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On the smoothness of the moduli space of mathematical instanton bundles

ROSA M. MIRO-ROIG* and JAVIER A. ORUS-LACORT*
Dept. Algebra y Geometría, Facultad de Matemáticas, Universidad de Barcelona, 08007 Barcelona, Spain, e-mail: miro at cerber.ub.es; e-mail: orus at cerber.ub.es

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Abstract. In this paper we prove that the moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$ of mathematical instanton bundles on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ with quantum number k are singular for $n \geqslant 2$ and $k \geqslant 3$, giving a positive answer to a conjecture made by Ancona and Ottaviani in 1993.

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Introduction

Throughout this paper **k** will be an algebraically closed field of characteristic zero and $\mathbf{P}^{2n+1}$ the (2n+1)-dimensional projective space over the field **k**. Let $MI_{2n+1}(k)$ be the moduli space of all mathematical instanton bundles over $\mathbf{P}^{2n+1}$ with second Chern class $c_2 = k$. Related to the smoothness of $MI_{2n+1}(k)$ there are two important conjectures:

CONJECTURE 1. *The moduli spaces $MI_3(k)$ are smooth of dimension* $8k - 3$.

It is well known that the moduli space $MI_3(k)$ is nonsingular of dimension 8k-3 for any $k \leqslant 4$ (See [B1] for $k = 1$, [H2] for $k = 2$, [ES] for $k = 3$, and [B2] or [LP] for $n = 4$); and as far as we know Conjecture 1 remains open for $k > 4$. For $n \geqslant 2$ the situation is quite different and we have:

CONJECTURE 2. *For all integers $n \geqslant 2$ and $k \geqslant 3$ the moduli spaces $MI_{2n+1}(k)$ are singular.*

In [AO1], Ancona and Ottaviani have recently proved: (1) the moduli spaces $MI_{2n+1}(2)$ are smooth, irreducible of dimension $4n^2 + 2n - 3$; and (2) the moduli spaces $MI_5(3)$ and $MI_5(4)$ are singular. The main goal of this paper is to show that Conjecture 2 is true.

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OUTLINE OF THE PROOF

First of all we observe that for any instanton bundle E on $\mathbf{P}^{2n+1}$, $H^i\mathcal{E}nd(E) = 0$ for $i \geqslant 3$, $H^0\mathcal{E}nd(E) = \mathbf{k}$ (E is simple) and $h^1\mathcal{E}nd(E) - h^2\mathcal{E}nd(E) = -k^2\binom{2n-1}{2} + k(8n^2) + 1 - 4n^2$ (by Hirzebruch-Riemann-Roch). Let $M^0 := MI^0_{2n+1}(k)$ be the irreducible component of $MI_{2n+1}(k)$ containing special instanton bundles. In [AO; Theorem 3.7], Ancona and Ottaviani proved that special symplectic instanton bundles are stable and the generic special instanton bundle is stable. Therefore the Zariski tangent space of $MI_{2n+1}(k)$ at special symplectic instanton bundles can be identified with $\mathrm{Ext}^1(E, E)$.

Very recently Ottaviani and Trautmann have proved that for any special symplectic instanton bundle $E \in MI^0_{2n+1}(k)$, $h^2\mathcal{E}nd(E) = (k-2)^2\binom{2n-1}{2}$ [OT]. In this paper, for all integers $n \geqslant 2$ and $k \geqslant 3$, we will construct deformations E' of special symplectic instanton bundles in $MI^0_{2n+1}(k)$ satisfying (Cf. Theorem 3.1):

$$h^2\mathcal{E}nd(E') < h^2\mathcal{E}nd(E) = (k-2)^2\binom{2n-1}{2}.$$

Putting altogether we get that for all integers $n \geqslant 2$ and $k \geqslant 3$, the moduli spaces $MI_{2n+1}(k)$ are singular at least in special symplectic bundles.

This paper was written in the context of the group ‘Vector Bundles on higher dimensional varieties’ of Europroj.

Notation

Throughout this paper $\mathbf{k}$ will be an algebraically closed field of characteristic zero and $\mathbf{P}^{2n+1}$ the (2n+1)-dimensional projective space over the field $\mathbf{k}$. We denote by $\mathcal{O}(d)$ the invertible sheaf of degree d on $\mathbf{P}^{2n+1}$. For any coherent sheaf F on $\mathbf{P}^{2n+1}$ we use the abbreviations $F(d) := F \otimes \mathcal{O}(d)$, $H^iF := H^i(\mathbf{P}^n, F)$ and $h^iF = \dim_k H^i(\mathbf{P}^n, F)$. The terms vector bundle and locally free sheaf are used synonymously. We will use the definiton of stable and semistable due to Mumford-Takemoto ([OSS]).

1. Instanton bundles

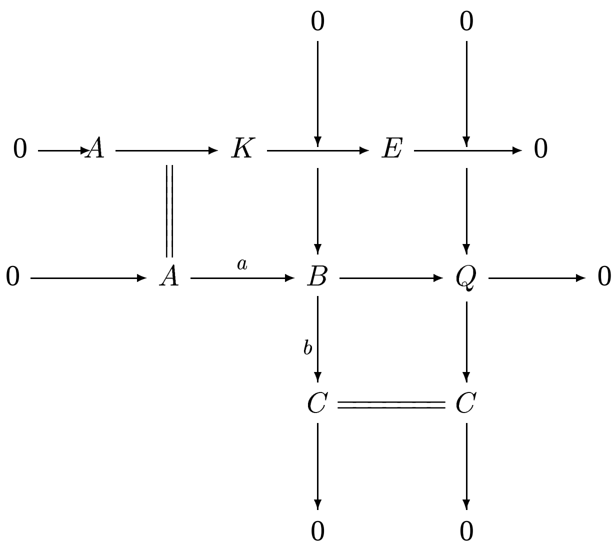
We will begin this section recalling the notion of monad and the basic facts on Instanton bundles needed in the sequel.

DEFINITION 1.1. A monad over $\mathbf{P}^{2n+1}$ is a complex of vector bundles:

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

with $ab = 0, a\colon A \to B$ an injective bundle-map and $b\colon B \to C$ surjective. The vector bundle $E := \mathrm{Ker}(b)/\mathrm{Im}(a)$ is called the cohomology bundle of the monad.

A monad $0 \to A \overset{a}{\to} B \overset{b}{\to} C \to 0$ has a so-called display: this is a commutative diagram with exact rows and columns:



where $K := \text{Ker}(b)$ and $Q := \text{Coker}(a)$. From the display one easily deduces that $rk(E) = rk(B) - rk(A) - rk(C)$ and $c_t(E) = c_t(B)c_t(A)^{-1}c_t(C)^{-1}$.

DEFINITION 1.2. A mathematical instanton bundle on $\mathbf{P}^{2n+1}$ with quantum number k is a rank 2n vector bundle E on $\mathbf{P}^{2n+1}$ satisfying:

(1) E has Chern polynomial $c_t(E) = (1 - t^2)^{-k} = 1 + kt^2 + \cdots$,
(2) E has natural cohomology in the range $-2n - 1 \leqslant d \leqslant 0$, that is for any d in that range $h^i E(d) \neq 0$ for at most one i, and
(3) the restriction of E to a general line is trivial.

Remark 1.3. In the original definition in [OS] the additional condition E is simple is imposed. However, it has been shown in [AO] that this last condition is already a consequence of (i) and (ii).

As an easy consequence of the above definition we get:

(1.4) Let E be an instanton bundle on $\mathbf{P}^{2n+1}$ with second Chern class k. Then, E is the cohomology bundle of a monad of the following type:

$$0 \to k\mathcal{O}(-1) \overset{A}{\to} (2n + 2k)\mathcal{O} \overset{B}{\to} k\mathcal{O}(1) \to 0. \qquad (*)$$

Moreover, any vector bundle E on $\mathbf{P}_{2n+1}$ which appears as the cohomology bundle of a monad of the type (*) is a rank 2n vector bundle on $\mathbf{P}^{2n+1}$ verifying the conditions (1) and (2) of definiton 1.2 [AO1].

With respect to a fixed system of homogeneous coordinates $X_0, \dots, X_{2n+1}$ of $\mathbf{P}^{2n+1}$ the morphism A (resp. B) of the monad (*) can be identified with a $(k)x(2n+2k)$ (resp. $(2n+2k)x(k)$) matrix whose entries are homogeneous linear polynomials of $\mathbf{k}[X_0, \dots, X_{2n+1}]$. Then the conditions that (*) is a monad are equivalent to: A, B have rank k at every point $x \in \mathbf{P}^{2n+1}$ and $AB = 0$.

DEFINITION 1.4. An instanton bundle E is called symplectic if there is an isomorphism $E \xrightarrow{\varphi} E^v$ satisfying $\varphi^v = -\varphi$.

We recall from [AO1] the following definitions:

DEFINITION 1.5. A bundle S appearing in an exact sequence:

$$0 \to S^* \to \mathcal{O}^d \xrightarrow{B} \mathcal{O}(1)^c \to 0 \qquad (**)$$

is called a Schwarzenberger type bundle (STB).

A particular class of STB that we are going to use in the sequel are the generalized Schwarzenberger bundles on $\mathbf{P}^{2n+1}$ introduced in [ST]. Set $d = 2n + 2k, c = k$ and

$$B^t := \begin{pmatrix} x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n & & & \\ & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n & & \\ & & & & & & & & & & & & & \\ & & & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \end{pmatrix},$$

where $(x_0, \dots, x_n, y_0, \dots, y_n)$ are homogeneous coordinates on $\mathbf{P}^{2n+1}$. B defines (as in (**)) a $(2n+k)$-bundle $\mathcal{S}_n^k$ on $\mathbf{P}^{2n+1}$. We call a generalized Schwarzenberger bundle, and we denote it by S_n^k, any bundle of the form $S_n^k = g^* \mathcal{S}_n^k$ for some $g \in \mathrm{Aut}(\mathbf{P}^{2n+1})$.

DEFINITION 1.6. An instanton bundle arising from a monad (*) where the kernel Ker$B$ is a generalized Schwarzenberger bundle is called a special instanton bundle.

In [ST], Spindler and Trautmann proved that there exists a coarse moduli space for special instanton bundles on $\mathbf{P}^{2n+1}$ of quantum number k. Its dimension is $2n^2 + 3n$ for $k = 1$ and $2nk + 4(n+1)^2 - 7$ for $k \geqslant 2$.

2. Determination of $\mathrm{Ext}^2(E, E)$

The aim of this section is to compute $\mathrm{Ext}^2(E, E)$. The method is essentially the one used in [AO] and [AO1] but we include it for helping the reader.

Remark 2.1. Let E be an instanton bundle arising from a monad (*). We can easily check that $\mathrm{Ext}^i(E, E) = 0$ for $i \geqslant 3$, $\mathrm{Hom}(E, E) = \mathbf{k}$ (E is simple) and dim $Ext^1(E, E)$ – dim $Ext^2(E, E) = -k^2\binom{2n-1}{2} + k(8n^2) + 1 - 4n^2$.

The following Lemma is the key point of our computations:

LEMMA 2.2. *Let E be an instanton bundle on $\mathbf{P}^{2n+1}$ with second Chern class k arising from the monad:*

$$0 \to k\mathcal{O}(-1) \xrightarrow{A} (2n+2k)\mathcal{O} \xrightarrow{B} k\mathcal{O}(1) \to 0. \qquad (*)$$

Then, $H^2(\mathcal{E}nd(E)) = \mathrm{Coker}(d_0)$ *where*

$$d_0 \colon \mathrm{Hom}(\mathcal{O}(-1)^k, \mathcal{O}^{2n+2k}) x\ \mathrm{Hom}(\mathcal{O}^{2n+2k}, \mathcal{O}(1)^k) \to \mathrm{Hom}(\mathcal{O}(-1)^k, \mathcal{O}(1)^k)$$

is the morphism given by $d_0(a,b) = Ab + aB$.

Proof. Using Künneth Theorem (Cf. [G]; p. 100) we get that $F = E \otimes E^v$ is the cohomology bundle of the simple complex:

$$\begin{aligned} 0 &\to k^2\mathcal{O}(-2) \to (4k^2+4nk)\mathcal{O}(-1) \xrightarrow{\beta} (6k^2+8nk+4n^2)\mathcal{O} \\ &\xrightarrow{\alpha} (4k^2+4nk)\mathcal{O}(1) \to k^2\mathcal{O}(2) \to 0 \end{aligned}$$

associated to the double complex:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1)^k \otimes \mathcal{O}(-1)^k & \longrightarrow & \mathcal{O}^{2n+2k} \otimes \mathcal{O}(-1)^k & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}(-1)^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1)^k \otimes \mathcal{O}^{2n+2k} & \longrightarrow & \mathcal{O}^{2n+2k} \otimes \mathcal{O}^{2n+2k} & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}^{2n+2k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1)^k \otimes \mathcal{O}(1)^k & \longrightarrow & \mathcal{O}^{2n+2k} \otimes \mathcal{O}(1)^k & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}(1)^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The above complex is exact except in the midle term and there we have $E \otimes E^v = \mathrm{Ker}(\alpha)/\mathrm{Im}(\beta)$. Therefore, we have the exact sequences:

$$0 \to k^2\mathcal{O}(-2) \to (4k^2+4nk)\mathcal{O}(-1) \xrightarrow{\beta} \mathrm{Im}(\beta) \to 0,$$

$$0 \to \mathrm{Im}(\beta) \to \mathrm{Ker}(\alpha) \to E \otimes E^v \to 0,$$

$$0 \to \mathrm{Ker}(\alpha) \to (6k^2+8nk+4n^2)\mathcal{O} \to \mathrm{Im}(\alpha) \to 0,$$

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$$0 \to \mathrm{Im}(\alpha) \to (4k^2 + 4nk)\mathcal{O}(1) \to k^2\mathcal{O}(2) \to 0,$$

and we obtain $H^2(E \otimes E^v) = H^2(\mathrm{Ker}\,\alpha) = H^1(\mathrm{Im}\,\alpha) = \mathrm{Coker}(d_0)$.

Using the above notation, we have:

$$\dim\ \mathrm{Ext}^2(E,E) = \dim\ \mathrm{Coker}(d_0) = \binom{2n+3}{2}k^2 - \dim\ \mathrm{Im}(d_0)$$
$$= \binom{2n+3}{2}k^2 - 2k(2n+2k)(2n+2) + \dim\ \mathrm{Ker}(d_0).$$

Assume that A (resp. B^t) is the matrix which presents a module P (resp. Q). We define $M = M(A, B^t) := M(A \otimes \mathrm{id}, \mathrm{id} \otimes B^t)$ as the matrix which presents the tensor product of P and Q; and we denote by syz_1M the dimension of the **k**-vector space of the syzygies of M of degree 1. We have $\dim\ \mathrm{Ker}(d_0) = syz_1M$ and we obtain the following useful formula:

$$\dim\ \mathrm{Ext}^2(E,E) = k(2n^2k - 3nk - 5k - 8n^2 - 8n) + syz_1M \qquad (***)$$

that will be used in the sequel.

3. Main theorem

Now we are ready for proving that for all integers $k \geqslant 3$ and $n \geqslant 2$, the moduli spaces $MI_{2n+1}(k)$ are singular at least in special symplectic instanton bundles.

THEOREM 3.1. *For all integers $k \geqslant 3$ and $n \geqslant 2$, the moduli spaces $MI_{2n+1}(k)$ are singular.*

Proof. Let E_0 be a special symplectic instanton bundle on $\mathbf{P}^{2n+1}$ with second Chern class k. By [AO; Theorem 3.7] any symplectic special instanton bundle on $\mathbf{P}^{2n+1}$ is stable; hence using deformation theory the Zariski tangent space to the moduli space $MI_{2n+1}(k)$ at the point corresponding to E_0 is isomorphic to the vector space $\mathrm{Ext}^1(E_0, E_0)$ and the obstructions to extending an infinitesimal deformation lie in $\mathrm{Ext}^2(E_0, E_0)$.

From [OT; Theorem 4.1], we get:

$$\dim\ \mathrm{Ext}^1(E_0, E_0) = 4k(3n-1) + (2n-5)(2n-1)$$

and

$$\dim\ \mathrm{Ext}^2(E_0, E_0) = (k-2)^2\binom{2n-1}{2}.$$

Let $M^0 \subset MI_{2n+1}(k)$ be the irreducible component containing special instanton bunles on $\mathbf{P}^{2n+1}$ with second Chern class k. By [AO; Corollary 3.10] a generic

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vector bundle $E \in M^0$ is stable. Hence for proving Theorem 3.1 it is enough to construct a vector bundle $E \in M^0$ with $\dim\ \mathrm{Ext}^1(E, E) < \dim\ \mathrm{Ext}^1(E_0, E_0)$. Let us denote by E_u the special instanton bundle on $\mathbf{P}^{2n+1}$ with second Chern class k defined as the cohomology bundle of the following monad:

$$0 \to k\mathcal{O}(-1) \stackrel{A_u}{\to} (2n+2k)\mathcal{O} \stackrel{B}{\to} k\mathcal{O}(1) \to 0, \qquad (*)$$

where

$$B^t := \begin{pmatrix} x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n & & & \\ & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n & & \\ & & & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \end{pmatrix}$$

and

$$A_u := \begin{pmatrix} & & & y_n & \dots & y_1 & y_0 & & & & -x_n & \dots & -x_1 & -x_0 \\ & & y_n & \dots & y_1 & y_0 & & & & -x_n & \dots & -x_1 & -x_0 & \\ M_n & \dots & M_1 & M_0 & & & M & N_n & \dots & N_1 & N_0 & & & N \end{pmatrix},$$

where $M_i = (1-u)y_i + uy_{i-1}$ for $i = 1, \dots, n$, $M_0 = (1-u)y_0$, $M = uy_n$ $N_i = (u-1)x_i - ux_{i-1}$ for $i = 1, \dots, n$, $N_0 = (u-1)x_0$ and $N = -ux_n$.

We have constructed a flat family $\{E_u\}_{u \in \mathbf{k}}$ of special instanton bundles on $\mathbf{P}^{2n+1}$ with second Chern class k which is a deformation of E_0 in M^0. Using lemma 3.2 and the above formula $(***)$ we get:

$$\dim\ \mathrm{Ext}^2(E_{u=1}, E_{u=1}) < \dim\ \mathrm{Ext}^2(E_0, E_0)$$

or, equivalently,

$$\dim\ \mathrm{Ext}^1(E_{u=1}, E_{u=1}) < \dim\ \mathrm{Ext}^1(E_0, E_0).$$

This gives us that $MI_{2n+1}(k)$ is singular at least in the special symplectic instanton bundle E_0.

