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Universal Power Series in $\mathbb{C}^N$

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Abstract. We establish the existence of power series in $\mathbb{C}^N$ with the property that the subsequences of the sequence of partial sums uniformly approach any holomorphic function on any well chosen compact subset outside the set of convergence of the series. We also show that, in a certain sense, most series enjoy this property.

1 Introduction

This paper is separated into two main sections. The first one deals with the existence of universal power series in several complex variables that converge on a given set, and are universal away from it. Our result is inspired by the following theorem, stated here in its improved form due to Nestoridis (see [5]), but first proved independently by Chui and Parnes (see [1]) and Luh (see [4]). Denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$ and, given a subset $E \subset \mathbb{C}$, denote by $A(E)$ the set of functions that are holomorphic on the interior of E and continuous on the boundary of E.

Theorem 1.1 *There exists a power series S with center at the origin and radius of convergence one with the property that for every compact subset $K \subset (\mathbb{C} \setminus \mathbb{D})$ with connected complement and every function $f \in A(K)$, there is a subsequence of the sequence of partial sums of S that converges to f uniformly on K.*

The second section investigates the size of the set of universal series. To this end, we use abstract characterizations of universality.

2 Existence of Universal Power Series

First recall that a set of the form $\{z \in \mathbb{C}^N : |p_1(z)| \leq 1, \dots, |p_m(z)| \leq 1\}$, where all the p_i are polynomials, is called a *polynomial polyhedron*. It is trivial to check that polynomial polyhedra are polynomially convex. The next lemma will allow us to restrict our attention to polynomial polyhedra most of the time that we are dealing with polynomially convex compact subsets of $\mathbb{C}^N$. See [3].

Lemma 2.1 *Let $K \subset \mathbb{C}^N$ be a polynomially convex compact subset and $U \subset \mathbb{C}^N$, an open neighbourhood of K. Then, there exist polynomials $p_1, \dots, p_m$ with rational coefficients (rational real and imaginary parts) such that*

$$K \subset \{z \in \mathbb{C}^N : |p_1(z)| \leq 1, \dots, |p_m(z)| \leq 1\} \subset U.$$

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In the same vein, a subset $X \subset \mathbb{C}^N$ such that there exist polynomials $p_1, \dots, p_J$ with the property that $X = \{z \in \mathbb{C}^N : p_j(z) = 0 \; \forall j = 1, \dots, J\}$ is said to be an *algebraic set*. We shall use the term *algebraic hypersurface* to distinguish the case where X is the zero set of a single polynomial. Note that every algebraic set can be written as a finite intersection of algebraic hypersurfaces: $X = \bigcap_{j=1}^{J} H_j$. The following lemma is the basic tool of this section.

Lemma 2.2 *Let $K \subset \mathbb{C}^N$ be a polynomially convex compact subset such that there exists an algebraic hypersurface H containing the origin that is disjoint from K. Then, given $k \in \mathbb{N}$ and $f \in \mathcal{O}(K)$, one can find a polynomial $r(z)$ with the following properties:*

- *the monomial of lowest degree in $r(z)$ is of degree at least k;*
- *$r(z)$ vanishes on H;*
- $\sup_K |r(z) - f(z)| < 2^{-(k+1)}$.

Proof Set $H = \{z \in \mathbb{C}^N : p(z) = 0\}$. Since K is disjoint from H, we have that for every $c \in \mathbb{C}$,

$$\frac{f(z) - c(p(z))^k}{(p(z))^{k+1}} \in \mathcal{O}(K).$$

Set $M = \sup_K |p(z)| > 0$. Then, by the Oka–Weil Theorem, there exists a polynomial q such that

$$\sup_K \Big| q(z) - \frac{f(z) - c(p(z))^k}{(p(z))^{k+1}} \Big| \le \frac{1}{(2M)^{k+1}}.$$

