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THE TOPOLOGICAL CENTERS OF $LUC(S)^*$ AND $M_a(S)^{**}$ OF CERTAIN FOUNDATION SEMIGROUPS

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Abstract. In the present paper, for a large family of topological semigroups namely, compactly cancellative and right cancellative foundation semigroups S, we study the topological centers of the Banach algebras $LUC(S)^*$ and $M_a(S)^{**}$. We also give a generalization of a known result of Lau and Lorsert by showing that for such topological semigroups the topological center of $LUC(S)^*$ ($M_a(S)^{**}$, respectively) is the same as $M(S)$ ($M_a(S)$, respectively).

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In recent years there has been shown considerable interest by harmonic analysts in the topological center problem of $LUC(G)^*$ and $L^1(G)^{**}$ of a locally compact group G.

In [**18**] Zappa proved that, for $G = \mathbb{R}$, $\tilde{Z} = M(\mathbb{R})$ (the measure algebra of $\mathbb{R}$), where $\tilde{Z}$ denote the topological center $LUC(G)^*$. This result was extended to abelian locally campact groups by Grosser and Losert in [**6**], and to all locally compact groups by Lau in [**11**]. In [**14**] for a certain discrete group G, Parsons proved that $Z_1 = \ell^1(G)$, where Z_1 denotes the topological center of $\ell^1(G)^{**}$. In [**10**] Isik, Pym and Ülger proved that for any compact group G the topological center of $L^1(G)^{**}$ is the same as $L^1(G)$. This result was generalized to all locally compact groups by Lau and Losert in [**12**] and again through a different proof by Lau and Ülger in [**13**].

It seems to the author that the topological center problem of corresponding algebras of topological semigroups has not been touched so far. It is the aim of this paper to generalize these results to an extensive class of topological semigroups namely, compactly cancellative and right cancellative foundation semigroups, for which topological groups and cancellative discrete semigroups are elementary examples.

1. Notation and preliminaries. In this section we have collected some notation and results which are needed for the subsequent sections. For any Banach algebra A with a bounded approximate identity we denote by A^* and A^{**} its first dual and second dual, respectively. The *first Arens multiplication* on A^{**} is defined in three steps as follows. For a, b in A, f in A^* and m, n in A^{**}, the elements $f.a$, $m.f$ of A^* and $m.n$ of A^{**} are defined by

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$$\langle f.a,\ b\rangle = \langle f,\ ab\rangle, \langle m.f,\ a\rangle = \langle m,\ f.a\rangle, \langle m.n,\ f\rangle = \langle m, n.f\rangle.$$

The basic propeties of this multiplication are as follows. For a fixed n in A^{**}, the mapping $m \mapsto m.n$ is weak*–weak*-continuous. For fixed m in A^{**}, the mapping $n \mapsto m.n$ is in general not weak*–weak* continuous unless m is in A, and so Z the topological center of A^{**} with respect to this multiplication, is defined as follows.

$$Z_1 = \left\{m \in A^{**} : \text{ the mapping } n \mapsto m.n \text{ is weak}^*\text{–weak}^* \text{ continuous on } A^{**}\right\}.$$

The *second Arens multiplication* is defined as follows. For a, b in A, f in A^* and m, n in A^{**}, the elements $a\Delta f, f\Delta m$ of A^* and $m\Delta n$ of A^{**} are defined by the equations

$$\langle b,\ a\Delta f\rangle = \langle ba,\ f\rangle, \langle a,\ f\Delta m\rangle = \langle a\Delta f,\ m\rangle, \langle f,\ m\Delta n\rangle = \langle f\Delta m,\ n\rangle.$$

For m fixed in A^{**}, the mapping $n \mapsto m\Delta n$ is weak*–weak*-continuous on A^{**}. For n fixed in A^{**}, the mapping $m \mapsto m\Delta n$ is in general not weak*–weak* continuous on A^{**} unless n is in A. Hence the topological center of A^{**} with respect to this multiplication is defined as follows.

$$Z_2 = \left\{n \in A^{**} : \text{ The mapping } m \mapsto m\Delta n \text{ is weak}^*\text{–weak}^*\text{-continuous on } A^{**}\right\}.$$

We note that with either multiplications A^{**} defines a Banach algebra [**1**]. Furthermore for a in A and $m \in A^{**}$, $a.m. = a\Delta m$ and $m.a. = m\Delta a$. It is clear that $A \subseteq Z_1 \cap Z_2$ and Z_i $(i = 1, 2)$ is a closed subalgebra of the Banach algebra A^{**} endowed with the first (second) Arens multiplication.

