Canad. Math. Bull. Vol. 62 (1), 2019 pp. 109–118
http://dx.doi.org/10.4153/CMB-2017-068-1

Irregular Weight One Points with D_4 Image

Hao Lee

Abstract. Darmon, Lauder, and Rotger conjectured that the relative tangent space of an eigencurve at a classical, ordinary, irregular weight one point is of dimension two. This space can be identified with the space of normalized overconvergent generalized eigenforms, whose Fourier coefficients can be conjecturally described explicitly in terms of p-adic logarithms of algebraic numbers. This article presents the proof of this conjecture in the case where the weight one point is the intersection of two Hida families of Hecke theta series.

1 Introduction

Fix a prime number p. Let $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ denote a fixed choice of algebraic closure of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively. Fix an embedding $i_p: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. This embedding induces an injection of the absolute Galois groups

$$G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

which identifies $G_{\mathbb{Q}_p}$ with the decomposition group of $\mathbb{Q}$ at the prime p.

Let g be a classical cuspidal newform of weight one, level N, and nebentypus character χ. Suppose g has q-expansion

$$g(q) = \sum_{n=1}^{\infty} a_n q^n.$$

Let

$$\rho_g: G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_p}(V_g) \cong GL_2(\overline{\mathbb{Q}}_p)$$

denote the associated odd, irreducible Artin representation given by Deligne and Serre [7]. Let α and β denote the roots of the p-th Hecke polynomial $x^2 - a_p x + \chi(p)$. The modular form g is said to be *regular* at p if the roots are distinct, and *irregular* at p if the roots are equal.

Assume that the level N is coprime to p. Then the p-stabilizations of g are defined to be

$$g_\alpha(z) = g(z) - \beta g(pz) \qquad \text{and} \qquad g_\beta(z) = g(z) - \alpha g(pz),$$

which have U_p-eigenvalues α and β, respectively. Since α and β are roots of unity, the p-stabilizations of g are ordinary at the prime p.

Let $S_1(Np, \chi)$ (resp. $S_1^{(p)}(N, \chi)$) denote the space of classical (resp. p-adic overconvergent) modular forms of weight one, level Np (resp. tame level N), nebentypus

Received by the editors September 21, 2017; revised October 30, 2017.
Published electronically January 23, 2018.
This work was partially supported by NSERC CGS-Master's Program.
AMS subject classification: **11F33**, 11F80.
Keywords: weight one points, irregular, dihedral image, generalized eigenform, eigencurve, tangent space.

character χ, and Fourier coefficients in $\overline{\mathbb{Q}}_p$. Let $\mathbb{T}$ be the Hecke algebra of level Np spanned over $\mathbb{Q}$ by the Hecke operators T_ℓ for primes $\ell \nmid Np$ and U_q for primes $q \mid Np$. Through the action of the Hecke operators on the space of modular forms, the p-stabilization g_α induces an algebra homomorphism $\lambda_{g_\alpha}: \mathbb{T} \to \overline{\mathbb{Q}}_p$. Let I_{g_α} be the kernel of this homomorphism. Let

$$S_1(Np,\chi)[g_\alpha] = S_1(Np,\chi)[I_{g_\alpha}]$$

be the space of I_{g_α}-torsion elements of $S_1(Np,\chi)$. Similarly, let

$$S_1^{(p)}(N,\chi)[\![g_\alpha]\!] = S_1^{(p)}(N,\chi)[I_{g_\alpha}^2]$$

be the space of $I_{g_\alpha}^2$-torsion elements of $S_1^{(p)}(N,\chi)$. Elements belonging to the latter space are called *overconvergent generalized eigenforms* attached to g_α.

If g is regular at p, the space $S_1(Np,\chi)[g_\alpha]$ is one dimensional and generated by g_α. If g is irregular, the space $S_1(Np,\chi)[g_\alpha]$ is two dimensional. Furthermore, the Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q \mid N$ act semi-simply and admit the decomposition

$$S_1(Np,\chi)[g_\alpha] = \overline{\mathbb{Q}}_p g_\alpha \oplus \overline{\mathbb{Q}}_p \widehat{g}, \tag{1.1}$$

where $\widehat{g}(q) = g(q^p)$. However, the action of U_p on this space is not semi-simple, as

$$U_p g_\alpha = \alpha g_\alpha \qquad \text{and} \qquad U_p \widehat{g} = g_\alpha + \alpha \widehat{g}.$$

The "natural" complement with respect to the natural inclusion of $S_1(Np,\chi)[g_\alpha]$ inside $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]$, consisting of non-classical overconvergent generalized eigenforms, is the main study of this paper.

