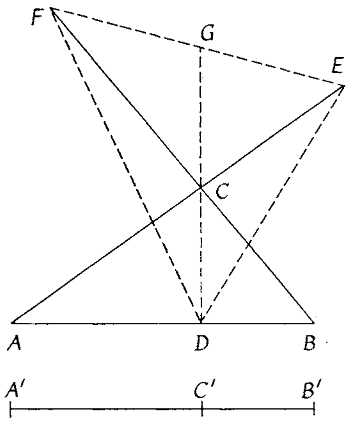


FIGURE 3B.

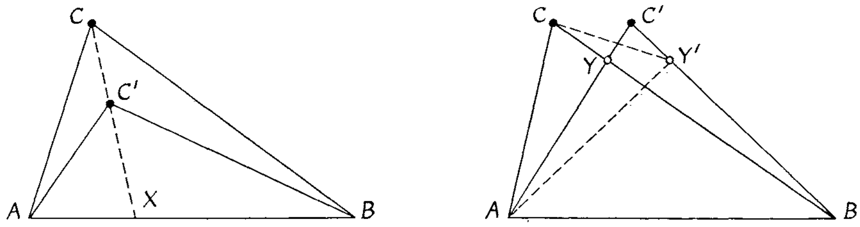
3.2. *Three collinear points cannot be congruent to a triangle*; *i.e. if ABC is a triangle and if $[A'C'B']$, then we cannot have $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$.*

Proof (Figure 3B). Suppose such a situation occurs. There exists D on ray AB such that $AD \equiv AC$. Then $AD \equiv A'C'$, so $DB \equiv C'B'$ and $[ADB]$. Hence $BD \equiv B'C' \equiv BC$.

There exists E such that $[ACE]$ and $AE \equiv AB$; hence $EC \equiv BD$. There exists F such that $[BCF]$ and $BF \equiv BA$; hence $FC \equiv AD$. Hence $ED \equiv BC$ and $FD \equiv AC$ (**C5b**). Hence $EC \equiv BD \equiv BC \equiv ED$ and $FC \equiv AD \equiv AC \equiv FD$. Using the transversal axiom and the transversal theorem (see Introduction) we easily prove that DC meets EF at G, say, where $[GCD]$. Now $\triangle EFC \equiv \triangle EFD$, so $GC \equiv GD$ (**C5a**). Hence $C = D$ (**C1***), a contradiction.

For the usual proof of 3.2 using $\mathbf{C5}_{2,1}$, see (**6**, p. 98; **7**, 6; **14**, Lemma 1). The next result will not be used until § 6.

3.3 (**C6**). *If* $\triangle ABC \equiv \triangle ABC'$, *with* A, B, C, C' *coplanar and with* C, C' *lying on the same side of* AB, *then* $C = C'$.



(i) (ii)

FIGURE 3C.

Proof (**7**, 5, 7). Suppose $C \neq C'$.

(i) (Figure 3C, i). If CC' meets AB at X, say, then $XC \equiv XC'$ (**C5a**) and $[XCC']$ or $[XC'C]$, which contradicts **C1***.

(ii) (Figure 3C, ii) If CC' does not meet AB, we may easily show that either the segments AC', BC meet or the segments AC, BC' meet. Suppose the former without loss of generality, and let $AC' \cap BC = Y$. There exists $Y' \in$ ray BC' such that $BY' \equiv BY$ (**C1***). Then $C'Y \equiv CY'$ (**C5b**) and $AY \equiv AY'$ (3.1). Hence the collinear points A, Y, C' are congruent to the triangle $AY'C$, which contradicts 3.2. Hence $C = C'$.

Definition. An *isometry* of an absolute space is a one–one mapping of the space onto itself that maps each point-pair onto a congruent point-pair. (No confusion is likely to arise with the relation of isometry between isometric points.)

3.4. *Any isometry maps collinear points onto collinear points and preserves order; hence it maps lines onto lines.*

Proof. Use 3.2 and 2.5.

Definition. Let O, P be distinct points. If P' is the unique point such that O bisects PP', then P' is the *reflection* or *image* of P in O. The reflection of O in O is defined to be O itself. The mapping $P \to P'$ (for fixed O) is called the *reflection* in O.

3.5. *The reflection in* O *is an isometry.*

Proof (**7**, 4) (Figure 3D). Let P, Q be any two distinct points. If $O = P$ or if $O = Q$ or if O, P, Q are collinear, then clearly $PQ \equiv P'Q'$ (1.8, 2.2). If not, then there exists $X \in$ ray OP such that $OQ \equiv OQ' \equiv OX$, and there exists $Y \in$ ray OQ such that $OP \equiv OP' \equiv OY$. Then $YX \equiv PQ$ and $YX \equiv P'Q'$ (**C5b**). Hence $PQ \equiv P'Q'$.

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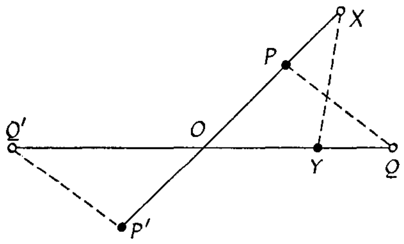

FIGURE 3D.

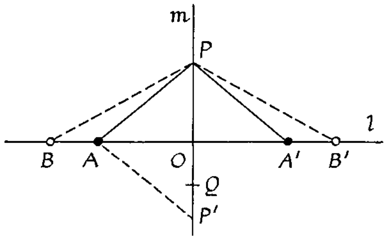

FIGURE 4A.

4. Perpendiculars; constructions.

Definition. (Figure 4A). Let the lines l, m intersect in O. If, for all points $P \in m$, and for all pairs of points $A, A' \in l$ such that O bisects AA', we have $PA \equiv PA'$, then m is *perpendicular to* l, and we write $m \perp l$.