It is also easy to see that

$$Z_1 = \{n \in A^{**} : m.n = m\Delta n \quad \forall n \in A^{**}\}$$

and

$$Z_2 = \{m \in A^{**} : n.m = n\Delta m \quad \forall n \in A^{**}\}.$$

An element E of A^{**} is said to be a *mixed unit* if $m.E = E\Delta m = m$, for all m in A^{**}. Note that E in A^{**} is a mixed unit if and only if it is a weak* cluster point of some bounded approximate identity in A; see [**3**: p. 146].

We also define the subspaces A^*A and $A\Delta A^*$ of A^* as follows:

$$A^*A = \left\{f.a : f \in A^* \text{ and } a \in A\right\},$$

$$A\Delta A^* - \left\{a\Delta f : f \in A^* \text{ and } a \in A\right\}.$$

Analoguous to A^{**}, the topological center of $(A^*A)^*$ is defined as follows.

$$\tilde{Z}_1 = \left\{\mu \in (A^*A)^*\text{: The mapping } n \mapsto \mu.n \text{ is weak}^*\text{–weak}^* \text{ continuous on } (A^*A)^*\right\}.$$

Convention. Throughout the paper, A^{**} will denote the Banach algebra of A^{**} equipped with the first Arens multiplication.

Recall also that on a locally compact Hausdorff and jointly continuous topological semigroup S, $M_a(S)$ (or $\tilde{L}(S)$) [**2**], [**5**], [**7**] denotes the space of all measures

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$\mu \in M(S)$ (the space of all bounded complex Radon measures on S) for which the mappings $x \mapsto |\mu| * \delta_x$ (where δ_x denotes the Dirac measure at x) and $x \mapsto \delta_x * |\mu|$ from S into $M(S)$ are weakly continuous. It is well known that $M_a(S)$ is a closed two-sided L-ideal of $M(S)$. A topological semigroup S is called a *foundation semigroup* if $\cup\{\text{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S. We also note that if S is a foundation semigroup with identity, then $M_a(S)$ has a bounded approximate identity and, for every μ in $M_a(S)$, both mappings $x \mapsto |\mu| * \delta_x$ and $x \mapsto \delta_x * |\mu|$ from S into $M_a(S)$ are norm continuous; (see [**5**]).

Let $LUC(S)$ denote the space of bounded left uniformly continuous complex-valued functions on S; i.e. all f in $C_b(S)$ (the space of complex-valued bounded continuous functions on S with the supremum norm) such that the map $x \mapsto \ell_x f$ of S into $C_b(S)$ is continuous when $C_b(S)$ has the sup-norm topology, where $(\ell_x f)(y) = f(xy)(x, y \in S)$. Then $LUC(S)$ is a closed subalgebra of $C_b(S)$ invariant under translation. The space of bounded right uniformly continuous functions $RUC(S)$ on S is defined similarly. It should be noted that in standard books on harmonic analysis in the case where S is a locally compact group, the space of bounded right uniformly continuous complex-valued functions on S is denoted by $LUC(S)$. It is well known that if S is a foundation semigroup with identity, then for $A = M_a(S)$, $A^*.A = LUC(S)$ and $A \Delta A^* = RUC(S)$; (see Lemma 2.1 of [**9**]). When S is a foundation semigroup with identity we shall identify $M_a(S)^*$ with $L^\infty(S, M_a(S))$, the Banach space of all bounded complex-valued μ-measurable ($\mu \in M_a(S)$) functions on S with the sup-norm, via the identification: $f \mapsto \tau_f$ where

$$\tau_f(\mu) = \int_S f(x)d\mu(x)(f \in L^\infty(S, M_a(S)), \mu \in M_a(S));$$

(see Proposition 3.6 of [**16**]). Note that if we denote again τ_f by f, then two functions f and g are identical in $L^\infty(S, M_a(S))$ whenever $f(x) = g(x)$ a.e. μ for every μ in $M_a(S)$. Since, by Lemma 2.5 of [**2**], we have