Define $r(z) = c(p(z))^k + (p(z))^{k+1} q(z)$, which obviously vanishes on H and also satisfies the first property, since p has zero constant coefficient. Finally, we have

$$\begin{aligned}\sup_K |r(z) - f(z)| &= \sup_K |c(p(z))^k + (p(z))^{k+1} q(z) - f(z)| \\ &\le \sup_K \big| (p(z))^{k+1} \big| \frac{1}{(2M)^{k+1}} \le \frac{1}{2^{k+1}}. \qquad \blacksquare\end{aligned}$$

With this lemma at our disposal, it is possible to construct universal power series that converge on prescribed algebraic sets. We shall denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$ and, given a multi-index $\nu \in \mathbb{N}_0^N$, let $|\nu| = \nu_1 + \cdots + \nu_N$ be its length.

Theorem 2.3 *Let $X = \bigcap_{j=1}^{J} H_j \subset \mathbb{C}^N$ be an algebraic set containing the origin. Then there exists a series $\sum_{\nu \in \mathbb{N}_0^N} a_\nu z^\nu$ converging on X with the property that given a polynomially convex compact subset $K \subset \mathbb{C}^N$ disjoint from at least one of the H_j and given a function $f \in \mathcal{O}(K)$, there exists a sequence $\{d_s\}_s \subset \mathbb{N}$ such that $\sum_{|\nu| \le d_s} a_\nu z^\nu \to f$ uniformly on K as $s \to \infty$.*

Proof Consider $\{K_m^j\}_m$ an enumeration of the polynomial polyhedra that are disjoint from H_j and defined by polynomials with rational coefficients, along with $\{\theta_i\}$ an enumeration of the polynomials with rational coefficients. Then, choose $\{(g_n, L_n)\}_n$ an enumeration of the pairs (θ_i, K_m^j), where $j = 1, \dots, J$ and $i, m \in \mathbb{N}$, and such that each pair appears infinitely many times. Set

$$\sigma_1(z) = g_1(z) = \sum_{|\nu| \le d_1} a_\nu z^\nu,$$

which is trivially such that $\sup_{L_1} |\sigma_1 - g_1| < 1$. By choosing $k = d_1 + 1$ in Lemma 2.2, we get

$$\sigma_2(z) = \sum_{d_1+1 \le |\nu| \le d_2} a_\nu z^\nu$$

vanishing on X such that $\sup_{L_2} |\sigma_2 - (g_2 - \sigma_1)| < 1/2$. Similarly, one can find

$$\sigma_n(z) = \sum_{d_{n-1}+1 \le |\nu| \le d_n} a_\nu z^\nu$$

vanishing on X such that $\sup_{L_n} |\sigma_n - (g_n - \sigma_1 - \cdots - \sigma_{n-1})| < 1/n$.

We shall show that the series $\sum_{n=1}^\infty \sigma_n(z)$ has all the required properties. First notice that for $n \ge 2$, every σ_n vanishes on X, hence the series converges everywhere on X. In order to show the approximation property, let $K \subset \mathbb{C}^N \setminus H_{j_0}$ be a polynomially convex compact subset and $f \in \mathcal{O}(K)$. By the Oka–Weil Theorem, there exists a sequence of polynomials $\{q_r\}$ such that $\sup_K |f - q_r| < 1/r$. Moreover, there exists a subsequence $\{\theta_{i_r}\}$ such that $\sup_K |\theta_{i_r} - q_r| < 1/r$. On the other hand, since $f \in \mathcal{O}(K)$, there exists an open set U such that $K \subset U \subset \mathbb{C}^N \setminus H_{j_0}$ and $f \in \mathcal{O}(U)$. By Lemma 2.1, there is $m_0 \in \mathbb{N}$ such that $K \subset K_{m_0}^{j_0} \subset U$ and so $f \in \mathcal{O}(K_{m_0}^{j_0})$. Now, the pair $(\theta_{i_r}, K_{m_0}^{j_0})$ appears infinitely many times in $\{(g_n, L_n)\}$, therefore $g_{n_1} = \cdots = g_{n_s} = \cdots = \theta_{i_r}$ and $L_{n_1} = \cdots = L_{n_s} = \cdots = K_{m_0}^{j_0}$. Hence, we have that