LEMMA 3.2. *For all integers $k \geqslant 3$ and $n \geqslant 2$, we take the (k)x(2n+2k) matrices:*

$$B(k)^t := \begin{pmatrix} x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n & & & \\ & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n & & \\ & & & x_0 & x_1 & \dots & x_n & & & & y_0 & y_1 & \dots & y_n \end{pmatrix}$$

and

$$A_u(k) := \begin{pmatrix} & & & y_n & \dots & y_1 & y_0 & & & & -x_n & \dots & -x_1 & -x_0 \\ & & y_n & \dots & y_1 & y_0 & & & & -x_n & \dots & -x_1 & -x_0 & \\ M_n & \dots & M_1 & M_0 & & & M & N_n & \dots & N_1 & N_0 & & & N \end{pmatrix},$$

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where $M_i = (1-u)y_i + uy_{i-1}$ *for* $i = 1, \dots, n$, $M_0 = (1-u)y_0$, $M = uy_n$, $N_i = (u-1)x_i - ux_{i-1}$ *for* $i = 1, \dots, n$, $N_0 = (u-1)x_0$ *and* $N = -ux_n$.

Assume that $A_u(k)$ *(resp.* $B(k)^t$*) is the matrix which presents a module* $P_u(k)$ *(resp.* $Q(k)$*). We define* $M_u(k) := M(A_u(k), B(k)^t)$ *as the matrix which presents the tensor product of* $P_u(k)$ *and* $Q(k)$. *Then, we have:*

(1) $syz_1(M_{u=0}(k)) = (k-2)^2(2n-1)(n-1) - k(2n^2k - 3nk - 5k - 8n^2 - 8n)$;
(2) $syz_1(M_{u=1}(k)) < syz_1(M_{u=0}(k))$.

Proof. (1) It follows from [OT; Theorem 4.1] and the above formula (***).

(2) We denote by $S_1(M_u(k))$ the **k**-vector space of the syzygies of $M_u(k)$ of degree 1. We consider the **k**-linear map

$$\Psi_k \colon S_1(M_{u=1}(k)) \to S_1(M_{u=0}(k-1))$$

which sends $v = (P_1^1, \dots, P_1^{k+n}, P_1^{k+n+1}, \dots, P_1^{2k+2n}; \dots; P_k^1, \dots, P_k^{k+n}, P_k^{k+n+1}, \dots, P_k^{2k+2n}; P_{k+1}^1, \dots, P_{k+1}^{k+n}, P_{k+1}^{k+n+1}, \dots, P_{k+1}^{2k+2n}; \dots; P_{2k}^1, \dots, P_{2k}^{k+n}, P_{2k}^{k+n+1}, \dots, P_{2k}^{2k+2n}) \in S_1(M_{u=1}(k))$ to $\Psi_k(v) := (P_1^2, \dots, P_1^{k+n}, P_1^{k+n+2}, \dots, P_1^{2k+2n}; \dots; P_{k-1}^2, \dots, P_{k-1}^{k+n}, P_{k-1}^{k+n+2}, \dots, P_{k-1}^{2k+2n}; P_{k+1}^1, \dots, P_{k+1}^{k+n-1}, P_{k+1}^{k+n+1}, \dots, P_{k+1}^{2k+2n-1}; \dots; P_{2k-1}^1, \dots, P_{2k-1}^{k+n-1}, P_{2k-1}^{k+n+1}, \dots, P_{2k-1}^{2k+2n-1}) \in S_1(M_{u=0}(k-1))$ (The indexing P_i^j has been chosen accordingly to the shape of the matrix M_u).

CLAIM. $\dim \operatorname{Ker}\Psi_k < 12k + 20n - 10$.