$$f(\mu * \nu) = \int_S f(\mu * \delta_x)d\nu(x) \text{ and } f(\nu * \mu) = \int f(\delta_x * \mu)d\nu(x),$$

for every $f \in L^\infty(S, M_a(S))$ and $\nu \in M(S)$, it follows that

$$\begin{aligned} \langle f.\mu, \nu\rangle = f(\mu * \nu) &= \int_S \int_S f(xy)d\mu(x)d\nu(y) \\ &= \int_S \int_S f(xy)d\nu(y)d\mu(x), \end{aligned} \tag{1}$$

for every $\mu, \nu \in M_a(S)$. Similarly

$$\begin{aligned} \langle \mu\Delta f, \nu\rangle = f(\nu * \mu) &= \int_S \int_S f(xy)d\nu(x)d\mu(y) = \int_S \int_S f(xy)d\nu(y)d\mu \\ &= \int_S \int_S f(xy)d\mu(y)d\nu(x)(\mu, \nu \in M_a(S)). \end{aligned} \tag{2}$$

If we consider $M_a(S) \subseteq Z_1$, then for every $\mu, \nu \in M_a(S)$ and $f \in L^\infty(S, M_a, (S))$, $\langle \mu.f, \nu\rangle = \langle \mu, f.\nu\rangle = f(\nu * \mu)$. Therefore, $\mu\Delta f = \mu.f$ and moreover from (1) and (2) it

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follows that if both μ and f have compact supports, then the supports of $f.\mu$ and $\mu\Delta f(=\mu.f)$ are compact.

A topological semigroup S is called a $*$-*semigroup* if there exists a continuous map $*: S \to S$ such that $x^{**} = x$ and $(xy)^* = y^*x^*$ for all x, y in S. Finally, a topological semigroup S is called *compactly cancellative* if CD^{-1} and $C^{-1}D$ are compact subsets of S for every two compact subsets C and D of S, where

$$CD^{-1} = \{z \in S : zd \in C \text{ for some } d \text{ in } D\},$$
$$C^{-1}D = \{z \in S : cz \in D \text{ for some } c \text{ in } D\}.$$

The set $C\{y\}^{-1}$ ($y \in S$) will be denoted by Cy^{-1}.

Throughout the rest of this paper we shall denote the topological center of $M_a(S)^{**}$ *and* $LUC(S)^*$ *by* Z_1 *and* $\tilde{Z}_1$, *respectively.*

2. The topological center of $LUC(S)^*$. The aim of the present section is to extend a result of Lau [**11**] from locally compact groups to compactly cancellative and right cancellative foundation semigroups with identity by proving that for such semigroups the topological center of $LUC(S)^*$ is the same as $M(S)$.

It should be noted that Lau's proof for locally compact groups depends heavily on the existence of the inverse operation on groups.

Our starting point of this section is the following lemma whose proof is straightforward.

Lemma 2.1. *Let S be a foundation semigroup with identity such that $C^{-1}D$ is a compact subset of S for every two compact subsets C and D* of *S. Let f* in *$L^\infty(S, M_a(S))$ and μ* in *$M_a(S)$ both have compact supports. Then the support of $f.\mu$ is compact.*

Notation. We denote by $L_0^\infty(S, M_a(S))$ the sup-norm closure of the space of all functions in $L^\infty(S, M_a(S))$ with compact support. We also denote by $M_a^K(S)$ the space of all measures in $M_a(S)$ with compact support.

The proof of the following lemma is omitted, since it is straightforward.

Lemma 2.2. *Let S be a foundation semigroup with identity e. Let U_0 be a fixed neighbourhood of e with compact closure and let Λ denote the set of all neighbourhoods of e contained in U_0. Suppose that Λ is directed downwards and, for each $\lambda \in \Lambda$, μ_λ is chosen so that $\mu_\lambda \geq 0$, $\|\mu_\lambda\| = 1$* and *$\mu_\lambda(S \setminus \lambda) = 0$. Let g in $L^\infty(S, M_a(S))$ be continuous at e. Then* $\lim_\lambda \langle \mu_\lambda, g\rangle = g(e)$.

Remark 2.3. The net (μ_λ) in the statement of the preceding lemma defines a bounded approximate identity for $M_a(S)$ (see, Proposition 5.16 of [**15**]).

Lemma 2.4. *Let S be a foundation semigroup with identity and let $m \in Z_1$. Then there exists a net (μ_a)* in *$M_a^K(S)$ such that for every f in $L^\infty(S, M_a(S))$,* $\|f\Delta\mu_\alpha - f\Delta m\|_\infty \to 0$. In particular, $f\Delta m$ is in $LUC(S)$ and $\langle x, f\Delta m\rangle = \langle m, r_x f\rangle$, *for every $x \in S$.*

Proof. Let C denote the convex set of all $v \in M_a^K(S)$ with $\|v\| \leq \|m\|$. By Goldstine's theorem and the norm density of $M_a^K(S)$ in $M_a(S)$, there exists a net (v_β) in C

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that converges to m in the weak*-topology of $M_a(S)^{**}$. For every n in $M_a(S)^{**}$ and f in $L^\infty(S, M_a(S))$, since $M_a(S) \subseteq Z_1$, we have

$$\begin{aligned}\lim_\beta \langle n, f\Delta v_\beta\rangle &= \lim_\beta \langle v_\beta \Delta n, f\rangle = \lim_\beta \langle v_\beta . n, f\rangle \\ &= \lim_\beta \langle v_\beta, n.f\rangle = \langle m, n.f\rangle \\ &= \langle m.n, f\rangle = \langle m\Delta n, f\rangle \\ &= \langle n, f\Delta m\rangle .\end{aligned}$$

Now, for every $m' \in L^\infty(S, M_a(S))^*$ we define $\bar{m}' : L^\infty(S, M_a(S)) \to L^\infty(S, M_a(S))$ by $\bar{m}'(f) = f\Delta m'$. Then $\bar{m}'$ is in $\mathcal{B}(L^\infty(S, M_a(S)))$ (the space of bounded linear operators on $L^\infty(S, M_a(S))$ with $\|\bar{m}'\| \le \|m\|$. From the above equalities it follows that $\bar{v}_\beta \to \bar{m}$ in the weak operator topology of $\mathcal{B}(L^\infty(S, M_a(S)))$. Since $\tilde{C} = \{\tilde{v} : v \in C\}$ is also convex, from Corollary 5 on page 477 of [**4**] it follows that $\tilde{m}$ is in the closure of $\tilde{C}$ with respect to the strong operator topology. Hence there is a net (μ_α) in C such that $\|f\Delta\mu_\alpha - f\Delta m\|_\infty \to 0$. Since for every $\mu, v \in M_a(S), \mu\Delta v = \mu.v$, it follows that $f\Delta\mu = f.\mu$, for every $f \in L^\infty(S, M_a(S))$. Therefore $f\Delta m \in LUC(S)$, by what was mentioned in the preliminaries. Let (μ'_λ) be a net as in the statement of Lemma 2.2. Then (μ'_λ) converges to a right identity E of $M_a(S)^{**}$ in the weak*-topology, so that

$$\begin{aligned}\langle e, f\Delta m\rangle &= \lim_\lambda \langle \mu'_\lambda, f\Delta m\rangle = \lim_\lambda \langle m\Delta\mu'_\lambda, f\rangle \\ &= \lim_\lambda \langle m.u'_\lambda, f\rangle = \langle m.E, f\rangle \\ &= \langle m, f\rangle .\end{aligned}$$

By a similar argument one can easily prove that $\langle x, f\Delta m\rangle = \langle m, r_x f\rangle (x \in S)$.

Lemma 2.5. Let S *be a compactly cancellative foundation semigroup with identity. Then* $Z_1 \cap C_0(S)^\perp \subseteq L_0^\infty(S, M_a(S))^\perp$.

