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THE OSCILLATORY BEHAVIOR OF A FIRST ORDER NON-LINEAR DIFFERENTIAL EQUATION WITH DELAY

BY

FORBES J. BURKOWSKI AND PETER J. PONZO(1)

SYNOPSIS. This paper establishes the existence of an infinite set $\{z_n\}_{n=1}^{\infty}$ of zeros for the solution of a certain functional differential equation. The primary condition assuring this oscillatory behavior is expressed in terms of the magnitude of the delay.

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The equation to be considered is

(1) $$x'(t)+F(t, x_t) = 0.$$

In conjunction with (1), it is assumed that we are given two functions $g(t)$ and $r(t)$ continuous on the real half line $[0, \infty)$, and such that

(2) $$g(t) \leq r(t) \leq t$$

for all $t>0$ the initial time. Both $g(t)$ and $r(t)$ are to be monotonically increasing, in fact, we assume the existence of $g^{-1}(t)$ and $r^{-1}(t)$ their respective inverse functions. Given a value t, it is to be considered that $g(t)$ represents the maximum retardation and $r(t)$ the minimum retardation associated with the delay equation (1). For each fixed $t>0$, the symbol x_t denotes a continuous function with domain $[-\infty, 0]$ such that its graph on $[g(t)-t, 0]$ coincides with the graph of $x(t)$ on the interval $[g(t), t]$. Hence $z_t \in C=C[-\infty, 0]$ the family of all curves continuous on the interval $[-\infty, 0]$ and thus F has as its domain the space $[0, \infty]\times C$. Due to the restrictions on $g(t)$ and $r(t)$, F effectively operates on a finite segment of the solution prior to t although this segment is not bounded in length for all t. We assume that the functional F is well enough behaved to guarantee the existence of a continuous solution for all $t>0$ when any continuous initial function is specified on the initial set $[g(0), 0]$. In addition, we assume the existence of a positive integrable function $h(t)$ and a time $T>0$ such that for all $t>T$ we have

(3) $$F(t, y_t) \geq h(t)y(r(t))$$

for any continuous $y(t)$ such that $y(t)$ is positive and monotone decreasing on the domain $[g(t), r(t)]$. Similarly,

(4) $$F(t, y_t) \leq h(t)y(r(t))$$

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for any continuous $y(t)$ such that $y(t)$ is negative and monotone increasing on $[g(t), r(t)]$. Finally,

$$F(t, y_t) = |F(t, y_t)|\,(s) \tag{5}$$

where (s) is $+1$ whenever $y(t)$ is positive on $[g(t), r(t)]$ and -1 whenever $y(t)$ is negative on $[g(t), r(t)]$.

THEOREM 1. *If the above conditions are satisfied and if*

$$\int_{r(t)}^{t} h(s)\,ds \geq 1 \tag{6}$$

for all large t, say $t \geq T$, then all solutions of (1) *are oscillatory.*

Proof. It can be demonstrated that for any $T_0 \geq T$, a zero of $x(t)$ must occur in the interval $(T_0, r^{-1}g^{-1}g^{-1}(T_0)]$. Let $T_1 = g^{-1}(T_0)$, $T_2 = g^{-1}(T_1)$ and $T_3 = r^{-1}(T_2)$. We obtain a proof by contradiction by assuming that $x(t) > 0$ for all $t \in (T_0, T_3]$ (a parallel demonstration holds for the case when $x(t) < 0$). This assumption implies that for $t \in (T_1, T_3]$, we have $x(t) > 0$ on the domain $[g(t), r(t)]$ and hence by (5) $x'(t) = -F(t, x_t) \leq 0$ indicating that $x(t)$ is monotone decreasing on $(T_1, T_3]$. Thus, for $t \in (T_2, T_3]$, $x(t)$ is monotone decreasing on the domain $[g(t), r(t)]$. Hence $t \in (T_2, T_3]$ implies

$$\dot{x}(t) = -F(t, x_t) \leq -h(t)x(r(t))$$

by (3) and thus

$$x(t) \leq x(T_2) - \int_{T_2}^{t} h(s)x(r(s))\,ds. \tag{7}$$

Now for $s \in (T_2, T_3]$, $r(s) \leq T_2$ and since $x(t)$ is monotone decreasing on $(T_1, T_3]$, we have $x(r(s)) \geq x(T_2)$ for $s \in (T_2, T_3]$. Hence

$$x(t) \leq x(T_2)\left\{1 - \int_{T_2}^{t} h(s)\,ds\right\}. \tag{8}$$

Setting $t = T_3$ in (8) and considering (6), one may obtain $x(T_3) \leq 0$ in contradiction of the fact that $x(t) > 0$ on $(T_0, T_3]$ and so the theorem is valid.

COROLLARY. *There exists a sequence of zeros of $x(t)$, $\{z_n\}_{n=0}^{\infty}$ which satisfies the recursive inequality $z_{n+1} \leq r^{-1}g^{-1}g^{-1}(z_n)$ for $z_0 \geq T$. It is possible that this set is part of a larger perhaps nondenumerable set of zeros.*

EXAMPLE. Consider

$$x'(t) + \sum_{i=1}^{n} h_i(t)x(g_i(t)) = 0 \tag{9}$$

where $h_i(t)$ is continuous and positive and $g_i(t)$ is a continuous monotone increasing retardation for any $1 \leq i \leq n$. Let us also assume there exists some $k > 0$

such that $g_i(t) \leq t-k$ for all $t \geq T$ and $i=1, 2, \ldots, n$. In this case, we may consider $r(t)=t-k$ and thus (3) and (4) are valid. Hence, if

$$\int_{t-k}^{t} \sum_{i=1}^{n} h_i(s)\, ds = \sum_{i=1}^{n} \int_{0}^{k} h_i(t-s)\, ds \geq 1 \tag{10}$$

for all t larger than some value T, then all solutions of (9) are oscillatory.

Oscillation theorems for linear differential-difference equations have also been presented by Lillo [1] and Myshkis [2]. In these cases, only one retardation was present and it was bounded. In [3] and [4], there are treatments of equations such as

$$x'(t)+A(t)x(g(t)) = 0$$

where $0 \leq g(t) \leq t$ and hence the initial data is a point. Under the assumption that solutions are oscillatory, various properties of the zeros are presented. In [5], equation (9) is studied with $n=1$ and the criterion expressed in (10) has been extended to accommodate the consideration of differential-difference equations of higher order.

As a final comment, we present the following result.

LEMMA. *If condition* (6) *is replaced by the condition*

$$\int^{\infty} h(s)\, ds = \infty \tag{11}$$

then nonoscillatory solutions of (1) *tend to zero as t approaches infinity.*

Proof. If $x(t)$ is eventually of constant sign, say $x(t)>0$ for all $t \geq T_0$, then we may derive as in Theorem 1 the inequality (7). Since $x(t)$ is decreasing beyond T_2, we may write $x(t) \leq X(T_2) - x(t) \int_{T_2}^{t} h(s)\, ds$ and hence

$$x(t) \leq \frac{x(T_2)}{1+\int_{T_2}^{t} h(s)\, ds}.$$

Thus, $\lim_{t\to\infty} x(t)=0$ as required.

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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA

DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO