APPLICATION OF BAECKLUND TRANSFORMATIONS TO THE STOKES-BELTRAMI EQUATIONS

C. ROGERS and J. G. KINGSTON

(Received 2 June 1970; revised 11 June 1971)

Communicated by W. W. Wood

1. Introduction

The Stokes-Beltrami equations, being applicable to many classes of physical problems, have for a long time received a great deal of attention from authors investigating their various aspects. One may cite, for example, Weinstein [1] and more recently Ranger [2] as just two of many research papers into this topic.

Baecklund transformations have, in previous work, been applied to hodograph-type equations (Loewner [3], Power, Rogers and Osborn [4], Rogers [5, 6]) and reduction to appropriate canonical forms in elliptic, parabolic, and hyperbolic regimes has been achieved.

This present paper is concerned with a new approach, i.e., of applying Baecklund transformations to the Stokes-Beltrami equations, in the matrix form

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_y = \begin{pmatrix}0 & -y^{-p}\\ y^{p} & 0\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_x \equiv \Lambda \begin{pmatrix}\phi\\ \psi\end{pmatrix}_x \tag{1}$$

in such a way that the character of these equations is preserved. That is, transformations between ϕ, ψ and ϕ', ψ' are sought which have the property of transforming equation (1) to another Stokes-Beltrami system

$$\begin{pmatrix}\phi'\\ \psi'\end{pmatrix} = \begin{pmatrix}0 & -y^{-q}\\ y^{q} & 0\end{pmatrix}\begin{pmatrix}\phi'\\ \psi'\end{pmatrix}_x \equiv \Lambda' \begin{pmatrix}\phi'\\ \psi'\end{pmatrix}_x \tag{2}$$

Explicit transformations are found in the six cases (a) $q = p$, (b) $q = -p$, (c) $q = p + 2$, $p \neq -1$, (d) $q = p - 2$, $p \neq 1$, (e) $q = -p - 2$, $p \neq -1$, and (f) $q = -p + 2$, $p \neq 1$. In particular it is shown that (a) generalises results obtained by Parsons [7].

2. The Baecklund transformations

The present investigation is concerned with the Baecklund transformations

$$\Omega'_x = \tilde{A}\Omega_x + \tilde{B}\Omega + \tilde{C}\Omega', \tag{3}$$

$$\Omega'_y = A\Omega_y + B\Omega + C\Omega', \tag{4}$$

$$\Omega = \begin{pmatrix}\phi\\ \psi\end{pmatrix}, \qquad \Omega' = \begin{pmatrix}\phi'\\ \psi'\end{pmatrix} \tag{5),(6}$$

which transform the Stokes-Beltrami system (1) to the associated system (2) where $\tilde{A}, \tilde{B}, \tilde{C}, A, B, C$ are in turn the 2×2 matrices $[\tilde{a}^i_j], [\tilde{b}^i_j], [\tilde{c}^i_j], [a^i_j], [b^i_j], [c^i_j]$, $(i, j = 1, 2)$ with entries functions of x and y.

It is required then to construct matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, A, B, and C such that the system (1) is transformed into system (2) subject also to the commutativity conditions

$$\Omega'_{xy} = \Omega'_{yx}, \qquad \Omega_{xy} = \Omega_{yx},$$

being satisfied.

Thus, employing the former commutativity condition in equations (3) and (4) and also making use of the latter condition,

$$\begin{aligned}&(A - \tilde{A})\Omega_{xy} + (A_x - \tilde{B} - \tilde{C}A)\Omega_y - (\tilde{A}_y - B - C\tilde{A})\Omega_x \\ &\quad + (B_x - \tilde{B}_y + C\tilde{B} - \tilde{C}B)\Omega + (C_x - \tilde{C}_y + C\tilde{C} - \tilde{C}C)\Omega' = 0,\end{aligned} \tag{7}$$

which, for $\Omega_y = \Lambda\Omega_x$ is identically satisfied by setting

$$A = \tilde{A}, \tag{8}$$

$$B_x - \tilde{B}_y + C\tilde{B} - \tilde{C}B = 0, \tag{9}$$

$$C_x - \tilde{C}_y + C\tilde{C} - \tilde{C}C = 0, \tag{10}$$

$$(A_x - \tilde{B} - \tilde{C}A)\Lambda = \tilde{A}_y - B - C\tilde{A}. \tag{11}$$