Proof. Let $m \in Z_1 \cap C_0(S)^\perp$. To show that $m \in L_0^\infty(S, M_a(S))^\perp$ we only need to prove that $m(f) = 0$, for every f in $L^\infty(S, M_a(S))$ with compact support. Fix such an f in $L^\infty(S, M_a(S))$. Then, as in the proof of the preceeding lemma,

$$\begin{aligned}\langle m, f\rangle = \langle e, f\Delta m\rangle &= \lim_\lambda \langle \mu'_\lambda, f\Delta m\rangle == \lim_\lambda \langle m\Delta\mu'_\lambda, f\rangle \\ &= \lim_\lambda \langle m.\mu'_\lambda, f\rangle = \lim_\lambda \langle m, \mu'_\lambda . f\rangle = 0,\end{aligned}$$

since $\mathrm{supp}(\mu'_\lambda . f)$ is compact and $\mu'_\lambda . f = \mu'_\lambda \Delta f \in RUC(S)$. □

The proof of the next result is similar to that of Lemma 1 of [**11**] and therefore it is omitted.

Lemma 2.6. *Let S be a non-compact locally compact semigroup such that $C^{-1}D$ and Dy^{-1} are compact subsets of S for every two compact subsets C, D* of *S and y* in *S. Then there is a net $\{K_a : \alpha \in I\}$ of compact subsets of S which is closed under the formation of finite unions of its members and such that $S = \bigcup_{\alpha\in I} K_\alpha^0$ (K_α^0 denotes the*

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interior of K_α). Furthermore, there are two nets $(y_\alpha)_{\alpha\in I}$ and $(z_\alpha)_{\alpha\in I}$ in S such that the families $\{K_\alpha y_\alpha : \alpha \in I\}$ and $\{K_\alpha z_\alpha : \alpha \in I\}$ are pairwise disjoint.

REMARK. The following result is an analogue of Theorem 1 of Lau [**11**]. It should be noted that Lau's proof is not valid for weakly cancellative discrete semigroups as claimed, since the two functions f' and f'' in his proof are not well-defined for such semigroups.

THEOREM 2.7. *Let S be a right cancellative foundation semigroup with identity such that $C^{-1}D$ and Cy^{-1} are compact subsets of S, for any $y \in S$ and any two compact subsets C and D of S. Then $Z_1 \cap C_0(S)^\perp = \{0\}$.*

Proof. Let $m \in Z_1 \cap C_0(S)^\perp$. If $m \neq 0$, then we may assume that $\|m\| = 1$. Suppose $0 < \varepsilon < \frac{1}{6}$ is given. Choose $f \in L^\infty(S, M_a(S))$ such that $\|f\|_\infty = 1$ and $|\langle m, f\rangle| > 1 - \varepsilon$. Choose the family $\{K_\alpha : \alpha \in I\}$ of compact subsets of S, the nets $(y_\alpha)_{\alpha\in I}$ and $(z_\alpha)_{\alpha\in I}$ as in the statement of Lemma 2.6. Define the two functions f' and f'' on S by $f'(xy_\alpha) = f''(xz_\alpha) = f(x)$ if $x \in K_\alpha$ and zero otherwise. Since the families $\{K_\alpha y_\alpha : \alpha \in I\}$ and $\{K_\alpha z_\alpha : \alpha \in I\}$ are pairwise disjoint and S is also right cancellative, we infer that f' and f'' are well defined. By Lemma 2.4 there exists a measure μ in $M_a^K(S)$ such that $\|\mu\| \leq 1$ with $\|f\Delta\mu - f\Delta m\|_\infty < \varepsilon$, $\|f'\Delta\mu - f'\Delta m\|_\infty < \varepsilon$, and $\|f''\Delta\mu - f''\Delta m\|_\infty < \varepsilon$. Thus there exists $\alpha_0 \in I$ such that $\mathrm{supp}(\mu) \subseteq K_{\alpha_0}$. Let $g' = r_{y_{\alpha_0}} f'$ and $g'' - r_{y_{\alpha_0}} f''$. Since $g'(x) = f'(xy_{\alpha_0}) = f(x)$, for all $x \in K_{\alpha_0}$, we have

$$|\langle f'\Delta\mu,\ y_{\alpha_0}\rangle - \langle f'\Delta m,\ y_{\alpha_0}\rangle| = |\langle \mu, f\rangle - \langle m, g'\rangle| > \varepsilon.$$

Also, since

$$\begin{aligned}|\langle m, f\rangle; -\langle \mu, f\rangle;| &= |\langle f\Delta m,\ e\rangle - \langle f\Delta\mu,\ e\rangle| \\ &\leq \|f\Delta m - f\Delta\mu\|_\infty < \varepsilon,\end{aligned}$$

it follows that $|\langle \mu, f\rangle| > 1 - 2\varepsilon$. Consequently $|m(g')| > 1 - 3\varepsilon$. Similarly $|m(g'')| > 1 - 3\varepsilon$.