$$\sup_{K_{m_0}^{j_0}} \Big| \sum_{n=1}^{n_s} \sigma_n - \theta_{i_r} \Big| = \sup_{L_{n_s}} \Big| \sum_{n=1}^{n_s} \sigma_n - g_{n_s} \Big| \le 1/n_s$$

and as such,

$$\begin{aligned} \sup_K \Big| \sum_{n=1}^{n_s} \sigma_n - f \Big| &\le \sup_K \Big| \sum_{n=1}^{n_s} \sigma_n - \theta_{i_r} \Big| + \sup_K |\theta_{i_r} - q_r| + \sup_K |q_r - f| \\ &\le \sup_{K_{m_0}^{j_0}} \Big| \sum_{n=1}^{n_s} \sigma_n - \theta_{i_r} \Big| + 1/r + 1/r \le 1/n_s + 2/r \to 0 \end{aligned}$$

as $s, r \to \infty$.

It is now clear that $\sum_{n=1}^\infty \sigma_n(z)$ diverges outside of X. Indeed, if $z \notin X$, there exists $1 \le j_0 \le J$ such that $\{z\} \subset \mathbb{C}^N \setminus H_{j_0}$. Since $\{z\}$ is a polynomially convex compact set, there exist two subsequences of the sequence of partial sums of $\sum_{n=1}^\infty \sigma_n(z)$ that do not have the same limit on $\{z\}$. ∎

Before moving on to prove the other important result of this section, we need a property of polynomially convex compact subsets of $\mathbb{C}^N$.

Lemma 2.4 *Let $K \subset \mathbb{C}^N \setminus \{0\}$ be a polynomially convex compact subset. Then there exists an algebraic hypersurface $H_K = \{z \in \mathbb{C}^N : q(z) = 0\}$ containing the origin and disjoint from K, where q is a polynomial with rational coefficients.*

Proof Since $0 \notin K$ and K is polynomially convex, there is a polynomial p such that $|p(0)| > \sup_K |p|$. Set $\epsilon = |p(0)| - \sup_K |p| > 0$ and $L = K \cup \{0\}$. Now it is easy to find a polynomial p_1 with rational coefficients such that $\sup_L |p_1 - p| < \epsilon/2$. But then we have that $|p_1(0)| > \sup_K |p_1|$ by the choice of ϵ. Write $p_1(z) = p_1(0) + q(z)$ with $q(0) = 0$ and set $H_K := \{z \in \mathbb{C}^N : q(z) = 0\}$. Notice that $p_1|_{H_K} = p_1(0)$ and so the previous inequality shows that H_K is disjoint from K. It is clear that $0 \in H_K$, and the proof is complete. ∎

The next theorem is very similar to a theorem of Seleznev that covers the case where the radius of convergence is zero in Theorem 1.1 (see [7]). Its proof is almost identical to that of Theorem 2.3, so we shorten it quite a bit.

Theorem 2.5 *There exists a series $\sum_{\nu \in \mathbb{N}_0^N} a_\nu z^\nu$ with the property that given a polynomially convex compact subset $K \subset \mathbb{C}^N$ that is disjoint from the origin and given a function $f \in \mathcal{O}(K)$, there exists a sequence $\{d_s\}_s \subset \mathbb{N}$ such that $\sum_{|\nu| \le d_s} a_\nu z^\nu \to f$ uniformly on K as $s \to \infty$.*