Returning to equations (3) and (4), it follows that if A is non-singular

$$\begin{aligned}A^{-1}(\Omega_y - A\Lambda A^{-1}\Omega'_x) &= (\Omega_y - \Lambda\Omega_x) + (A^{-1}B - \Lambda A^{-1}\tilde{B})\Omega \\ &\quad + (A^{-1}C - \Lambda A^{-1}\tilde{C})\Omega',\end{aligned} \tag{12}$$

so that on setting

$$A\Lambda A^{-1} = \Lambda' \tag{13}$$

$$B = \Lambda'\tilde{B}, \tag{14}$$

$$C = \Lambda'\tilde{C}, \tag{15}$$

https://doi.org/10.1017/S1446788700012945 Published online by Cambridge University Press

equation (12) reduces, since $\Omega_y = \Lambda\Omega_x$ to the associated Stokes-Beltrami system

$$\Omega'_y = \Lambda'\Omega'_x$$

It is noted that isolating Ω_x and Ω_y in equations (3) and (4) and employing the commutativity condition $\Omega_{xy} = \Omega_{yx}$ yields no further conditions since the same equation, namely (7), results.

Summarising, it has been shown that, provided matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, A, B, and C can be found such that conditions (8)–(11), (13)–(15) are satisfied together with $|A| \neq 0$, the Baecklund transformations defined by (3) and (4) transform the system $\Omega_y = \Lambda\Omega_x$ into an associated system.

The converse is also true, since, subject to the same conditions, equation (12) shows that $\Omega'_y = \Lambda'\Omega'_x$ implies $\Omega_y = \Lambda\Omega_x$ while (7) is again satisfied.

3. A class of transformations of the Stokes-Beltrami equations

We now investigate the conditions on the six matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, A, B, and C obtained in the previous section with the object of finding explicit expressions for the elements of these matrices. Firstly we note that the elements of $\tilde{A}$, B, and C are known from equations (8), (14) and (15) when A, $\tilde{B}$ and $\tilde{C}$ are known, and also that equations (9), (10), and (11) are considerably simplified when

$$\tilde{C}\Lambda' = \Lambda'\tilde{C}. \tag{16}$$

Thus, eliminating $\tilde{A}$, B, and C and employing this further condition, equations (9), (10), and (11) become

$$(\Lambda'\tilde{B})_x - \tilde{B}_y = 0, \tag{17}$$

$$(\Lambda'\tilde{C})_x - \tilde{C}_y = 0, \tag{18}$$

$$(A_x - \tilde{B})\Lambda = A_y - \Lambda'\tilde{B}. \tag{19}$$

The equation

$$A\Lambda A^{-1} = \Lambda', \quad |A| \neq 0, \tag{20}$$

completes the set of conditions on $\tilde{B}$, $\tilde{C}$, and A.

The equations involving $\tilde{C}$, (16) and (18), may be isolated from those involving A and $\tilde{B}$ and yield

$$\tilde{c}_1^1 = \tilde{c}_2^2, \qquad \tilde{c}_2^1 = -y^{-2q}\,\tilde{c}_1^2,$$

where $\nabla^2\tilde{c}_1^1 = \nabla^2\tilde{c}_2^1 = 0$ when $q = 0$ and $\tilde{c}_1^1 = \text{constant} = c$, $\tilde{c}_2^1 = 0$ when $q \neq 0$. Only the latter values of $\tilde{c}_1^1$ and $\tilde{c}_2^1$ are considered here but we note that they are valid for the $q = 0$ case also. Hence

$$\tilde{C} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, C = \begin{pmatrix} 0 & -cy^{-q} \\ cy^{q} & 0 \end{pmatrix}$$

From (20), a_1^2 and a_2^2 may be found in terms of a_1^1 and a_2^1, that is

(21), (22) $$a_1^2 = -y^{p+q}a_2^1, \qquad a_2^2 = y^{-p+q}a_1^1.$$

It thus remains to determine six unknowns, namely the elements a_1^1, a_2^1, and the four elements of $\tilde{B}$, while it is observed that the remaining two equations (17) and (19) lead to eight equations containing these six terms.

Firstly, condition (19), together with (21), (22), requires that

(23) $$(a_1^1)_y + \tilde{b}_1^2 y^{-q} = [(a_2^1)_x - \tilde{b}_2^1]y^p,$$

(24) $$(a_2^1)_y + \tilde{b}_2^2 y^{-q} = -[(a_1^1)_x - \tilde{b}_1^1]y^{-p},$$

(25) $$(a_2^1 y^{p+q})_y + \tilde{b}_1^1 y^q = -[(a_1^1 y^{-p+q})_x - \tilde{b}_2^2]y^p,$$

(26) $$(a_1^1 y^{-p+q})_y - \tilde{b}_2^1 y^q = [(a_2^1 y^{p+q})_x + \tilde{b}_1^2]y^{-p}.$$

It is seen that equations (23) and (26) combine to give

(27) $$\tfrac{1}{2}(p-q)a_1^1 y^{-1} + \tilde{b}_1^2 y^{-q} + \tilde{b}_2^1 y^p = 0,$$

(28) $$(a_1^1)_y - \tfrac{1}{2}(p-q)a_1^1 y^{-1} - (a_2^1)_x y^p = 0,$$

while (24) and (25) together yield the relations

(29) $$\tfrac{1}{2}(p+q)a_2^1 y^{-1} + \tilde{b}_1^1 y^{-p} - \tilde{b}_2^2 y^{-q} = 0,$$

(30) $$(a_1^1)_x y^{-p} + \tfrac{1}{2}(p+q)a_2^1 y^{-1} + (a_2^1)_y = 0.$$

Secondly, condition (17), together with (27), (29), shows that

(31) $$[\tfrac{1}{2}(p-q)a_1^1 y^{-1} + \tilde{b}_2^1 y^p]_x - (\tilde{b}_1^1)_y = 0,$$

(32) $$[\tfrac{1}{2}(p+qa_2^1 y^{-1} + \tilde{b}_1^1 y^{-p}]_x + (\tilde{b}_2^1)_y = 0,$$

(33) $$[\tfrac{1}{2}(p-q)a_1^1 y^{q-1} + \tilde{b}_2^1 y^{p+q}]_y + (\tilde{b}_1^1)_x y^q = 0,$$

(34) $$[\tfrac{1}{2}(p+q)a_2^1 y^{q-1} + \tilde{b}_1^1 y^{-p+q}]_y - (\tilde{b}_2^1)_x y^q = 0.$$

Of these eight equations, (27)–(34), two equations, (27), (29), give $\tilde{b}_1^2$ and $\tilde{b}_2^2$ in terms of $\tilde{b}_1^1$ and $\tilde{b}_2^1$ and also, interestingly, the two equations (28) and (30) are the Cauchy-Riemann equations

$$[a_1^1 y^{\frac{1}{2}(q-p)}]_y - [a_2^1 y^{\frac{1}{2}(q+p)}]_x = 0,$$
$$[a_2^1 y^{\frac{1}{2}(q+p)}]_y + [a_1^1 y^{\frac{1}{2}(q-p}]_x = 0,$$

showing that $a_2^1 y^{\frac{1}{2}(q+p)} + i\, a_1^1 y^{\frac{1}{2}(q-p)}$ is an analytic function, $2f$ say, of $z = x + iy$.

Thus

$$\begin{cases} a_1^1 = -i[f(z) - \overline{f(z)}]y^{\frac{1}{2}(p-q)}, \\ a_1^1 = [f(z) + \overline{f(z)}]y^{\frac{1}{2}(p+q)}. \end{cases} \tag{35}$$

In the following work

$$f' \equiv f'(z) \equiv \frac{d}{dz}[f(z)],$$

$$\bar{f}' \equiv \bar{f}'(\bar{z}) \equiv \frac{d}{d\bar{z}}[\bar{f}(\bar{z})] \equiv \frac{d}{d\bar{z}}[\overline{f(z)}].$$

Considering now the remaining four equations (31)–34) we substitute the values obtained for a_1^1, a_2^1, $\tilde{b}_1^2$ and $\tilde{b}_2^2$ in terms of $\tilde{b}_1^1$, $\tilde{b}_2^1$ and $f(z)$. This will leave four equations connecting $\tilde{b}_1^1$, $\tilde{b}_2^1$ and $f(z)$. Employing the relations (35),

$$(a_1^1)_x = -i(f' - \bar{f}')y^{\frac{1}{2}(p-q)},$$

$$(a_2^1)_x = (f' + \bar{f}')y^{-\frac{1}{2}(p+q)},$$

$$(a_1^1)_y = (f' + \bar{f}')y^{\frac{1}{2}(p-q)} - \tfrac{i}{2}(p-q)(f-\bar{f})y^{\frac{1}{2}(p-q-2)},$$

and

$$(a_2^1)_y = i(f' - \bar{f}')y^{-\frac{1}{2}(p+q)} - \tfrac{1}{2}(p+q)(f+\bar{f})y^{-\frac{1}{2}(p+q+2)},$$

and making the above substitution, equations (31)–(34) become

$$(\tilde{b}_1^1)_y - (\tilde{b}_2^1)_x y^p + \frac{i}{2}(p-q)(f' - \bar{f}')y^{\frac{1}{2}(p-q-2)} = 0, \tag{36}$$

$$(\tilde{b}_1^1)_x + (\tilde{b}_2^1)_y y^p + \tfrac{1}{2}(p+q)(f' + \bar{f}')y^{\frac{1}{2}(p-q-2)} = 0, \tag{37}$$

$$\begin{aligned}(\tilde{b}_1^1)_x + (\tilde{b}_2^1)_y y^p + (p+q)\tilde{b}_2^1 y^{p-1} + \tfrac{1}{2}(p-q)(f' + \bar{f}')y^{\frac{1}{2}(p-q-2)} \\ - \frac{i}{4}(p+q-2)(p-q)(f-\bar{f})y^{\frac{1}{2}(p-q-4)} = 0,\end{aligned} \tag{38}$$

$$\begin{aligned}(\tilde{b}_1^1)_y - (p-q)\tilde{b}_1^1 y^{-1} - (\tilde{b}_2^1)_x y^p + \frac{i}{2}(p+q)(f' - \bar{f}')y^{\frac{1}{2}(p-q-2)} \\ - \frac{1}{4}(p-q+2)(p+q)(f+\bar{f})y^{\frac{1}{2}(p-q-4)} = 0.\end{aligned} \tag{39}$$

Combining (37), (38) and (36), (39) respectively yields

$$(p+q)\tilde{b}_2^1 = q(f' + \bar{f}')y^{-\frac{1}{2}(p+q)} + \frac{i}{4}(p+q-2)(p-q)(f-\bar{f})y^{-\frac{1}{2}(p+q+2)}, \tag{40}$$

$$(p-q)\tilde{b}_1^1 = iq(f' - \bar{f}')y^{\frac{1}{2}(p-q)} - \tfrac{1}{4}(p-q+2)(p+q)(f+\bar{f})y^{\frac{1}{2}(p-q-2)}, \tag{41}$$

https://doi.org/10.1017/S1446788700012945 Published online by Cambridge University Press

The necessary forms of $\tilde{b}_2^1$ and $\tilde{b}_1^1$ are given in terms of $f(z)$ by equations (40) $(q \neq -p)$ and (41) $(q \neq p)$ and we are finally left with two equations, (36) and (37), which will determine the necessary restrictions on $f(z)$. In fact it will be shown later that for an $f(z)$ to exist, certain relations must exist between p and q. However, before proceeding, we dispose of the special cases $q = p$, $q = -p$. It should also be remembered that in each case it must be verified that $|A| \neq 0$ (see condition (20)) to ensure the validity of the transformation obtained.

(a) $q = p$.

Equation (41) now reduces to

$$(z - \bar{z})(f' - \bar{f}') - 2(f + \bar{f}) = 0, \qquad p \neq 0$$

By successive differentiation and manipulation of this equation it is easy to see that

$$f(z) = i(\tfrac{1}{2}az^2 + bz + e)$$

where a, b and e are real constants.

Relation (40) then yields for $p \neq 0$

$$\tilde{b}_2^1 = -ay^{-p+1},$$

while (36) shows that $\tilde{b}_1^1$ is a function of x, $g(x)$ say, whence, on substitution in (37),

$$g'(x) = a(p+1).$$

Thus

$$\tilde{b}_1^1 = a(p+1)x + d,$$

where d is a real constant of integration. The remaining elements of the matrices A, B, $\tilde{A}$, and $\tilde{B}$ may now be evaluated. Equation (35) gives a_1^1 and a_2^2, equations (21), (22), (27), and (29) then give a_1^2, a_2^2, $\tilde{b}_1^2$, and $\tilde{b}_2^2$ respectively, and finally equations (8) and (14) give the matrices $\tilde{A}$ and B.