Let $K = (K_{\alpha_0} y_{\alpha_0}) y_{\alpha_0}^{-1} \bigcup (K_{\alpha_0} z_{\alpha_0}) z_{\alpha_0}^{-1}$. Then K is compact. Since the support of $g'g''$ is contained in K, from Lemma 2.5 it follows that $m(g'g'') = 0$. Hence

$$\langle |g'(1 - g'')|, |m|\rangle \geq |\langle g'(1 - g''), m\rangle| = |m(g')| > 1 - 3\varepsilon,$$

and

$$\langle |g''(1 - g')|, |m|\rangle \geq |\langle g''(1 - g'), m\rangle| = |m(g'')| > 1 - 3\varepsilon;$$

(see [**17** p. 40]). Adding the above equations we obtain

$$\langle |g' + g''||1 - g'g''|, |m|\rangle > 2 - 6\varepsilon > 1,$$

since $0 < \varepsilon < \frac{1}{6}$. But $\| |g' + g''||1 - g'g''| \| \leq 1$. This contradicts the assumption $\|m\| = \||m|\| = 1$. □

REMARK. We remark that the hypothesis of the above theorem does not force S to be a sub-semigroup of any group. To see this, let $S' = \{1, 2, \ldots, n\}(n \in \mathbb{N})$. Define

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the multiplication on S' by $1k = k1 = k$ for every $k \in S'$ and $k\ell = k$ for $k \neq 1$ and $\ell \neq 1$. Let G be any locally compact group. Then $S = S' \times G$ with the product topology and coordinatewise multiplication defines a foundation semigroup (see [**5**, p. 43] that satisfies the hypothesis of the preceding theorem. It is also clear that S is not a subset of any group.

We also note that if S is a right cancellative foundation semigroup, then the conclusion of the above theorem is not valid. To see this, let S be an infinite set, for a fixed element e in S we define $ex = xe = x$, for all $x \in S$, and for all $x, y \in S\backslash\{e\}$ we define $xy = x$. Then S defines a right cancellative foundation semigroup with identity for which $C^{-1}D$ and Cy^{-1} are not in general compact subsets of S for every two compact subsets C, D of S and $y \in S$. But it is clear that $C_0(S)^{\perp}$ is a non-zero subspace of Z_1.

The following is the main result of this paper.

THEOREM 2.8. *Let S be a right cancellative foundation semigroup with identity such that, for every two compact subsets C and D of S and $y \in S$, $C^{-1}D$* and *Cy^{-1} are* compact. Then $\tilde{Z}(LUC(S)^*) = M(S)$.

Proof. It is clear that $M(S) \subseteq \tilde{Z}$, by Lemma 3 of [**12**]. If S is compact, then $M(S) = LUC(S)^*$, and thus $\tilde{Z} = M(S)$. Hence we may assume that S is not compact. By Theorem 2 of [**8**], $LUC(S)^* = M(S) \oplus C_0(S)^{\perp}$ and $C_0(S)^{\perp}$ is also a left ideal in $LUC(S)^*$. Let $m \in \tilde{Z}$. Then $m = \mu + m_1$ for some $\mu \in M(S)$ and $m_1 \in C_0(S)^{\perp}$. Since $C_0(S)^{\perp}$ is a left ideal in $LUC(S)^*$, it follow that $v.m_1$ is $C_0(S)^{\perp}$, for every $v \in M_a(S)$. On the other hand $v.m_1 \in Z_1(M_a(S)^{**})$, by part (c) of Lemma 3.1 of [**13**]. Thus $v.m_1 \in C_0(S)^{\perp} \cap Z_1(M_a(S)^{**})$. Therefore $v.m_1 = 0$, by Theorem 2.6. Hence $\langle v.m_1, h\rangle = 0$, for all $h \in L^{\infty}(S, M_a(S))$ and $v \in M_a(S)$. Let $f \in LUC(S)$. Then $f = h.v$, for some $h \in L^{\infty}(S, M_a(S))$ and $v \in M_a(S)$. Hence

$$\begin{aligned}\langle m_1, f\rangle; &= \langle m_1,\ h.v\rangle = \langle m_1,\ h\Delta v\rangle \\ &= \langle v\Delta m_1,\ h\rangle = \langle v.m_1,\ h\rangle = 0\end{aligned}$$

and so $m_1 = 0$. This completes the proof.