Proof Consider $\{A_l\}_l$ an enumeration of the algebraic hypersurfaces in $\mathbb{C}^N$ containing the origin that are zero sets of polynomials with rational coefficients. Also consider $\{K_k^l\}_k$ an enumeration of the polynomial polyhedra that are disjoint from A_l and defined by polynomials with rational coefficients, and finally consider $\{\theta_i\}$ an enumeration of the polynomials with rational coefficients. Choose $\{(g_n, L_n)\}_n$ an enumeration of the pairs (θ_i, K_k^l), where $i, k, l \in \mathbb{N}$ and such that each pair appears infinitely many times. As before, we obtain $\sigma_n(z) = \sum_{d_{n-1} \le |\nu| \le d_n} a_\nu z^\nu$ such that $\sup_{L_n} |\sigma_n - (g_n - \sigma_1 - \cdots - \sigma_{n-1})| < 1/n$ and $\sigma_n(0) = 0$ for $n \ge 2$.

Now, consider the series $\sum_{n=1}^\infty \sigma_n(z)$ that converges at $z = 0$. In order to show the approximation property, let $K \subset \mathbb{C}^N$ be a polynomially convex compact subset disjoint from the origin and $f \in \mathcal{O}(K)$. Using Lemma 2.4, one finds A_{l_0} such that $K \subset \mathbb{C}^N \setminus A_{l_0}$ and then proceeds as in the proof of Theorem 2.3 to show that

$$\sup_K \left| \sum_{n=1}^{n_s} \sigma_n - f \right| \to 0$$

as $s \to \infty$.

If $z \neq 0$, then the singleton $\{z\}$ is a polynomially convex compact subset of $\mathbb{C}^N$ disjoint from the origin, and there exist two subsequences of the sequence of partial sums of $\sum_{n=1}^\infty \sigma_n(z)$ that do not have the same limit on $\{z\}$. Thus, the series diverges outside the origin. ∎

Note that aside from establishing the existence of universal power series, the proofs of the preceding theorems give us relative control over which powers of $z_1, \dots, z_N$ appear in such series.

3 How Many Such Series are There?

It is now natural to ask ourselves just how large the set of universal series is. As is often the case with such phenomena, the answer is: very large! We shall now proceed to

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show this in two different settings: first by considering series as sequences of complex numbers, and then by identifying them with families of complex numbers indexed by multi-indices.

Let X be a complex vector space endowed with a metric that is compatible with the vector space operations and invariant under translation. Given a sequence $x = \{x_k\}_k \subset X$, define $U(x)$ to be the set of sequences of complex numbers $\{a_k\}_k$ such that the set of partial sums $\{\sum_{k=1}^{n} a_k x_k, n \in \mathbb{N}\}$ is dense in X. Endow $\mathbb{C}^{\mathbb{N}}$, the set of all complex sequences, with the product topology. We shall first present an abstract characterization of universality (see [2, Proposition 7] along with [6, Theorem 1.2]).

Theorem 3.1 *The following assertions are equivalent:*

- $U(x) \neq \varnothing$;
- *span*$\{x_n, x_{n+1}, \dots\}$ *is dense in* X *for all* $n \in \mathbb{N}$;
- $U(x)$ *is a dense* G_δ *set in* $\mathbb{C}^{\mathbb{N}}$ *and contains a dense subspace of* $\mathbb{C}^{\mathbb{N}}$, *except for the zero sequence.*

Our strategy here is to cautiously choose the vector space X to which we apply this theorem. To this end, first note that the series of Theorems 2.3 and 2.5 are completely and uniquely determined by the choice of the numbers $\{a_\nu\}$. Now, in order to use Theorem 3.1, pick an enumeration $\{\nu_\lambda\}_\lambda$ of the set of multi-indices $\nu \in \mathbb{N}_0^N$ with the following property: for all $d \in \mathbb{N}_0$, there exists $d' \in \mathbb{N}$ such that $\{\nu : |\nu| \leq d\} = \{\nu_\lambda\}_{\lambda=1}^{d'}$. Such an enumeration is easily seen to exist.