Summarising, it has been shown that the Stokes-Beltrami system (1) is transformed to the new such system (2) by the Baecklund transformations

$$\begin{pmatrix}\phi' \\ \psi'\end{pmatrix}_x = \begin{pmatrix} a(x^2-y^2)+2(bx+e) & -2y^{-p+1}(ax+b) \\ 2y^{p+1}(ax+b) & a(x^2-y^2)+2(bx+e)\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}_x$$

$$+ \begin{pmatrix} a(p+1)x+d & -ay^{-p+1} \\ ay^{p+1} & a(1-p)x-2pb+d\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c\end{pmatrix}\begin{pmatrix}\phi' \\ \psi'\end{pmatrix}$$

$$\begin{pmatrix}\phi' \\ \psi'\end{pmatrix}_y = \begin{pmatrix} a(x^2-y^2)+2(bx+e) & -2y^{-p+1}(ax+b) \\ 2y^{p+1}(ax+b) & a(x^2-y^2)+2(bx+e)\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}_y$$

$$+ \begin{pmatrix} -ay & -y^{-p}[a(1-p)x-2pb+d] \\ y^p[a(p+1)x+d] & -ay\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} + \begin{pmatrix} 0 & -cy^{-p} \\ cy^p & 0\end{pmatrix}\begin{pmatrix}\phi' \\ \psi'\end{pmatrix},$$

where the case $a = b = e = 0$ is excluded to ensure $|A| \neq 0$. It is of interest to observe that the two transformations given by Parsons [7] are extracted by setting in turn

$$p = 1, \qquad a = c = e = 0, \qquad 2b = d = 1/l$$

and

$$p = 1, \qquad a = c = e = 0, \qquad 4b = d = 2/m.$$

(b) $q = -p$.

Here, (40) becomes

$$(z - \bar{z})(f' + \bar{f}') - 2(f - \bar{f}) = 0, \qquad p \neq 0,$$

the most general solution of which is

$$f(z) = \tfrac{1}{2}az^2 + bz + e$$

where a, b, and e are real constants.

Equation (41) now yields, for $p \neq 0$,

$$\tilde{b}_1^1 = ay^{p+1},$$

while (37) shows that $\tilde{b}_2^1 = g(x)$. Employing (36), it is seen that

$$g'(x) = a(1 - p),$$

so that

$$\tilde{b}_2^1 = a(1 - p)x + d.$$

The remaining matrix elements may now be calculated as before showing that the Stokes-Beltrami equations (1) are transformed to the associated system (2) by the Baecklund transformations

$$\begin{pmatrix}\phi' \\ \psi'\end{pmatrix}_x = \begin{pmatrix} 2y^{p+1}(ax + b) & a(x^2 - y^2) + 2(bx + e) \\ -a(x^2 - y^2) - 2(bx + e) & 2y^{-p+1}(ax + b)\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}_x$$

$$+ \begin{pmatrix} ay^{p+1} & a(1 - p)x + d \\ -a(p + 1)x - d - 2pb & ay^{-p+1}\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c\end{pmatrix}\begin{pmatrix}\phi' \\ \psi'\end{pmatrix},$$

$$\begin{pmatrix}\phi' \\ \psi'\end{pmatrix}_y = \begin{pmatrix} 2y^{p+1}(ax + b) & a(x^2 - y^2) + 2(bx + e) \\ -a(x^2 - y^2) - 2(bx + e) & 2y^{-p+1}(ax + b)\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}_y$$

$$+ \begin{pmatrix} y^p[a(p + 1)x + d + 2pb] & -ay \\ ay & y^{-p}[a(1 - p)x + d]\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -cy^p \\ cy^{-p} & 0\end{pmatrix}\begin{pmatrix}\phi' \\ \psi'\end{pmatrix},$$

https://doi.org/10.1017/S1446788700012945 Published online by Cambridge University Press

where, here again, $a = b = e = 0$ is excluded so that $|A| \neq 0$.