3. The topological center of $M_a(S)^{**}$ of certain foundation $*$-semigroups. In the present section we shall generalize Lau's result of the topological center $L^1(G)^{**}$ of locally compact groups G to cancellative foundation $*$-semigroups S with identity for which $C^{-1}D$ and Cy^{-1} are compact for every two compact subsets C and D of S and $y \in S$. We have concluded this section with an example of such a $*$-semigroup which is not a subsemigroup of any locally group.

We start with the following result which is a generalization of Lemma 2.3 of [**10**].

LEMMA 3.4. *Let S be a foundation semigroup with identity. Let $f \in L^{\infty}(S, M_a(S))$ be such that for every two pair $(e_i), (e'_j)$ of positive bounded approximate identities in $M_a(S)$, $\lim_i\langle f, e_i\rangle = \lim_j\langle f, e'_j\rangle$. Then f is identical to a function g in $L^{\infty}(S, M_a(S))$ that is continuous at the identity.*

Proof. Without loss of generality we may assume that f is real. Suppose f is not (apart from the zero function in $L^{\infty}(S, M_a(S))$ continuous at e. Then every neigh-

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bourhood V of e contains two sets V' and V'' which are not $M_a(S)$ negligible (i.e. there exist μ' and μ'' in $M_a(S)$ such that $\mu'(V') \neq 0$ and $\mu''(V'') \neq 0$) with $f \geq 1$ on V' and $f \leq 1$ on V''. Choose probability measures $\mu_{V'}$ and $\mu_{V''}$ in $M_a(S)$ such that $\mu_{V'}(S \backslash V') = 0$ and $\mu_{V''}(S \backslash V'') = 0$. By Proposition 4.16 of [**15**] each of the nets $(\mu_{V'})$ and $(\mu_{V''})$ defines a bounded approximate identity, whenever each of the collections of the sets V''s and V'''s is directed downwards. Therefore $(\liminf)_{V'} \langle f, \mu_{V'} \rangle \geq 1$ and $(\limsup)_{V'} \langle f, \mu_{V''} \rangle \leq 0$. This contradicts the hypothesis.

As an application of the above lemma, by a method similar to that of Theorem 5.4 of [**13**] one can easily obtain the following generalization of that theorem. The details are omitted.

THEOREM 3.2. *Let S be a foundation semigroup with identity and let $A = M_a(S)$. Then for m in A^{**}, the following are equivalent.*

(a) *m is in A.*

(b) (i) *$Am \subseteq A$. (ii) For each $E.m = m$. (iii) For each f in A^*, $m.f$ is in AA^*.*

An argument similar to the proof of Corollary 5.5 of [**13**] with the aid of Theorem 2.7 and Theorem 2.8 gives the following generalization of that corollary.

COROLLARY 3.2. *Let S be a cancellative foundation *-semigroup with identity such that $C^{-1}D$ is a compact subset of S for every two compact subsets C and D of S.* Then *$Z(M_a(S)^{**}) = M_a(S)$.*

REMARK. *Let S* be the set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with a, b, c in $\{0, 1, 2, \ldots\}$ and $a \neq 0, c \neq 0$. Then with the usual multiplication, the involution $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^* = \begin{bmatrix} c & b \\ 0 & a \end{bmatrix}$ and the discrete topology, S defines a non-commutative cancellative foundation $*$-semigroup with the identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that $C^{-1}D$ is a compact subset of S for every two compact subsets C and D of S. It is clear that with this involution S is not a subsemigroup of any group G such that $x^* = x^{-1}$ for every $x \in S$, where x^{-1} denotes the inverse of x in G. Note that if G is any locally compact group, then $S \times G$ (with the product topology and the involution $(s, g)^* = (s^*, g^{-1})(s \in S, g \in G)$) also satisfies the hypothesis of the preceding theorem.

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