Let U be the set of sequences $\{a_\lambda\}_\lambda$ with the property that given a polynomially convex compact subset $K \subset \mathbb{C}^N$ disjoint from the origin and given a function $f \in \mathcal{O}(K)$, there exists a sequence $\{d_s\} \subset \mathbb{N}$ such that $\sum_{\lambda=1}^{d_s} a_\lambda z^{\nu_\lambda} \to f$ uniformly on K as $s \to \infty$. Note that for every sequence $\{a_\lambda\} \in U$, the associated series $\sum_{\lambda=1}^{\infty} a_\lambda z^{\nu_\lambda}$ obviously converges at the origin.

Theorem 3.2 *U is a dense G_δ in $\mathbb{C}^{\mathbb{N}}$.*

Proof As in the proof of Theorem 2.5, let $\{A_l\}_l$ be an enumeration of the algebraic hypersurfaces in $\mathbb{C}^N$ containing the origin and that are zero sets of polynomials with rational coefficients, and let $\{K_k^l\}_k$ be an enumeration of the polynomial polyhedra that are disjoint from A_l and defined by polynomials with rational coefficients. Define $U_{l,k}$ to be the subset of $\mathbb{C}^{\mathbb{N}}$ of sequences $\{b_\lambda\}_\lambda$ such that $\{\sum_{\lambda=1}^{n} b_\lambda z^{\nu_\lambda} : n \in \mathbb{N}\}$ is dense in $\mathcal{O}(K_k^l)$. By Theorem 3.1 applied to $X = \mathcal{O}(K_k^l)$ and $x = \{z^{\nu_\lambda}\}_\lambda$, we get that $U_{l,k}$ is a dense G_δ set in $\mathbb{C}^{\mathbb{N}}$, since it is non-empty by Theorem 2.5 and by the choice of the enumeration $\{\nu_\lambda\}$. The result will thus follow from the fact that $U = \bigcap_{l,k} U_{l,k}$, since $\mathbb{C}^{\mathbb{N}}$ is a Baire space.

Let $\{a_\lambda\}_\lambda \in \bigcap_{l,k} U_{l,k}$ and consider a polynomially convex compact subset $K \subset \mathbb{C}^N \setminus \{0\}$, along with $f \in \mathcal{O}(K)$. Just like we did in the proof of Theorem 2.5, we invoke Lemmas 2.1 and 2.4 to find K_k^l such that $K \subset K_k^l$ and $f \in \mathcal{O}(K_k^l)$. But then we know that there exists a sequence $\{d_s\}$ in $\mathbb{N}$ such that

$$\sup_K \left| \sum_{\lambda=1}^{d_s} a_\lambda z^{\nu_\lambda} - f \right| \leq \sup_{K_k^l} \left| \sum_{\lambda=1}^{d_s} a_\lambda z^{\nu_\lambda} - f \right| \to 0$$

as $s \to \infty$. Hence, $\{a_\lambda\}_\lambda \in U$ and the proof is complete, since the reverse inclusion is obvious. ■

Although this is all well and good, we notice that unlike the series obtained in Theorems 2.3 and 2.5, we currently have no control regarding the length of the multi-indices that appear in the series associated with the sequences of U. This arises because the correspondence between series of the form $\sum_{|\nu|\leq d} b_\nu z^\nu$ and those of the form $\sum_{\lambda=1}^{d'} a_\lambda z^{\nu_\lambda}$ is not a bijection. To recover this control, it is necessary to use a slightly modified version of Theorem 3.1, which is better suited for our needs.