Let $\mathcal{C}$ denote the eigencurve of level Np generated by the Hecke algebra $\mathbb{T}$. Bellaïche and Dimitrov [1] proved that if g is regular at the prime p, then the eigencurve $\mathcal{C}$ is smooth at g_α. Additionally, the weight map is étale if and only if g is not the theta series attached to a finite order character on a real quadratic field K in which p splits. In the latter case, Cho and Vatsal [2] showed that the weight map is not étale, and the space $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]$ is two dimensional. Moreover, there exists a unique non-classical overconvergent generalized eigenform whose first Fourier coefficient is zero (*normalized*). Darmon, Lauder and Rotger [4] were able to explicitly describe the Fourier coefficients of this overconvergent eigenform in terms of p-adic logarithms of algebraic numbers in a ring class field of K.

Darmon, Lauder and Rotger [3] further conjectured that in the case where g_α is irregular, the eigencurve $\mathcal{C}$ is not smooth at g_α. The dimension of the vector space of overconvergent generalized eigenforms is conjecturally four. By the decomposition given in equation (1.1), since

$$\big(a_1(g_\alpha), a_p(g_\alpha)\big) = (1,\alpha) \qquad \text{and} \qquad \big(a_1(\widehat{g}), a_p(\widehat{g})\big) = (0,1),$$

there is a natural linear complement of $S_1(Np,\chi)[g_\alpha]$ inside $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]$, consisting of eigenforms whose first and p-th Fourier coefficients are zero. Let $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ denote the space of such forms; its elements are said to be *normalized*. This space is conjecturally of dimension two. Similar to the regular setting, the authors conjectured a formula describing the Fourier coefficients of its elements in

https://doi.org/10.4153/CMB-2017-068-1 Published online by Cambridge University Press

terms of p-adic logarithms of algebraic numbers [3]. This explicit formula will be described in detail as Conjecture 3.2.

Suppose g is a Hecke theta series attached to a character χ defined over a quadratic imaginary field K. In this setting, Darmon, Lauder, and Rotger [3] gave a simplified expression of the Fourier coefficients of two canonical basis elements of $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$, which will be presented as Conjecture 3.3. The goal of this paper is to study a special setting where g_α is the intersection of two Hida families of Hecke theta series at weight one. A proof of Conjecture 3.3 in this scenario will be presented in Section 4. This result is relevant in the study of the generalization of Darmon, Lauder, and Rotger's proposed elliptic Stark conjecture [5] to the irregular case.

2 Preliminaries

This section will be devoted to recalling the necessary background required to present the conjectures of Darmon, Lauder, and Rotger [3] in the irregular case. Assume henceforth that g is irregular at the prime p. Since ρ_g is an Artin representation, it factors through a finite quotient $\mathrm{Gal}(L/\mathbb{Q})$ for some field extension L of K. Let $W_g = \mathrm{Ad}\,\rho_g = \mathrm{End}(V_g)$ denote the space of endomorphisms of V_g, also called the adjoint representation of ρ_g. Let $W_g^\circ = \mathrm{Ad}^\circ\rho_g$ denote the subspace of W_g consisting of trace zero endomorphisms. An element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ acts on an element $w \in W_g$ by conjugation in the following way:

$$\sigma \star w = \rho_g(\sigma) \circ w \circ \rho_g^{-1}(\sigma).$$

The adjoint representation also factors through a finite quotient $G = \mathrm{Gal}(H/\mathbb{Q})$, where H is a subfield of L.

There is a canonical exact sequence of G-modules

$$0 \longrightarrow W_g^\circ \longrightarrow W_g \longrightarrow \overline{\mathbb{Q}}_p \longrightarrow 0,$$

with a canonical G-equivariant splitting $\varphi\colon W_g \to W_g^\circ$ given by

$$\varphi(A) = A - \tfrac{1}{2}\,\mathrm{Tr}(A)\cdot I.$$

By enlarging L, we can assume that $\rho_g(L[G_\mathbb{Q}]) \cong M_2(L)$ in order to get a two dimensional L-vector space $\widetilde{V}_g$ and an identification $\iota\colon \widetilde{V}_g \otimes_L \overline{\mathbb{Q}}_p \to V_g$. Similarly, there are G-stable L-vector spaces

$$\widetilde{W}_g = \mathrm{Ad}\,\widetilde{V}_g \quad \text{and} \quad \widetilde{W}_g^\circ = \mathrm{Ad}^\circ\,\widetilde{V}_g$$

with identifications

$$\iota\colon \widetilde{W}_g \otimes_L \overline{\mathbb{Q}}_p \longrightarrow W_g \quad \text{and} \quad \iota\colon \widetilde{W}_g^\circ \otimes_L \overline{\mathbb{Q}}_p \longrightarrow W_g^\circ.$$

By the natural duality between eigenforms and Hecke algebras, there are identifications

$$S_1(Np,\chi)[g_\alpha] \cong \hom(\mathbb{T}/I_{g_\alpha}, \overline{\mathbb{Q}}_p) \quad \text{and} \quad S_1^{(p)}(N,\chi)[\![g_\alpha]\!] \cong \hom(\mathbb{T}/I_{g_\alpha}^2, \overline{\mathbb{Q}}_p).$$

There is also a natural exact sequence

$$0 \longrightarrow I_{g_\alpha}/I_{g_\alpha}^2 \longrightarrow \mathbb{T}/I_{g_\alpha}^2 \longrightarrow \mathbb{T}/I_{g_\alpha} \longrightarrow 0.$$

By applying the contravariant $\hom(\,\cdot\,,\overline{\mathbb{Q}}_p)$ functor on the above exact sequence, we obtain the following proposition.

Proposition 2.1 *There is a natural isomorphism of vector spaces*

$$S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0 \cong \hom(I_{g_\alpha}/I_{g_\alpha}^2, \overline{\mathbb{Q}}_p).$$

This proposition shows that the space $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ can be identified as the relative tangent space of the eigencurve at g_α. Following the general $R = \mathbb{T}$ philosophy, it should be expected that the space $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ is isomorphic to the set of isomorphism classes of p-adic deformations of ρ_g to the ring of dual numbers $\overline{\mathbb{Q}}_p[\epsilon]/(\epsilon^2)$ with constant determinant. By the works of Mazur [8], the space of such deformations can be characterized by the set of cocycles $H^1(G_{\mathbb{Q}}, W_g^\circ)$.

More specifically, for each $g' \in S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$, the form $\widetilde{g} = g_\alpha + \epsilon g'$ is an eigenform for $\mathbb{T}$ with values in $\overline{\mathbb{Q}}_p[\epsilon]/(\epsilon^2)$. It has an associated Galois representation

$$\rho_{\widetilde{g}}\colon G_{\mathbb{Q}} \longrightarrow GL_2\big(\overline{\mathbb{Q}}_p[\epsilon]/(\epsilon^2)\big)$$

satisfying $\rho_{\widetilde{g}} = \rho_g \mod \epsilon$ and $\det(\rho_{\widetilde{g}}) = \chi$. In other words, $\rho_{\widetilde{g}}$ is a deformation of ρ_g with constant determinant. Every such deformation $\rho_{\widetilde{g}}$ is of the form $(1+\epsilon\cdot c_{g'})\rho_g$ for some cocycle $c_{g'} \in H^1(G_{\mathbb{Q}}, W_g^\circ)$.

Conjecture 2.2 *The map $g' \mapsto c_{g'}$ is an isomorphism between $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ and $H^1(G_{\mathbb{Q}}, W_g^\circ)$.*

From a cocycle $c_{g'} \in H^1(G_{\mathbb{Q}}, W_g^\circ)$ obtained from some $g' \in S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$, its corresponding modular form g' can be recovered in the following canonical way. For every prime $\ell \nmid Np$, the ℓ-th Fourier coefficient g' is given by

$$a_\ell(g') = \mathrm{Tr}\left(c_{g'}(\sigma_\ell)\rho_g(\sigma_\ell)\right),$$

where σ_ℓ is a Frobenius element corresponding to the prime ℓ. Thus, the map $g' \mapsto c_{g'}$ is injective.

Remark 2.3 Conjecturally, this construction can be generalized to give an inverse map

$$H^1(G_{\mathbb{Q}}, W_g^\circ) \longrightarrow S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0.$$

However, this map may not be well defined. In particular, if we had started with an arbitrary cocycle $c \in H^1(G_{\mathbb{Q}}, W_g^\circ)$, it is difficult to justify that its corresponding formal q-series g' must be a modular form satisfying all the necessary properties to be an element of $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$.

Theorem 2.4 *The space $H^1(G_{\mathbb{Q}}, W_g^\circ)$ is two dimensional.*

https://doi.org/10.4153/CMB-2017-068-1 Published online by Cambridge University Press

Proof By the inflation-restriction exact sequence and the fact that G_H acts trivially on W_g°, there are isomorphisms

$$H^1(G_{\mathbb{Q}}, W_g^\circ) \overset{\mathrm{re}_{G_H}}{\cong} H^1(G_H, W_g^\circ) = \hom_G(G_H, W_g^\circ).$$

By class field theory, the space $H^1(G_{\mathbb{Q}}, W_g^\circ)$ can further be identified as

$$\hom_G(G_H, W_g^\circ) \cong \ker(\hom_G\big((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, W_g^\circ) \xrightarrow{\mathrm{re}} \hom_G(\mathcal{O}_H^\times \otimes \mathbb{Z}_p, W_g^\circ)\big).$$

The space $\hom_G((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \overline{\mathbb{Q}}_p)$ is spanned by morphisms of the form

$$\log_p\big((i_p \circ \sigma) \otimes 1\big)_{\sigma \in G},$$

where $\log_p: \overline{\mathbb{Q}}_p^\times \to \overline{\mathbb{Q}}_p$ is the standard p-adic logarithm sending p to 0 and i_p is the previously fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Therefore, there is another isomorphism

$$H^1(G_H, W_g^\circ) \cong \ker\big(\hom_G(H \otimes \mathbb{Q}_p, W_g^\circ) \xrightarrow{\mathrm{re}} \hom_G(U, W_g^\circ)\big), \tag{2.1}$$

where U is the image of $\mathcal{O}_H^\times \otimes \mathbb{Z}_p$ inside $H \otimes \mathbb{Q}_p$ under the p-adic logarithm. By Dirichlet's unit theorem,

$$H \otimes \mathbb{Q}_p \cong \mathrm{Ind}_1^G \mathbb{Q}_p \qquad \text{and} \qquad U \cong \mathrm{Ind}_{\langle c\rangle}^G \mathbb{Q}_p - 1_G,$$

where $c \in G$ is a complex conjugation and 1_G is the trivial representation. Since ρ_g is odd, the action of complex conjugation on W_g° has eigenvalues 1 and −1. By Frobenius reciprocity and the fact that every irreducible component of W_g° is non-trivial and self-dual,

$$\dim_{\overline{\mathbb{Q}}_p} \hom_G(H \otimes \mathbb{Q}_p, W_g^\circ) = 3 \qquad \text{and} \qquad \dim_{\overline{\mathbb{Q}}_p} \hom_G(U, W_g^\circ) = 1. \tag{2.2}$$

In summary,

$$\dim_{\overline{\mathbb{Q}}_p} H^1(G_{\mathbb{Q}}, W_g^\circ) = 2. \qquad \blacksquare$$

Corollary 2.5 *The dimension of the space $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ is at most two, and equality holds if and only if Conjecture 2.2 is true.*

Proof This follows from the injectivity of the map

$$S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0 \to H^1(G_{\mathbb{Q}}, W_g^\circ). \qquad \blacksquare$$

3 Main Conjecture

Since the eigenform g is irregular at p, the p-th Frobenius map acts as the identity map on W_g. Equivalently, the prime p splits completely in H. With the fixed embedding $H \hookrightarrow \overline{\mathbb{Q}}_p$ and a chosen prime $\mathfrak{p}_0$ of H above p, let $\log_{\mathfrak{p}_0}: H_p \to \mathbb{Q}_p$ denote the $\mathfrak{p}_0$-adic logarithm that factors through $\log_p$.

For all $\mathfrak{u} \in \mathcal{O}_H^\times$, $\omega \in \widetilde{W}_g^\circ$, define

$$\xi(\mathfrak{u}, \omega) = \frac{1}{|G|}\sum_{\sigma \in G} (\sigma \mathfrak{u}) \otimes (\sigma \star \omega) \in (\mathcal{O}_H^\times \otimes \widetilde{W}_g^\circ)^G.$$

https://doi.org/10.4153/CMB-2017-068-1 Published online by Cambridge University Press

By equation (2.2), the elements

$$\xi_{\mathfrak{p}_0}(\mathfrak{u},\omega) = (\log_{\mathfrak{p}_0} \otimes \mathrm{id})\xi(\mathfrak{u},\omega) = \frac{1}{|G|}\sum_{\sigma\in G} \log_{\mathfrak{p}_0}(\sigma\mathfrak{u})\cdot(\sigma\star\omega) \in W_g^\circ$$

span a 1-dimensional L-vector subspace of W_g°. Let $\omega(1)$ be any generator of this line.

For each prime $\ell \nmid Np$, choose a prime λ of H above ℓ. Suppose h is the class number of H. Let $\widetilde{\mathfrak{u}}_\lambda$ be a generator of the principal ideal λ^h, and let

$$\mathfrak{u}_\lambda = \widetilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in \left(\mathcal{O}_H[1/\ell]^\times\right) \otimes_{\mathbb{Z}} L.$$

Suppose $\sigma_\lambda \in G$ is a Frobenius map associated with λ. Set

$$\widetilde{\omega}_\lambda = \rho_g(\sigma_\lambda) \in \widetilde{W}_g \quad \text{and} \quad \omega_\lambda = \varphi(\widetilde{\omega}_\lambda) \in \widetilde{W}_g^\circ.$$

With this notation, denote

$$\xi(\mathfrak{u}_\lambda,\omega_\lambda) = \frac{1}{|G|}\sum_{\sigma\in G}(\sigma\mathfrak{u}_\lambda)\otimes(\sigma\star\omega_\lambda) \in (\mathcal{O}_H[1/\ell]^\times \otimes \widetilde{W}_g^\circ)^G,$$

$$\omega(\ell) = \xi_{\mathfrak{p}_0}(\mathfrak{u}_\lambda,\omega_\lambda) = \frac{1}{|G|}\sum_{\sigma\in G}\log_{\mathfrak{p}_0}(\sigma\mathfrak{u}_\lambda)\cdot(\sigma\star\omega_\lambda) \in W_g^\circ.$$

Finally, let

$$\mathfrak{M}(\ell) = \left[\,\omega(1),\omega(\ell)\right] = \omega(1)\omega(\ell) - \omega(\ell)\omega(1) \in W_g^\circ.$$

Since $\mathfrak{u}_\lambda$ is well defined up to some non-zero scalar multiple of $\mathcal{O}_H^\times$, the above elements are well defined up to an element of $(\mathcal{O}_H^\times \otimes W_g^\circ)^G$ and $L\cdot\omega(1)$, respectively. Additionally, the image of $\omega(\ell)$ inside $W_g/(L\cdot\omega(1))$ does not depend on the choice of λ above ℓ. Hence, $\mathfrak{M}(\ell)$ is independent of all the choices made to define it.

Proposition 3.1 ([3, Theorem 5.3]) *Fix $\omega \in W_g^\circ$. Define a G-equivariant linear map*

$$\varphi_w : H\otimes\mathbb{Q}_p \longrightarrow W_g^\circ$$

as follows. For each $h = (h_\lambda)_{\lambda|p} \in H\otimes\mathbb{Q}_p = \bigoplus_{\lambda|p}\mathbb{Q}_p$, define

$$\varphi_\omega(h) = \frac{1}{|G|}\sum_{\sigma\in G}(\sigma^{-1}(h))_{\mathfrak{p}_0}\cdot(\sigma\star\omega).$$

Then φ_ω is the zero map on $U = \log_p(\mathcal{O}_H^\times) \subseteq H\otimes\mathbb{Q}_p$ if and only if ω is orthogonal to the line spanned by $\omega(1)$ in W_g°. Hence, the map $\omega \mapsto \varphi_{[\omega,\omega(1)]}$ is an isomorphism between $W_g^\circ/(L\cdot\omega(1))$ and $H^1(G_{\mathbb{Q}}, W_g^\circ)$ via the identification given by equation (2.1).

Applying Proposition 3.1 to the conjectural inverse map described in Remark 2.3, along with some simple calculations, we can restate Conjecture 2.2 in the following equivalent way.

Conjecture 3.2 *For each $\omega \in W_g^\circ$, there exists an associated normalized, overconvergent, generalized eigenform $g'_\omega \in S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ whose coefficients satisfy*

$$a_\ell(g'_\omega) = \mathrm{Tr}\left(\,\omega\cdot\mathfrak{M}(\ell)\right)$$

for all primes $\ell \nmid Np$. Additionally, the map $\omega \mapsto g'_\omega$ induces an isomorphism between $W_g^\circ/(L\cdot\omega(1))$ and $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$.

https://doi.org/10.4153/CMB-2017-068-1 Published online by Cambridge University Press

In the case where g is a CM form, Darmon, Lauder, and Rotger made the observation that some specific choices of $\omega(1)$, u_λ, and basis elements $\omega_1, \omega_2 \in W_g^\circ/(L\cdot\omega(1))$ can be picked to simplify Conjecture 3.2. We will now briefly describe this result.

Suppose g is a theta series associated with a finite order Hecke character

$$\psi_g: G_K = \mathrm{Gal}(\overline{K}/K) \longrightarrow L^\times$$

defined over a quadratic imaginary field K over $\mathbb{Q}$. Fix an element $\tau \in G_\mathbb{Q}\backslash G_K$. Let ψ'_g be a Galois character defined by

$$\psi'_g(\sigma) = \psi_g(\tau\sigma\tau^{-1}) \qquad \text{for all } \sigma \in G_K.$$

As a G_K-representation, $\widetilde{V}_g$ is a direct sum of two one-dimensional representations where G_K acts via the characters ψ_g and ψ'_g. Let e_1 and e_2 be a fixed choice of eigenvectors associated wiht ψ_g and ψ'_g, respectively.

Let $\psi = \psi_g/\psi'_g$ and $G_0 = \mathrm{Gal}(H/K)$. Fix $\mathfrak{u} \in \mathcal{O}_H^\times$. Let

$$e_\psi = \frac{1}{|G_0|}\sum_{\sigma\in G_0}\psi^{-1}(\sigma)\sigma \in L[G_0]$$

and define

$$\mathfrak{u}_\psi = e_\psi(\mathfrak{u}) \qquad \text{and} \qquad \tau\mathfrak{u}_\psi = e_{\psi^{-1}}(\tau\mathfrak{u}).$$

With this notation, we can now state the simplification of Conjecture 3.2 to the CM case.

Conjecture 3.3 *There exists a canonical basis (g_1, g_2) of $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ whose q-series is of the following form.*

(i) *The Fourier coefficients $a_\ell(g_1)$ of g_1 are supported on primes $\ell \nmid Np$ that split in K. Suppose ℓ splits into $(\ell) = \lambda\lambda'$ in K. Let $\widetilde{\mathfrak{u}}_\lambda$ be a generator of λ^h where h is the class number of K, and let*

$$\mathfrak{u}_\lambda = \widetilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in (\mathcal{O}_K[1/\ell]^\times)\otimes L.$$

Then the coefficients are given by

$$a_\ell(g_1) = (\psi_g(\sigma_\lambda) - \psi_g(\sigma_{\lambda'}))\cdot \log_{\mathfrak{p}_0}\Big(\frac{\mathfrak{u}_\lambda}{\tau\mathfrak{u}_{\lambda'}}\Big),$$

where σ_λ and $\sigma_{\lambda'}$ are a choice of Frobenius maps corresponding to λ and λ', respectively.

(ii) *The Fourier coefficients $a_\ell(g_2)$ of g_2 are supported on primes $\ell \nmid Np$ that are inert in K. Choose a prime λ above ℓ in H and suppose σ_λ is a Frobenius element associated with λ. Choose $\mathfrak{u}_\lambda \in (\mathcal{O}_H[1/\ell]^\times)\otimes L$ in the same way as described earlier in this section. With respect to the previously chosen basis e_1 and e_2, suppose*

$$\rho_g(\sigma_\lambda) = \begin{pmatrix} 0 & b_\lambda \\ c_\lambda & 0 \end{pmatrix}$$

for some $b_\lambda, c_\lambda \in L$. Pick

$$\widehat{\mathfrak{u}}_\lambda \in (\mathcal{O}_H[1/\ell]^\times/\mathcal{O}_H^\times)\otimes L \cong \bigoplus_{\lambda|\ell} L\cdot\lambda$$

https://doi.org/10.4153/CMB-2017-068-1 Published online by Cambridge University Press

so that it has prime factorization $\widehat{\mathfrak{u}}_\lambda = b_\lambda \lambda + c_\lambda(\tau\lambda)$*. Let* $\mathfrak{u}_\psi(\ell) = e_\psi(\widehat{\mathfrak{u}}_\lambda)$*. Then the coefficients are given by*

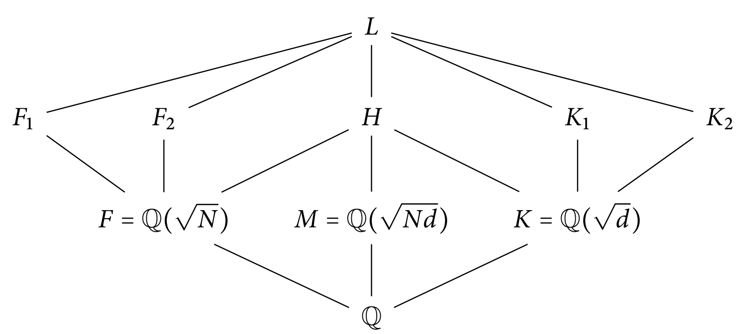
$$a_\ell(g_2) = \det\begin{pmatrix} \log_{\mathfrak{p}_0}(\mathfrak{u}_\psi) & \log_{\mathfrak{p}_0}(\mathfrak{u}_\psi(\ell)) \\ \log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\psi) & \log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\psi(\ell)) \end{pmatrix}.$$

4 Dihedral Image Conjecture

This section will be dedicated to describing a special scenario, where g is a CM form induced from two distinct imaginary quadratic fields, and the image of ρ_g is isomorphic to the dihedral group of order eight. A more detailed description of this setting along with various interesting properties can be found in [6]. The representation ρ_g is also known as a *Hecke–Shintani representation*, which is a two dimensional irreducible monomial Artin representation induced from multiple quadratic fields. Such representations play a significant role in Shintani's work on Stark's conjecture [10] and Rohrlich's study of almost abelian Artin representations of $\mathbb{Q}$ [9]. A proof of Conjecture 3.3 in this special setting will be presented at the end of the section.

By construction, g will be the intersection of two explicit Hida families of weight one. From these, we will be able to find two linearly independent generalized eigenforms that span $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$, which are the forms described in Conjecture 3.3. This method does not extend to the general dihedral group setting, where we have access to only one explicit Hida family. It is currently unknown how to directly show that the q-series defined in Conjecture 3.3 are modular forms with the desired properties.

Let $F = \mathbb{Q}(\sqrt{N})$ for some $N > 0$ be a real quadratic field. Let F_1 be an almost totally real extension of F. That is, F_1 is of the form $\mathbb{Q}(\sqrt{a - b\sqrt{N}})$ for some integers a, b such that $d = a^2 - Nb^2 < 0$. Since F_1 has two real places and one imaginary place, it is not Galois over $\mathbb{Q}$. Let L be its Galois closure. By construction, $L = F_1F_2$ where F_2 is the Galois conjugate of F_1 over $\mathbb{Q}$ and $\mathrm{Gal}(L/\mathbb{Q}) \cong D_4$ is the dihedral group of order eight. The following diagram summarizes all the subextensions of L over $\mathbb{Q}$, where every line indicates a field extension of degree two.



Let

$$\psi_K : G_K \longrightarrow \{\pm 1\} \qquad \text{and} \qquad \psi_M : G_M \longrightarrow \{\pm 1, \pm i\}$$

be the Galois characters of the quadratic imaginary fields K and M that cut out the extensions K_1 over K and L over M, respectively. Let θ_{ψ_K} denote the weight one Hecke theta series attached to ψ_K, and let $\rho_K = \operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \psi_K$ denote the associated Artin representation attached to θ_{ψ_K}. Adopt similar notation for the field M. Since D_4 has one unique irreducible two dimensional representation, $\rho_K \cong \rho_M$, which implies that $\theta_{\psi_K} = \theta_{\psi_M}$. In other words, there are two Hida families of theta series attached to the characters ψ_K and ψ_M that have the same specialization at weight one.

Remark 4.1 The fields are named to match previously used notations. That is, $G_L = \ker \rho_K$ and $G_H = \ker \operatorname{Ad}\rho_K$.

Let $g = \theta_{\psi_K}$. By the irregularity assumption, the prime p splits in H and hence splits in both K and M. Therefore, the Galois representations associated with the cuspidal family of ordinary theta series attached to ψ_K at weight k is given by

$$\operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \psi_K \nu_K^{k-1},$$

where ν_K is a canonical Hecke character. More specifically, as an idèle class character of K,

$$\nu_K((\alpha)) = \pm\alpha$$

for all $\alpha \equiv 1$ modulo the conductor of ν_K.

Consider k as a "weight variable" that can be manipulated analytically. Then the 1-cocycle $c \in H^1(G_\mathbb{Q}, W_g^\circ)$ corresponding to the p-adic deformation to weight $k = 1+\epsilon$ is given by

$$c(\sigma) = \varphi(\operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \log_{\mathfrak{p}_0} \nu_K)(\sigma) \in W_g^\circ,$$

where $\varphi\colon W_g \to W_g^\circ$ is the canonical splitting defined in Section 2. Therefore, the ℓ-th coefficient of the associated normalized generalized overconvergent eigenform is given by

$$\operatorname{Tr}\big((\rho_K \cdot \varphi(\operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \log_{\mathfrak{p}_0} \nu_K))(\sigma_\lambda)\big) \in \overline{\mathbb{Q}}_p,$$

where $\sigma_\lambda \in G_\mathbb{Q}$ is a Frobenius element of a choice of prime λ of K above ℓ. It is easy to see that the coefficients of this eigenform are supported on primes ℓ that split in K.

Suppose ℓ is a rational prime that splits into $(\ell) = \lambda\lambda'$ in K and let h be the class number of K. Suppose $\widetilde{\mathfrak{u}}_\lambda$ is a generator of λ^h. Then

$$\log_{\mathfrak{p}_0} \nu_K(\sigma_\lambda)^h = \log_{\mathfrak{p}_0} \nu_K((\widetilde{\mathfrak{u}}_\lambda)) = \log_{\mathfrak{p}_0} \widetilde{\mathfrak{u}}_\lambda,$$

which implies that

$$\log_{\mathfrak{p}_0} \nu_K(\sigma_\lambda) = h^{-1} \log_{\mathfrak{p}_0}(\widetilde{\mathfrak{u}}_\lambda) = \log_{\mathfrak{p}_0} \mathfrak{u}_\lambda,$$

where $\mathfrak{u}_\lambda = \widetilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in \mathcal{O}_K[1/\ell]^\times \otimes L$. Using this dictionary and the previously chosen basis, the ℓ-th coefficient can be simplified to

$$\operatorname{Tr}\left(\begin{pmatrix} \psi_K(\sigma_\lambda) & 0 \\ 0 & \psi_K(\sigma_{\lambda'}) \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} \log_{\mathfrak{p}_0}(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda) & 0 \\ 0 & \log_{\mathfrak{p}_0}(\mathfrak{u}'_\lambda/\mathfrak{u}_\lambda) \end{pmatrix}\right) = \frac{1}{2} \log_{\mathfrak{p}_0}(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda)\big(\psi_K(\sigma_\lambda) - \psi_K(\sigma_{\lambda'})\big).$$

This is exactly the eigenform described by Conjecture 3.3. Denote this normalized generalized eigenform by g_K, and the eigenform constructed from the theta series θ_{ψ_M} using the same method by g_M.

By construction, $H = KM$ is a biquadratic field. If ℓ splits completely in K and M, then ℓ splits in H and the Frobenius element is the identity for all primes above ℓ. Hence, both $a_\ell(g_K)$ and $a_\ell(g_M)$ are zero, because $\psi_K(\sigma_\lambda) = \psi_K(\sigma_{\lambda'})$ and similarly for M. Therefore, the Fourier coefficients of g_K are supported on the primes ℓ that are split in K, but are inert in M. For such a prime ℓ, we have $\psi_K(\sigma_\lambda) = -\psi_K(\sigma_{\lambda'})$, which shows that $a_\ell(g_K)$ is non-zero. This implies that g_K and g_M are linearly independent. By Corollary 2.5, they must form a basis of $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$. More importantly, we have proved Conjectures 2.2 and 3.3.

Acknowledgements I would like to thank my supervisor Professor Henri Darmon for introducing me to this problem and for providing me with some great insight and advice.

References

[1] J. Bellaïche and M. Dimitrov, *On the eigencurve at classical weight 1 points.* Duke Math. J. **165**(2016), no. 2, 245–266. http://dx.doi.org/10.1215/00127094-3165755

[2] S. Cho and V. Vatsal, *Deformations of induced Galois representations.* J. Reine Angew. Math. **556**(2003), 78–98. http://dx.doi.org/10.1515/crll.2003.025

[3] H. Darmon, A. Lauder, and V. Rotger, *First order p-adic deformations of weight one newforms.* https://mat-web.upc.edu/people/victor.rotger/docs/DLR4.pdf.

[4] ———, *Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields.* Adv. Math. **283**(2015), 130–142. http://dx.doi.org/10.1016/j.aim.2015.07.007

[5] ———, *Stark points and p-adic iterated integrals attached to modular forms of weight one.* Forum Math. Pi **3**(2015), e8, 95 pp. http://dx.doi.org/10.1017/fmp.2015.7

[6] H. Darmon, V. Rotger, and Y. Zhao, *The Birch and Swinnerton-Dyer conjecture for* $\mathbb{Q}$*-curves and Oda's period relations.* In: Geometry and analysis of automorphic forms of several variables, Ser. Number Theory Appl, 7, World Sci. Publ., Hackensack, NJ, 2012, pp. 1–40. http://dx.doi.org/10.1142/9789814355605_0001

[7] P. Deligne and J-P. Serre, *Formes modulaires de poids* 1. Ann. Sci. Éc. Norm. Sup. (4) **7**(1974), 507–530.

[8] B. Mazur, *Deforming Galois representations.* In: Galois Groups over $\mathbb{Q}$, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989, pp. 385–437. http://dx.doi.org/10.1007/978-1-4613-9649-9_7

[9] D. Rohrlich, *Almost abelian Artin representations of* $\mathbb{Q}$. http://math.bu.edu/people/rohrlich/aa.pdf

[10] T. Shintani, *On certain ray class invariants of real quadratic fields.* J. Math. Soc. Japan **30**(1978), no. 1, 139–167. http://dx.doi.org/10.2969/jmsj/03010139

Department of Mathematics and Statistics, McGill University, Montreal, Quebec
e-mail: hao.lee@mail.mcgill.ca