Proof of the claim. For all integers $i = 1, \dots, 2k$ and $j = 1, \dots, 2k+2n$, we consider the linear forms $P_i^j := X_{0,i}^j x_0 + \cdots + X_{n,i}^j x_n + Y_{0,i}^j y_0 + \cdots + Y_{n,i}^j y_n \in \mathbf{k}[x_0, \dots, x_n, y_0, \dots, y_n]$. Notice that $Ker(\Psi_k)$ is the **k**-vector space generated by the vectors $v = (P_1^1, \dots, P_1^{2k+2n}; P_2^1, \dots, P_2^{2k+2n}; \dots; P_{2k}^1, \dots, P_{2k}^{2k+2n})$ satisfying the following conditions:

(1) $P_i^j = 0$ for $1 \leqslant i \leqslant k-1$ and $j \in \{1, \dots, 2k+2n\} \setminus \{1, k+n+1\}$
(2) $P_i^j = 0$ for $k+1 \leqslant i \leqslant 2k-1$ and $j \in \{1, \dots, 2k+2n\} \setminus \{k+n, 2k+2n\}$
(3) $v \in S_1(M_{u=1}(k))$

On the other hand condition (3) is equivalent to condition:

$$(3')\, v(M_{u=1}(k)) = (0, \dots, 0)$$

Therefore, $Ker(\Psi_k)$ is the **k**-vector space generated by the vectors $v := (P_1^1, \dots, P_1^{2k+2n}; P_2^1, \dots, P_2^{2k+2n}; \dots; P_{2k}^1, \dots, P_{2k}^{2k+2n})$ verifying:

(1) $P_i^j = 0$ for $1 \leqslant i \leqslant k-1$ and $j \in \{1, \dots, 2k+2n\} \setminus \{1, k+n+1\}$
(2) $P_i^j = 0$ for $k+1 \leqslant i \leqslant 2k-1$ and $j \in \{1, \dots, 2k+2n\} \setminus \{k+n, 2k+2n\}$

and the following $2k-1$ equations:

(3a) $P_i^1 y_{n-1} + P_i^{k+n+1}(-x_{n-1}) + P_{2k}^i x_0 + \cdots + P_{2k}^{i+n} x_n + P_{2k}^{k+n+i} y_0 + \cdots + P_{2k}^{k+i+2n} y_n = 0$ for $i = 1, \ldots, k-1$;

(3b) $P_k^{k+1-i} y_n + \cdots + P_k^{k+n+1-i} y_0 + P_k^{2k+n+1-i}(-x_n) + \cdots + P_k^{2n+2k+1-i}(-x_0) + P_{k+i}^{k+n} x_n + P_{k+i}^{2k+2n} y_n = 0$ for $i = 1, \ldots, k-1$;

(3c) $P_k^1 y_{n-1} + P_k^2 y_{n-2} + \cdots + P_k^n y_0 + P_k^{n+k} y_n + P_k^{k+n+1}(-x_{n-1}) + P_k^{k+n+2}(-x_{n-2}) + \cdots + P_k^{k+2n}(-x_0) + P_k^{2n+2k}(-x_n) + P_{2k}^k x_0 + \cdots + P_{2k}^{n+k} x_n + P_{2k}^{2k+n} y_0 + \cdots + P_{2k}^{2k+2n} y_n = 0$.

The $k-1$ equations (3a) give rise to the following relations among the coefficients of the linear forms $P_i^j := X_{0,i}^j x_0 + \cdots + X_{n,i}^j x_n + Y_{0,i}^j y_0 + \cdots + Y_{n,i}^j y_n \in \mathbf{k}[x_0, \ldots, x_n, y_0, \ldots, y_n]$ $(i = 1, \ldots, k-1)$:

$$X_{\gamma,2k}^{i+\beta} + X_{\beta,2k}^{i+\gamma} = 0 \text{ for } 1 \leqslant \beta \leqslant \gamma \leqslant n, \beta \neq n-1 \quad \text{and} \quad \gamma \neq n-1;$$

$$-X_{\beta,i}^{k+n+1} + X_{n-1,2k}^{i+\beta} + X_{\beta,2k}^{i+n-1} = 0 \text{ for } 1 \leqslant \beta < n-1;$$