Excluding special cases (a) and (b), equations (40) and (41) will define $\tilde{b}_2^1$ and $\tilde{b}_1^1$ respectively, and substituting these values into (36) and (37) gives that

$$(f+\bar{f})M + i(f' - \bar{f}')Ly + (f'' + \bar{f}'')Ly^2 = 0$$

and

$$(f-\bar{f})N + i(f' + \bar{f}')Ly + (f'' - \bar{f}'')Ly^2 = 0,$$

where

$$L = -2pq/(p^2 - q^2), \qquad M = -(p-q+2)(p-q-2)(p+q)/[8(p-q)]$$

$$N = (p+q-2)\ (p+q+2)\ (p-q)/[8(p+q)].$$

Since the former equation has purely real terms, whereas the latter equation has purely imaginary terms, they are both completely represented by the single equation formed by their addition. That is

$$(42) \qquad f(M+N) + \bar{f}(M-N) + f'L(z-\bar{z}) - \tfrac{1}{2}f''L(z-\bar{z})^2 = 0.$$

Operating on (42) with $\dfrac{\partial^3}{\partial z \partial \bar{z}^2}$ shows that

$$Lf''' = 0.$$

The possibility $L = 0$ leads to transformations, involving a general $f(z)$, between Stokes-Beltrami systems with $p = \pm 2$ and systems with $q = 0$. However, since we are looking for more general relations between p and q, we take $f'''(z) = 0$ and, in fact, find eventually that we still obtain transformations between these two specialised systems although they will not of course be as general. Thus

$$f(z) = \tfrac{1}{2}\alpha z^2 + \beta z + \delta,$$

where α, β, and δ are complex constants and (42) becomes

$$\frac{\alpha}{2}z^2(M+N+L) + \tfrac{1}{2}\bar{z}^2(\bar{\alpha}(M-N) - \alpha L) + \beta z(M+N+L)$$
$$+\ \bar{z}(\bar{\beta}(M-N) - \beta L) + \delta(M+N) + \bar{\delta}(M-N) = 0.$$

which will be satisfied if and only if

$$\alpha(M+N+L) = 0, \qquad \bar{\alpha}(M-N) - \alpha L = 0$$
$$\beta(M+N+L) = 0, \qquad \bar{\beta}(M-N) - \beta L = 0$$
$$\delta(M+N) + \bar{\delta}(M-N) = 0.$$

Closer investigation shows that when neither M nor N is zero, $\alpha = \beta = \delta = 0$ so that $f(z) = 0$ and the matrix A is the zero matrix. This case must be excluded since $|A| \neq 0$. The only other possibilities are

(i) $M=0$, $N=-L$, $\alpha=a$, $\beta=b$, $\delta=d$, $(q=p+2$ or $q=p-2$, $q\neq\pm p)$

or (ii) $N=0$, $M=-L$, $\alpha=ia$, $\beta=ib$, $\delta=id$, $(q=-p-2$ or $q=-p+2$, $q\neq\pm p)$

where a, b, and d are real constants. These four relations between p and q are now considered separately.

(c) $q=p+2$, $p\neq-1$.

Here,

$$f(z)=\tfrac{1}{2}az^2+bz+d$$

and the Baecklund transformations are

$$\Omega'_x=\tilde{A}\Omega_x+\tilde{B}\Omega+\tilde{C}\Omega',$$

$$\Omega'_y=A\Omega_y+B\Omega+C\Omega',$$

where

$$A=\tilde{A}=\begin{pmatrix}2(ax+b) & y^{-p-1}[a(x^2-y^2)+2(bx+d)]\\ -y^{p+1}[a(x^2-y^2)+2(bx+d)] & 2y^2(ax+b)\end{pmatrix},$$

$$\tilde{B}=\begin{pmatrix}a(p+2) & 2y^{-p-1}(ax+b)\\ 0 & ay^2+(p+1)[ax^2+2(bx+d)]\end{pmatrix},$$

$$B=\begin{pmatrix}0 & -ay^{-p}-(p+1)y^{-p-2}[ax^2+2(bx+d)]\\ a(p+2)y^{p+2} & 2y(ax+b)\end{pmatrix},$$

$$\tilde{C}=\begin{pmatrix}c & 0\\ 0 & c\end{pmatrix},\qquad C=\begin{pmatrix}0 & -cy^{-p-2}\\ cy^{p+2} & 0\end{pmatrix}.$$

The non-singularity condition on the matrix A is satisfied by excluding the special case $a=b=d=0$.

(d) $q=p-2$, $p\neq1$.

The expression for $f(z)$ is the same as in (c) and the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, A, B, and C of the Baecklund transformations are given by

$$A=\tilde{A}=\begin{pmatrix}2y^2(ax+b) & y^{-(p+1)}[a(x^2-y^2)+2(bx+d)]\\ -y^{p-1}[a(x^2-y^2)+2(bx+d)] & 2(ax+b)\end{pmatrix},$$

$$\tilde{B}=\begin{pmatrix}ay^2-(p-1)(ax^2+2bx+2d) & 0\\ -2y^{p-1}(ax+b) & -a(p-2)\end{pmatrix},$$

$$B=\begin{pmatrix}2y(ax+b) & a(p-2)y^{-p+2}\\ ay^p-(p-1)y^{p-2}(ax^2+2bx+d) & 0\end{pmatrix},$$

https://doi.org/10.1017/S1446788700012945 Published online by Cambridge University Press

$$\tilde{C} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -cy^{-p+2} \\ cy^{p-2} & 0 \end{pmatrix}.$$

The matrix A is only singular when $a = b = d = 0$, so again, this case must be excluded.

(e) $q = -p-2$, $p \neq -1$

In this case

$$f(z) = i(\tfrac{1}{2}az^2 + bz + d),$$

and the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, A, B, and C are given by

$$A = \tilde{A} = \begin{pmatrix} y^{p+1}[a(x^2-y^2)+2(bx+d)] & -2y^2(ax+b) \\ 2(ax+b) & y^{-p-1}[a(x^2-y^2)+2(bx+d)] \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} 0 & -ay^2-(p+1)(ax^2+2bx+2d) \\ a(p+2) & 2y^{-p-1}(ax+b) \end{pmatrix}$$

$$B = \begin{pmatrix} -a(p+2)y^{p+2} & -2y(ax+b) \\ 0 & -ay^{-p}-(p+1)y^{-p-2}(ax^2+2bx+2d) \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -cy^{p+2} \\ cy^{-p-2} & 0 \end{pmatrix}$$

Here again it can be directly verified that if the case $a = b = d = 0$ is excluded then A is non-singular and the transformation is valid.

(f) $q = -p+2$, $p \neq 1$.

As in (e), $f(z) = i(\tfrac{1}{2}az^2 + bz + d)$ and the Baecklund transformations (3) and (4) are obtained by setting

$$A = \tilde{A} = \begin{pmatrix} y^{p-1}[a(x^2-y^2)+2(bx+d)] & -2(ax+b) \\ 2y^2(ax+b) & y^{-p+1}[a(x^2-y^2)+2(bx+d)] \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} 2y^{p-1}(ax+b) & a(p-2) \\ ay^2-(p-1)(ax^2+2bx+2d) & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -ay^p+(p-1)y^{p-2}(ax^2+2bx+2d) & 0 \\ 2y(ax+b) & a(p-2)y^{-p+2} \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -cy^{p-2} \\ cy^{-p+2} & 0 \end{pmatrix}$$

where we exclude $a = b = d = 0$ thereby ensuring that A is non-singular.

This completes the present discussion. It has been seen that Baecklund transformations lead to a number of new results concerning the Stokes-Beltrami equations. It is hoped that future papers will deal with physical applications of this work.

References

[1] A. Weinstein, 'Generalized axially symmetric potential theory', *Bull. Amer. Math. Soc.* 59 (1953), 20–28.

[2] K. B. Ranger, 'Some integral transformation formulae for the Stokes-Beltrami equations', *J. Math. Mech.* 12 (1963), 663–673.

[3] C. Loewner, 'A transformation theory of partial differential equations of gasdynamics', *NACA, Technical Note* 2065 (1950), 1–56.

[4] G. Power, C. Rogers and R. A. Osborn, 'Baecklund and generalized Legendre transformations in gasdynamics', *Z. angew. Math. Mech.* 49 (1969), 333–340.

[5] C. Rogers, 'Transformări Baecklund in magnetogazodinamica Liniară nedisipativă', presented to Conferinta Natională de Mecanică Aplicată, România (1969).

[6] C. Rogers, 'Application of Baecklund transformations in aligned magneto-gasdynamics', *Acta Physica Austriaca* 31 (1970), 80–88.

[7] D. H. Parsons, 'Irrotational flow of a liquid with axial symmetry', *Proc. Edin. Math. Soc.* 13 (1963) 201–204.

Department of Mathematics
The University, Nottingham, England