4.1. *If l, m intersect in O, and if there exist $P \in m$ and $A, A' \in l$ such that $P \neq O$, $A \neq A'$, O bisects AA' and $PA \equiv PA'$, then $m \perp l$.*

Proof (cf. **7**, 8, 12; **14**, p. 272) (Figure 4A). Let Q be any point on m, and let B, B' be any pair of distinct points on l such that O bisects BB'. Since $\triangle OPA \equiv \triangle OPA'$, we have $PB \equiv PB'$ (3.1). Since $\triangle POB \equiv \triangle POB'$, we have $QB \equiv QB'$ (**C5a**). Hence $m \perp l$ by definition.

4.2. *If $m \perp l$, then $l \perp m$.*

Proof (cf. **7**, 14; **14**, Theorem 1) (Figure 4A). There exist $A, A' \in l$ and $P, P' \in m$ such that O bisects both AA' and PP' (**C1***). Then $AP' \equiv A'P$ (3.5) $\equiv AP$ (since $m \perp l$.) Hence $l \perp m$ (4.1).

My proofs of the next two results are simpler than those usually given (**14**, Theorems 4, 6; **7**, 13).

4.3. *In any plane containing a line l, there exists at most one line perpendicular to l through a given point O on l.*

Proof (Figure 4B). Suppose that there exist two such lines, m and n, $m \neq n$. Take $P \in m$, $Q \in n$, on opposite sides of l. Then PQ meets l at R, say; $R \neq O$. There exists $A \in l$ such that $[ORA]$. There exists $B \in l$ such that O bisects AB. Since $m \perp l$ and $n \perp l$, we have $PA \equiv PB$ and $QA \equiv QB$. Hence $\triangle PQA \equiv \triangle PQB$, so $RA \equiv RB$. (**C5a**). Hence R bisects AB, contradicting 2.4.

In the above proof, we could also take $A = R$ and then use 3.2.

4.4. *Through any point P not on the line l, there exists at most one line perpendicular to l.*

Proof (Figure 4C). Suppose that PM, PN are both perpendicular to l, $M \neq N$. There exists P' such that M bisects PP'. There exists $M' \in l$ such

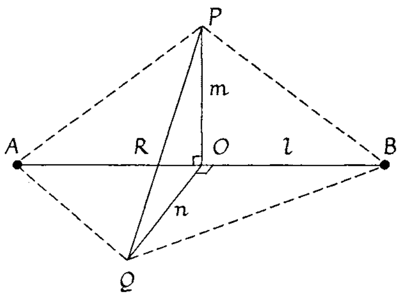

FIGURE 4B.

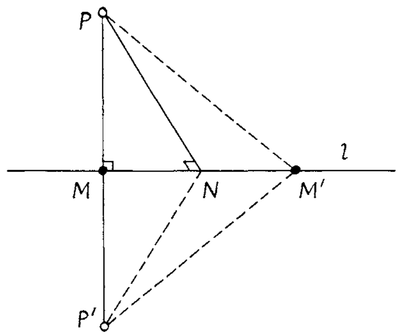

FIGURE 4C.

that N bisects MM'. Then $P'M' \equiv PM'$ (since $l \perp PP'$) $\equiv PM$ (since $PN \perp l$) $\equiv P'M$. Hence $P'N \perp l$ (4.1). This contradicts 4.3, since $P'N$, PN are distinct lines.

We could also end the above proof by saying that the collinear points P, M, P' are congruent to the triangle $PM'P'$, which contradicts 3.2.

Definition. Let l be a line and P a point. If $P \notin l$, and if there exists a perpendicular PM from P to l, unique by 4.4, where $M \in l$, let P' be the point such that M bisects PP'. If $P \in l$, let $P' = P$. Then P' is the *reflection* or *image of P in l.* If, for fixed l, every point P (or every point P in some plane containing l) has a reflection in l, then the mapping $P \to P'$ is called the *reflection in l.*

The reflection in a line, if it exists, is clearly one–one and onto; also $(P')' = P$, so the reflection is a mapping of period 2.

It is of interest to consider the constructibility of certain points and lines, as distinct from their mere existence. We assume that we can construct

(i) the ray AB, where A, B are given distinct points,

(ii) the point of intersection of two lines that are known to intersect,

(iii) the point C, on any given ray through A, such that $AB \equiv AC$, where A, B are given distinct points.

It follows from (i) that we can construct the line AB. We also assume that we can decide, in the course of a construction, whether or not a point coincides with another point or lies on a given line, ray, or segment; and that, in the course of a construction, we can make use of an arbitrary point, either on a given line, ray, or segment, or not on it.

We shall frequently prove the existence of perpendiculars etc. by constructing them. In these cases a statement such as "We can construct the unique line through P perpendicular to l" means "There exists a unique line through P perpendicular to l, and we can construct it."

4.5. *If we can construct one line m perpendicular to a given line l in a given plane π, then*

(i) *through any point $P \in \pi$, $P \notin l$, we can construct the unique line perpendicular to l;*

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(ii) *the reflection in l, confined to the points of π, exists, and the reflection of any given point is constructible.*

Proof. (i) (**7**, 20; **14**, Theorem 5) (Figure 4D). Let $l \cap m = 0$. If $P \in m$, there is nothing more to construct. If not, let R be any point of m on the same side of l as P. Construct $R' \in m$ such that O bisects RR'. PR' meets l at A, say; we can construct A. Construct $P^* \in$ ray A/R such that $AP^* \equiv AP$. PP^* meets l at M, say. Since $AR \equiv AR'$ by the definition of perpendicularity, we have $\triangle AOR \equiv \triangle AOR'$. Hence $OP^* \equiv OP$ (3.1). Hence $\triangle OAP \equiv \triangle OAP^*$. Hence $MP \equiv MP^*$ (**C5a**). Hence $l \perp PP^*$ (4.1). Hence $PP^* \perp l$ (4.2). The uniqueness of the perpendicular from P to l follows from 4.4.