Consider the vector space $\prod_{\nu\in\mathbb{N}_0^N} \mathbb{C}_\nu = \{\{a_\nu\}_{\nu\in\mathbb{N}_0^N} : a_\nu \in \mathbb{C}\,\forall\nu\}$, which may be endowed with the metric

$$\rho(a,b) = \sum_{j=0}^{\infty} \frac{1}{N_j 2^j} \sum_{|\nu|=j} \frac{|a_\nu - b_\nu|}{1+|a_\nu - b_\nu|},$$

where N_j is the number of multi-indices of length j. We obtain a complete vector space, and thus a Baire space. If X is a vector space equipped with a translation invariant metric which is compatible with the operations and if $x = \{x_\nu\} \in \prod_\nu X_\nu$, define $\tilde{U}(x)$ to be the set of families of complex numbers $\{a_\nu\} \in \prod_\nu \mathbb{C}_\nu$ such that $\{\sum_{|\nu|\leq n} a_\nu x_\nu, n \in \mathbb{N}\}$ is dense in X. By using essentially the same arguments as in the proof of [6, Theorem 1.2], one obtains the following.

Theorem 3.3 *$\tilde{U}(x) \neq \varnothing$ if and only if $\tilde{U}(x)$ is a dense G_δ set in $\prod_\nu \mathbb{C}_\nu$.*

It is now an easy matter to apply this to our case and obtain the desired result. Let $\tilde{U}$ be the set of families of complex numbers $\{a_\nu\}_\nu$ with the property that given a polynomially convex compact subset $K \subset \mathbb{C}^N$ that is disjoint from the origin and given a function $f \in \mathcal{O}(K)$, there exists a sequence $\{d_s\} \subset \mathbb{N}$ such that $\sum_{|\nu|\leq d_s} a_\nu z^\nu \to f$ uniformly on K as $s \to \infty$. By adapting the proof of Theorem 3.2 and by using the aforementioned theorem, we get the following.

Theorem 3.4 *$\tilde{U}$ is a dense G_δ set in $\prod_\nu \mathbb{C}_\nu$.*

As a closing remark, let us mention that it may seem as if we only did half of the job. Indeed, we showed that most series share the universal property of Theorem 2.5, but what about the series with properties of Theorem 2.3? However, one must take into consideration the fact that the approach used here is not very convergence-friendly. When forming the set U or $\tilde{U}$, one cannot ensure convergence anywhere except at the origin.

Hope may lie in the following direction. By setting

$$A = \Big\{ \{a_\lambda\} \in \mathbb{C}^{\mathbb{N}} : \sum_{\lambda=1}^{\infty} a_\lambda z^{\nu_\lambda} \text{ converges on } X \Big\},$$

one could try to apply a variant of Theorem 3.1 (in fact, this is the original statement of [6, Theorem 1.2]) to this vector subspace. In order to do so though, one needs to find a metric with respect to which A is complete. Unfortunately, the natural metric induced by that of $\mathbb{C}^{\mathbb{N}}$ does not work.

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References

[1] C. K. Chui and M. N. Parnes, *Approximation by overconvergence of a power series.* J. Math. Anal. Appl. **36**(1971), 693–696. doi:10.1016/0022-247X(71)90049-7

[2] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators.* Bull. Amer. Math. Soc. (N. S.) **36**(1999), no. 3, 345–381. doi:10.1090/S0273-0979-99-00788-0

[3] L. Hörmander, *An introduction to complex analysis in several variables.* D. Van Nostrand Co., Inc., Princeton-Toronto-London, 1966.

[4] W. Luh, *Approximation analytischer Funktionen durch überconvergente Potenzreiken und deren Matrix-Transformierten.* Mitt. Math. Sem. Giessen **88**(1970), 1–56.

[5] V. Nestoridis, *Universal Taylor series.* Ann. Inst. Fourier (Grenoble) **46**(1996), no. 5, 1293–1306.

[6] V. Nestoridis and C. Papadimitropoulos, *Abstract theory of universal series and an application to Dirichlet series.* C. R. Math. Acad. Sci. Paris **341**(2005), no. 9, 539–543.

[7] A. I. Seleznev, *On universal power series.* (Russian), Mat. Sbornik N. S. **28(70)**(1951), 453–460.

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