(ii) The existence and constructibility of the reflection follows immediately from (i).

4.6. *If l is a line in a plane π, and if the reflection in l, confined to the points of π, exists, then this reflection is an isometry.*

Proof (**7**, 10; **14**, Theorem 10). Let P, Q be distinct points of π. If either P or Q lies on l, then $PQ \equiv P'Q'$ by the definition of perpendicularity. Suppose that neither P nor Q lies on l. Let R be any point on the same side of l as P. We shall show that $PR \equiv P'R'$ and $PR' \equiv P'R$. If P, Q lie on the same side of l, the result then follows if we let $R = Q$. If P, Q lie on opposite sides of l, the result follows if we let $R = Q'$.

If $PR \perp l$, the result is immediate. If not, let $PR' \cap l = A$. (See Figure 4D, in which the line m is no longer a special line perpendicular to l, as in 4.5.) Define P^* as in the proof of 4.5, and let $PP^* \cap l = M$. Then $PP^* \perp l$ and M bisects PP^*, as in the proof of 4.5. Hence $P^* = P'$. Hence $PR' \equiv P^*R = P'R$. Finally $PR \equiv P^*R'$ (**C5b**) $= P'R'$.

The proof of the next result as given by Szász (**14**, Theorem 7) uses axiom $\mathbf{C5}_{2,2}$ twice (in the general case not covered by even **C5c**). Forder's proof (**7**, 22, using 17 and 21) uses the notion of the bisector of an angle, for which we need **C5c**; he inadvertently proves not 21 but its converse, but this is easily remedied. The following proof is shorter, and does not use **C5c**.

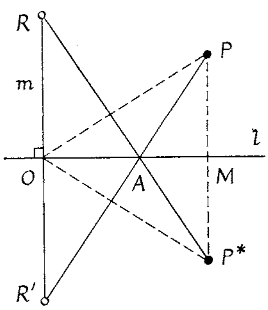

FIGURE 4D.

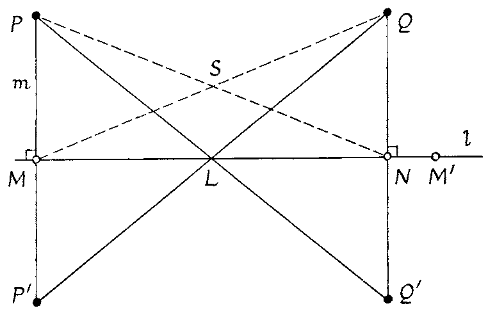

FIGURE 4E.

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4.7. *If we can construct one line m perpendicular to a given line l in a given plane π, then through any point $L \in l$ we can construct a line in π perpendicular to l.*

Proof (Figure 4E). Let $l \cap m = M$. If $L = M$, there is nothing more to construct. If not, let P be any point of m, $P \neq M$, and construct the reflection P' of P in l; $P' \in m$. Construct Q, Q' such that L bisects $P'Q$ and PQ'. Construct the point N where QQ' meets l. Then, as in the proof of 4.5(i), Q' is the reflection of Q in l. Let M' be the reflection of M in L. Then Q', M', Q (the reflections of P, M, P' in L) are collinear (3.4, 3.5), so $M' = N$. Hence L bisects MN.

Lines MP, NQ are both perpendicular to l, so they cannot meet (4.4); P, Q lie on the same side of l; hence segments PN, QM meet, at S, say; we can construct S. The product of the reflection in l (4.5, ii) and the reflection in L maps P, Q, M, N onto Q, P, N, M. This product is an isometry, so $S = PN \cap QM$ is mapped onto itself (3.4), and $SM \equiv SN$. Clearly $S \neq L$, so $LS \perp l$ (4.1).

4.8. *Any isometry maps perpendicular lines onto perpendicular lines.*

Proof (Figure 4A). Let l, m be perpendicular lines, and let $l \cap m = O$. Let A, A' be points on l such that O bisects AA', and let $P \in m$ (A, B, P all distinct from O). Let the isometry map l, m, O, A, A', P onto $l_1, m_1, O_1, A_1, A'_1, P_1$. Then $OA \equiv O_1A_1$, etc.; but $OA \equiv OA'$ and $PA \equiv PA'$; hence $O_1A_1 \equiv O_1A'_1$ and $P_1A_1 \equiv P_1A'_1$. Hence $m_1 \perp l_1$ (4.1).

5. Existence theorems. To establish the existence of perpendicular lines we now use axiom **C5c**. First we prove a special case of (**7**, 16 or **14**, Lemma 2), to which we can apply **C5c** rather than **C5**$_{2,2}$.

5.1. *If $OA \equiv OB$, if Y, Z lie on rays OA, OB and $OY \equiv OZ$, and if S denotes the point $AZ \cap BY$, then $AS \equiv BS$.*

Proof (Figure 5A). We know that S exists by the transversal theorem, and $[ASZ]$. Since A, B are isometric (1.5, cor.), there exists $S' \in$ ray BY such that $AS \equiv BS'$. Hence $BS \equiv AS'$ (**C5c**).

Since $BY \equiv AZ$ (**C5b**), we have $[BS'Y]$ and $YS' \equiv ZS$. Hence $ZS' \equiv YS$ (applying **C5c** with A, B interchanged with Y, Z). We now have $YB \equiv ZA$, $BS \equiv AS'$, $SY \equiv S'Z$. Hence A, S', Z are collinear (3.2). Hence $S' = S$, so $AS \equiv BS$.

5.2. *If A, B are distinct points, and if there is given a point O such that $OA \equiv OB$, then we can construct the mid-point of AB.*

Proof (**7**, 17; **14**, Theorem 2) (Figure 5B). If O lies on AB, then it is the mid-point of AB, If not, let Y be a point between O and A. Construct $Z \in$ ray OB such that $OZ \equiv OY$. Seg YB meets seg ZA at S, say; we can construct S. We have $AS \equiv BS$ (5.1); OS meets AB at M, say; we can construct M. Since $\triangle OSA \equiv \triangle OSB$, we have $MA \equiv MB$ (**C5a**), so M is the mid-point of AB.

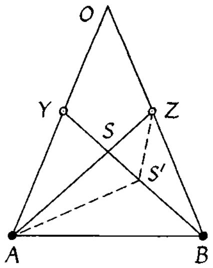

FIGURE 5A.

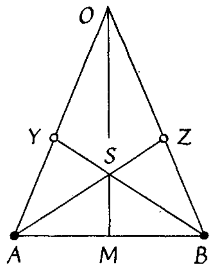

FIGURE 5B.

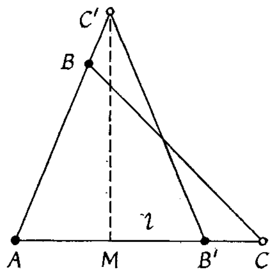

FIGURE 5C.

5.3. *Given a line l, we can construct a line perpendicular to it, in any plane through l.*

Proof (**14**, Theorem 3) (Figure 5C). Let A, C be distinct points of l. On any ray through C in the given plane, distinct from CA and C/A, construct B such that $CB \equiv CA$. On ray AB construct C' such that $AC' \equiv AC$. On ray AC construct B' such that $AB' \equiv AB$. We have $C'B' \equiv CB$ (**C5b**). Hence $C'A \equiv CA \equiv CB \equiv C'B'$. Hence we can construct the mid-point M of AB' (5.2). Then $C'M \perp l$ (4.1).

5.4. (i) *Through any point P, not on a line l, we can construct a unique line perpendicular to l.*

(ii) *The reflection in any line exists and is constructible.*

Proof. This follows from 5.3, 4.5.

5.5. *The reflection in a line, when confined to a plane through the line, is an isometry.*

Proof. This follows from 4.6.

5.6. *Through any point on a line l we can construct, in any plane containing l, a unique line perpendicular to l.*

Proof. This follows from 5.3, 4.7.

In (**14**, Theorem 8), Szász proves that every segment has a mid-point, using $\mathbf{C5}_{2,2}$. Forder, in his proof of this result (**7**, 23), makes an assumption that can only be justified by using $\mathbf{C5}_{2,2}$. If we were to assume $\mathbf{C5}_{2,2}$, we could adapt these proofs to show that any two isometric points have a mid-point, which is constructible. (In the proof of (**14**, Theorem 8), we must construct $AP \equiv AB$ and $BQ \equiv BA$.) I have found no way of adapting these proofs if we can use only **C5c**.

6. Absolute planes and metric planes. Throughout this section and the next π will denote an absolute plane. We shall show that π is a *metric plane* (to be defined later) and shall then apply various theorems about metric planes to show that every segment in π has a mid-point.

6.1. THE RIGIDITY THEOREM. *In π, if h is a ray from A, if h' is a ray from A', if l, l' are the lines of which h, h' are parts, if S denotes a side of l and S' a side of l', then there exists at most one isometry mapping A, h, S onto A', h', S'.*

Proof. Let α, β be two such isometries. Then $\alpha\beta^{-1}$ fixes A, h, S; it must therefore fix h pointwise. Let $B \in h$ and let $P \in S$. Let $P\alpha\beta^{-1} = P^*$. Since $\alpha\beta^{-1}$ is an isometry, we have $\triangle ABP \equiv \triangle ABP^*$. But $P^* \in S$, so P, P^* lie on the same side of l. Hence $P = P^*$ (3.3). Similarly, $\alpha\beta^{-1}$ fixes every point on the side of l opposite to S. Hence $\alpha\beta^{-1} = 1$, so $\alpha = \beta$.

If g is a line of π, we shall denote the reflection in g by σ_g. This notation will also apply later in any metric plane.

We shall not use the next result until the end of § 7.

6.2. *In π, if a, b are the lines perpendicular to a line l at the points A, B on l, and if there exists a line g such that $a\sigma_g = b$, then $A\sigma_g = B$ and g bisects AB.*

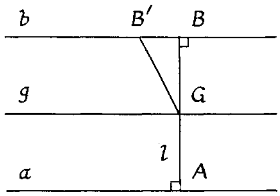

FIGURE 6A.

Proof (Figure 6A). If a met g, at C say, then b also would pass through C, contradicting 4.4. Hence a does not meet g, so a, b lie on opposite sides of g. Hence g meets seg AB, at G say. Let $A\sigma_g = B'$; $B' \in b$. Then $GB' \perp b$ (4.8), so $B = B'$ (4.4), and G bisects AB since σ_g is an isometry.

Suppose we have a geometric system S consisting of two sets P and L, the elements of P and L being called points and lines respectively, together with a relation of *incidence* between points and lines and a relation of *perpendicularity* between pairs of lines. A one–one mapping of P and L onto themselves that preserves incidence and perpendicularity is called an *orthogonal collineation* of S. An *involutory* orthogonal collineation (i.e. one of period 2) that leaves fixed every point incident with a line g is called a *generalized reflection* in g.

Definition. The system S is called a *generalized metric plane* if it satisfies the following axioms (**1**, p. 24):

1. Incidence axioms. *There exists at least one line; there exist at least three points incident with each line, Given two distinct points, there exists just one line incident with both.*

2. Orthogonality axioms. *If a is perpendicular to b, then b is perpendicular to a. Perpendicular lines are incident with a common point. Given any point and line, there exists at least one line through the given point perpendicular to the given line, and if the given point lies on the given line this perpendicular line is unique.*

3. The reflection axiom. *There exists at least one generalized reflection in each line.*

6.3. *The plane π is a generalized metric plane.*

Proof. This follows from the previous sections. Axiom 3 is satisfied since the reflection in any line is a generalized reflection.

6.4 (**1**, pp. 27, 30). *In a generalized metric plane there exists only one generalized reflection in each line.*

COROLLARY. *The generalized reflections in lines of π are just the reflections.*

In any generalized metric plane we can now refer to *the* reflection σ_g in the line g.

6.5. THE THEOREM OF THE THREE REFLECTIONS. *In the plane π,*

(i) *If a, b, c are three lines with a common point P, then there exists a line d through P such that $\sigma_a\sigma_b\sigma_c = \sigma_d$.*

(ii) *If a, b, c are three lines all perpendicular to a line g, then there exists a line $d \perp g$ such that $\sigma_a\sigma_b\sigma_c = \sigma_d$.*

The proof given in (**1**, p. 5, Theorem 5) is valid here, since this proof and the preceding ones depend only on various properties of absolute planes, and in particular on the rigidity theorem (6.1). The only new concept in the proof is that of an *angle bisector*. If h_1, h_2 are two rays from P, and if $Q_1 \in h_1$, $Q_2 \in h_2$ such that $PQ_1 \equiv PQ_2$, and if M is the mid-point of Q_1Q_2, then the *bisector of the angle h_1h_2* is the line PM. If $h_1 = h_2$, then the bisector is h_1; if h_1, h_2 are opposite rays, then the bisector is the perpendicular to h_1 through P. This bisector is independent of Q_1, Q_2, as can easily be seen by considering the reflection in PM. The reflection σ_{PM} maps h_1 onto h_2. We also use the fact that if A, B, C are collinear and if AB, BC have mid-points, then AC has a mid-point.†

Definition. A *metric plane* is a generalized metric plane satisfying the extra axiom:

4. The axiom of the three reflections. *The product of the three reflections in three lines a, b, c that have a common point or a common perpendicular line is equal to the reflection in a line d.*

†This is not obvious, since not all points are known to be isometric. See note at end of paper.

6.6. *The plane π is a metric plane.*

Proof. This follows from 6.3 and 6.5.

Definition. Let S be a metric plane. Let a, b be distinct lines of S. The *pencil* (a, b) consists of all lines c such that $\sigma_a \sigma_b \sigma_c$ is a line-reflection.

6.7 (**1**, p. 39, pp. 62–66). (i) $a \in (a, b)$, $b \in (a, b)$.

(ii) *If a, b have a common point P, then (a, b) consists of all lines through P.*

(iii) *If a, b have a common perpendicular g, then (a, b) consists of all lines perpendicular to g.*

(iv) *If $a' \in (a, b)$, $b' \in (a, b)$, and $a' \neq b'$, then $(a', b') = (a, b)$.*

Definition. If a, b have a common point, then (a, b) is a *proper* pencil; otherwise, the pencil is *improper.*

It follows from 6.7(ii) that there is a one–one correspondence between proper pencils and points.

6.8. THE ANGLE-BISECTOR THEOREM (**1**, p. 67) (Figure 6B). *If a, b, c do not belong to the same pencil, and if there exist u, v, w such that $a\sigma_w = b$, $b\sigma_u = c$, $v \in (a, c)$, and $v \in (u, w)$, then $c\sigma_v = a$.*

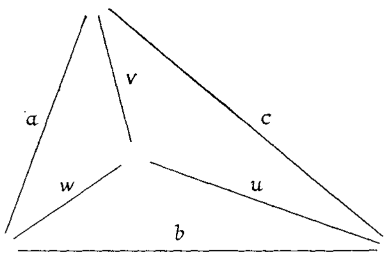

FIGURE 6B.

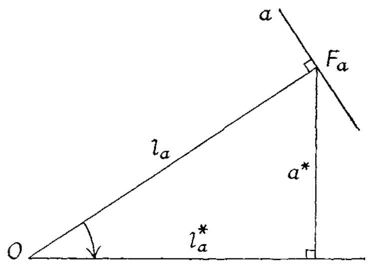

FIGURE 7A.

Note that if $a\sigma_w = b$, then $\sigma_b = \sigma_w \sigma_a \sigma_w$, so $\sigma_a \sigma_w \sigma_b = \sigma_w$. Hence a, w, b belong to the same pencil.

7. Semi-rotations. Let O be a given point of a metric plane S, and let u, v be given lines through O, not perpendicular. Let a be a line of S. Let l_a denote the perpendicular from O to a, meeting a at F_a (Figure 7A). We define the line a^* as follows (**1**, p. 94):

(i) if $O \in a$, then a^* is the line such that $\sigma_{a^*} = \sigma_a \sigma_u \sigma_v$;

(ii) if $O \notin a$, then a^* is the perpendicular from F_a to $l_a{}^*$ (defined as in (i)).

The mapping $a \to a^*$ is called *the semi-rotation about O determined by u and v.* In elementary language, $l_a{}^*$ is obtained by rotating l_a about O through an angle congruent to the angle between u and v.

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7.1 (**1**, pp. 94–98). *Semi-rotations have the following properties*:

(i) *if* $a \neq b$, *then* $a^* \neq b^*$;

(ii) *if* $c \in (a, b)$, then $c^* \in (a^*, b^*)$, *so that lines of a pencil are mapped onto lines of a pencil*;

(iii) *lines of a proper pencil are mapped onto lines of a proper pencil*;

(iv) *each pencil is the image of a pencil.*

Given a line $b \in S$, there does not necessarily exist a line a such that $b = a^*$, so (ii) does not state that the totality of all lines of a pencil is mapped onto the totality of all lines of a pencil. A similar remark applies to (iv). We can use (iii) to define the *image* P^* of a point P, so that a semi-rotation is also a point-to-point mapping and preserves incidence.

7.2. *Let* a, b *be the perpendiculars to a line* l *of* π *at two distinct points* $A, B \in l$. *Let* P, Q *be points of* a, b *on the same side of* l. *Let* p, q *bisect the angles* PAB, QBA. *Then the pencils* (p, q) *and* (a, b) *have a common line.*

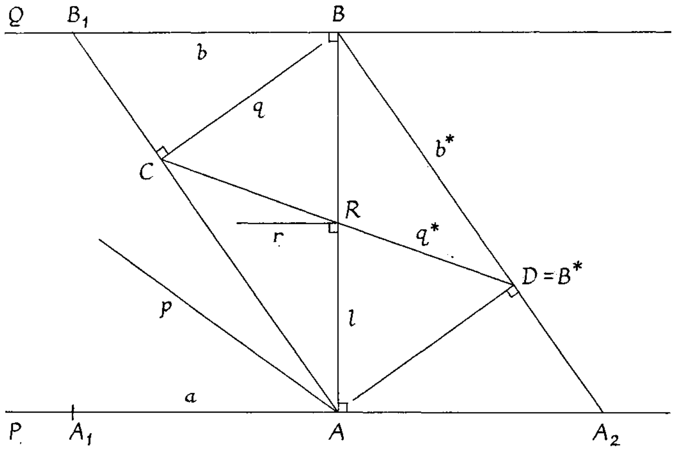

FIGURE 7B.

Proof (Figure 7B). Let $A_1 \in a$, $B_1 \in b$, on the same side of l as P, Q, such that $AB \equiv AA_1$, $BA \equiv BB_1$, and let $A_2 \in a$ on the opposite side of l, such that A bisects A_1A_2. Then q bisects AB_1, at C say, and $BC \perp AB_1$; also p bisects BA_1 and $p \perp BA_1$. Hence, if D bisects BA_2 (5.2), then $AD \perp BA_2$ and $AD = p\sigma_l$.

Consider the semi-rotation about A determined by p and l (which are not perpendicular). Since $\sigma_l = \sigma_p\sigma_p\sigma_l$ and $\sigma_{AD} = \sigma_l\sigma_p\sigma_l$, we have $p^* = l$, $l^* = AD$. Hence $b^* = BD$; $B = b \cap l$, so $B^* = b^* \cap l^* = D$. Now q^* passes through C by definition, and $B \in q$ so $B^* = D \in q^*$. Hence $q^* = CD$.

Now C, D lie on opposite sides of l, so CD meets l at R, say. Let r denote the perpendicular to l through R. Then r^* passes through R. Hence p^*, q^*, r^*, being concurrent, belong to a pencil. Hence p, q, r belong to a pencil (7.1, iv), i.e. $r \in (p, q)$. But a, b, r are all perpendicular to l, so $r \in (a, b)$ (6.7, iii). Thus (p, q) and (a, b) have a common line r.

7.3. *Every segment in* π *has a mid-point.*

Proof (Figure 7B). Let A, B be distinct points of π, and let $AB = l$. Define a, b, p, q, R, r, as in 7.2. Then a, b, l do not belong to a pencil, and $a\sigma_p = l$, $l\sigma_q = b$, $r \in (a, b)$, $r \in (q, p)$. Hence $b\sigma_r = a$ (6.7). Hence $B\sigma_r = A$ and R bisects AB (6.2).

7.4. *In any absolute space of dimension greater than* 1, *every segment has a mid-point.*

Proof. Any two points are contained in a plane, which is itself an absolute plane. Hence we can apply 7.3.

COROLLARY. *In any absolute space of dimension greater than* 1, *any two points are isometric* (1.5, cor.).

The geometry given at the end of § 1 shows that 7.4 is not true for a one-dimensional geometry. E.g. the segment joining 0 and $\frac{1}{3}$ has no mid-point. Even if we extend the definition of congruence in this example by saying that the point-pairs ab and cd are congruent if and only if $|a - b| = |c - d|$, so that axiom **C1** is satisfied, then the segment joining 0 and $\frac{1}{3}$ still has no mid-point, since $\frac{1}{6}$ is not a point of the geometry.

Bachmann gives an example of a metric plane in which not every segment has a mid-point (**1**, p. 281) but this plane is not an ordered plane. The proof of 7.2 breaks down in this example because we cannot say that CD meets AB, so that R may not exist.

8. Congruent triangles. It is not difficult now to prove **C5** and **C7** in a plane, but we need some further theory to prove them in spaces of higher dimension. An important result in spaces of dimension greater than 2, due to Dorroh (**5**), is that if two rays are both perpendicular to a third, all the rays having a common origin, then the right-angles so formed are congruent. (See (**6**, p. 97 et seq.) for the definition of congruent angles.) Dorroh's proof is still valid using our axioms (we need 3.1) but we give an alternative proof in 8.4, without using the terminology of congruent angles.

We assume that we can construct, in spaces of dimension greater than 2, the plane through three given non-collinear points. We also assume that, in the course of a construction, we can decide whether or not a point lies in a given plane, and can make use of an arbitrary point, either in a given plane or not in it.

8.1. *If a line* l, *through a point* O *of a plane* π, *is perpendicular to two distinct lines* a *and* b *of* π *through* O, *then* l *is perpendicular to every line of* π *through* O.

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Proof (cf. **6**, p. 121) (Figure 8A). If c is any other line of π through O, there exist $A \in a$, $B \in b$, on opposite sides of c. Hence AB meets c at C, say, and A, B, C are all distinct from O. There exist $P, P' \in l$, distinct from O, such that O bisects PP'. Since a, b are perpendicular to l, we have $AP \equiv AP'$, $BP \equiv BP'$. Hence $\triangle ABP \equiv \triangle ABP'$. Hence $CP \equiv CP'$ (**C5a**) so $c \perp l$ (4.1).

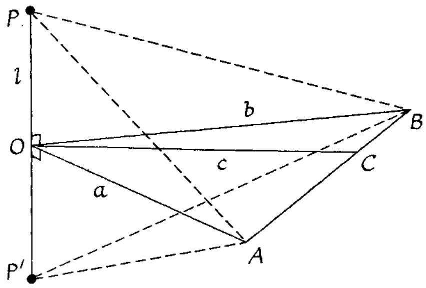

FIGURE 8A.

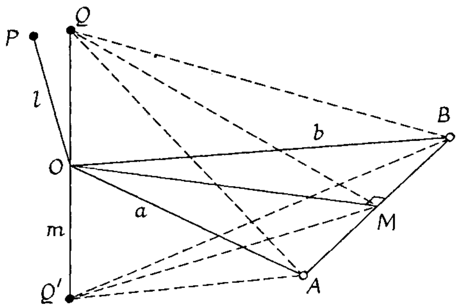

FIGURE 8B.

Definition. Under the conditions of 8.1, l and π are *perpendicular* to each other, and we write $l \perp \pi$ and $\pi \perp l$.

8.2. *If O is a given point on a line l in a 3-space $\mathscr{S}$, then the lines through O in $\mathscr{S}$ perpendicular to l all lie in a plane π, the unique plane through O in $\mathscr{S}$ perpendicular to l; π can be constructed.*

Proof. See (**6**, p. 122).

8.3. *Through a given point of a plane there exists, in any 3-space containing the plane, a unique line perpendicular to the plane; this line can be constructed.*

Proof. See (**6**, p. 123).

8.4. *If a, b are lines through a point O of a line l, both perpendicular to l, if $A \in a, B \in b$, and $OA \equiv OB$ (A, B distinct from O), and if $P \in l$, then $PA \equiv PB$.*

Proof (Figure 8B). The proof is trivial if $a = b$. Suppose then that $a \neq b$; AB has a mid-point $M \neq O$, and $OM \perp AB$ (5.2, 4.1). Let $\mathscr{S}$ be the 3-space defined by a, b, l. Let π be the unique plane through M in $\mathscr{S}$ perpendicular to AB; then MO lies in π (8.2). OA is not perpendicular to OM, so the plane through O perpendicular to OA in $\mathscr{S}$ is distinct from π and therefore meets π in a line m through O (**6**, p. 65); $m \perp a$. There exist $Q, Q' \in m$ such that O bisects QQ' and $OQ \equiv OQ' \equiv OP$. (We assume $P \neq O$; if $P = O$ there is nothing to prove.) Now $MQ, MQ' \subset \pi$, so $MQ \perp AB$, $MQ' \perp AB$ (8.1). Hence $QA \equiv QB$, $Q'A \equiv Q'B$. But $QA \equiv Q'A$ since $m \perp a$. Hence $QB \equiv Q'B$. Hence $m \perp b$ also. Hence $m = l$ (8.3, 8.1), so without loss of generality $P = Q$. Hence $PA \equiv PB$.

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8.5 (**C5**). *If $\triangle ABC \equiv \triangle A'B'C'$, if D, D' lie on rays $AB, A'B'$, and if $AD \equiv A'D'$, then $CD \equiv C'D'$.*

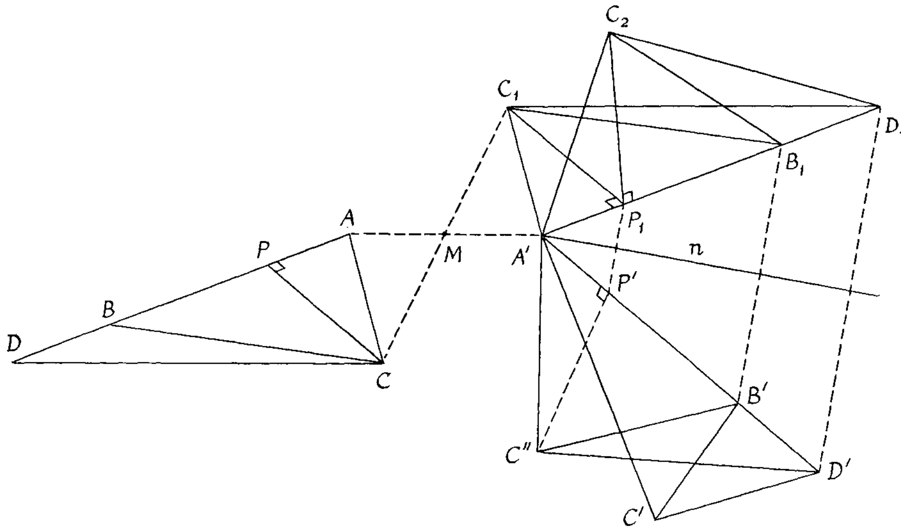

FIGURE 8C.

Proof (Figure 8C). Let CP be the perpendicular from C to AB, meeting AB at P (5.4, i). Let M denote the mid-point of AA' (7.4). Let the reflections of A, B, C, D, P in M be A', B_1, C_1 D_1, P_1. Then $AB \equiv A'B_1$, etc., D_1 lies on ray $A'B_1$, and $C_1P_1 \perp A'B_1$ (3.5, 3.4, 4.8).

If $B' \notin A'B_1$, let π denote the plane $A'B_1B'$. If $B' \in A'B_1$, let π denote any plane through $A'B_1$. On the line through P_1 in π perpendicular to $A'B_1$ (5.6) let C_2 be a point such that $P_1C_1 \equiv P_1C_2$. Then $A'C_1 \equiv A'C_2$, $B_1C_1 \equiv B_1C_2$, $D_1C_1 \equiv D_1C_2$ (8.4). If $A'B_1$, $A'B'$ are not opposite rays, let n be the bisector of the angle $B_1A'B'$; if these rays are opposite, let n be the line in π through A' perpendicular to $A'B_1$ (see the remarks on the proof of 6.5). Since $A'B' \equiv AB \equiv A'B_1$, $A'D' \equiv AD \equiv A'D_1$, and since D', D_1 lie on the rays $A'B'$, $A'B_1$, it follows that the reflections in n of A', B_1, D_1 are A', B', D'. Let the reflection in n of C_2 be C''. Then $\triangle A'B_1C_2 \equiv \triangle A'B'C''$ and $C_2D_1 \equiv C''D'$ (5.5 in the plane π).

Hence $\triangle A'B'C'' \equiv \triangle A'B_1C_2 \equiv \triangle A'B_1C_1 \equiv \triangle ABC \equiv \triangle A'B'C'$. Hence $C''D' \equiv C'D'$ (**C5a**). But $C''D' \equiv C_2D_1 \equiv C_1D_1 \equiv CD$. Hence $CD \equiv C'D'$.

8.6 (**C7**). *If ABC is a triangle, and if $AB \equiv A'B'$, then in any plane π' containing $A'B'$ there exist just two points C'_1, C'_2, one on either side of $A'B'$, such that $\triangle A'B'C'_1 \equiv \triangle A'B'C'_2 \equiv \triangle ABC$, and these points are constructible.*

Proof (Figure 8C). The points used in the proof of 8.5 are all constructible.

Construct the reflection P' of P_1 in n; $C''P' \perp A'B'$ (4.8). Construct the line through P' in π' perpendicular to $A'B'$, and on this line construct C'_1, C'_2 on opposite sides of P' such that $P'C'_1 \equiv P'C'_2 \equiv P'C''$. Then $A'C'_1 \equiv A'C'_2 \equiv A'C''$ and $B'C'_1 \equiv B'C'_2 \equiv B'C''$ (8.4.) Hence $\triangle A'B'C'_1 \equiv \triangle A'B'C'_2 \equiv \triangle A'B'C''$. But $\triangle A'B'C'' \equiv \triangle ABC$, as in the proof of 8.5. Hence the result; the uniqueness of C'_1, C'_2 follows from 3.3.

We have now shown that our axioms imply all the traditional axioms of congruence (3.3, 8.5, 8.6) with constructibility added to **C7**. Since we can construct the mid-point of any segment, it is easy to show that we can add constructibility to **C1**, in the sense that on any ray we can construct a segment congruent to a given segment.

Note added in proof. I am now able to show that axiom **C5c** is a consequence of the previous axioms. I hope to publish a further paper including proofs of this result and of the result mentioned in the footnote on p. 175.

References

1. F. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff* (Grundlehren der mathematischen Wissenschaften, 96; Berlin, 1959).

2. H. S. M. Coxeter, *Non-Euclidean geometry* (Toronto, 1957).

3. ——— *Introduction to geometry* (New York, 1961).

4. J. L. Dorroh, *Concerning a set of metrical hypotheses for geometry*, Ann. of Math. (2), *29* (1927), 229–231.

5. ——— *Concerning a set of axioms for the semi-quadratic geometry of a three-space*, Bull. Amer. Math. Soc., *36* (1930), 719–721.

6. H. G. Forder, *The foundations of Euclidean geometry* (London, 1927; New York, 1958).

7. ——— *On the axioms of congruence in semi-quadratic geometry*, J. London Math. Soc., *22* (1947), 268–275.

8. T. L. Heath, *The thirteen books of Eculid's elements*, Vol. I (Cambridge, 1908; New York, 1956).

9. D. Hilbert, *The foundations of geometry*, tr. E. J. Townsend (Chicago, 1910).

10. B. Kerékjártó, *Les fondaments de la geometrie*, Vol. I (Budapest, 1955).

11. R. L. Moore, *Sets of metrical hypotheses for geometry*, Trans. Amer. Math. Soc., *9* (1908), 487–512.

12. Z. Piesyk, *Uwagi o aksjomatyce geometrii Tarskiego*, Prace Mat., *11* (1965), 23–33.

13. G. de B. Robinson, *The foundations of geometry* (Toronto, 1940).

14. P. Szász, *On axioms of congruence due to H. G. Forder*, Monatsh. Math., *65* (1961), 270–276.

15. A. Tarski, *What is elementary geometry?* (*The axiomatic method, ed. L. Henkin, P. Suppes, A. Tarski*) (Amsterdam, 1959).

16. O. Veblen, *A system of axioms for geometry*, Trans, Amer. Math. Soc., *5* (1904), 343–384.

17. ——— *The foundations of geometry* (*Monographs on topics of modern mathematics, ed. J. W. Young, Chapter I*) (New York, 1911).

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