$$-X_{n-1,i}^{k+n+1} + X_{n-1,2k}^{i+n-1} = 0;$$

$$-X_{n,i}^{k+n+1} + X_{n,2k}^{i+n-1} + X_{n-1,2k}^{i+n} = 0;$$

$$Y_{\gamma,2k}^{i+\beta} + X_{\beta,2k}^{k+n+i+\gamma} = 0 \text{ for } 1 \leqslant \beta, \gamma \leqslant n, \beta \neq n-1 \quad \text{and} \quad \gamma \neq n-1;$$

$$-Y_{\gamma,i}^{k+n+1} + Y_{\gamma,2k}^{i+n-1} + X_{n-1,2k}^{k+n+i+\gamma} = 0 \text{ for } 1 \leqslant \gamma < n \quad \text{and} \quad \gamma \neq n-1;$$

$$X_{\beta,i}^1 + Y_{n-1,2k}^{i+\beta} + X_{\beta,2k}^{k+2n+i-1} = 0 \text{ for } 1 \leqslant \beta \leqslant n \quad \text{and} \quad \beta \neq n-1;$$

$$X_{n-1,i}^1 - Y_{n-1,i}^{k+n+1} + Y_{n-1,2k}^{i+n-1} + X_{n-1,2k}^{k+2n+i-1} = 0;$$

$$Y_{\gamma,2k}^{k+n+i+\beta} + Y_{\beta,2k}^{k+n+i+\gamma} = 0 \text{ for } 1 \leqslant \beta \leqslant \gamma \leqslant n, \beta \neq n-1 \quad \text{and} \quad \gamma \neq n-1;$$

$$Y_{\beta,i}^1 + Y_{n-1,2k}^{k+n+i+\beta} + Y_{\beta,2k}^{k+2n+i-1} = 0 \quad \text{for } 1 \leqslant \beta < n-1;$$

$$Y_{n-1,i}^1 + Y_{n-1,2k}^{k+2n+i-1} = 0;$$

$$Y_{n,i}^1 + Y_{n,2k}^{k+2n+i-1} + Y_{n-1,2k}^{k+2n+i} = 0;$$

The $k-1$ equations (3b) give rise to the following relations $(i = 1, \ldots, k-1)$:

$$X_{\gamma,k}^{2k+2n-i-\beta+1} + X_{\beta,k}^{2k+2n-i-\gamma+1} = 0 \quad \text{for } 1 \leqslant \beta \leqslant \gamma < n;$$

$$Y_{\gamma,k}^{k+n-i-\beta+1} + Y_{\beta,k}^{k+n-i-\gamma+1} = 0 \quad \text{for } 1 \leqslant \beta \leqslant \gamma < n;$$

$$X_{\beta,k}^{k+n-i-\gamma+1} - Y_{\gamma,k}^{2k+2n-i-\beta+1} = 0 \quad \text{for } 1 \leqslant \beta \leqslant \gamma < n;$$

$$-X_{\beta,k}^{2k+n+1-i} - X_{n,k}^{2k+2n-i-\beta+1} + X_{\beta,k+i}^{k+n} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$-X_{n,k}^{2k+n-i+1} + X_{n,k+i}^{k+n} = 0;$$

$$Y_{\beta,k}^{k+1-i} + Y_{n,k}^{k+n-i-\beta+1} + Y_{\beta,k+i}^{2k+2n} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$Y_{n,k}^{k-i+1} + Y_{n,k+i}^{2k+2n} = 0;$$

$$X_{\beta,k}^{k+1-i} - Y_{n,k}^{2k+2n-i-\beta+1} + X_{\beta,k+i}^{2k+2n} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$X_{n,k}^{k+n-\beta+1-i} - Y_{\beta,k}^{2k+n-i+1} + Y_{\beta,k+i}^{k+n} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$X_{n,k}^{k+1-i} - Y_{n,k}^{2k+n-i+1} + Y_{n,k+i}^{k+n} + X_{n,k+i}^{2k+2n} = 0.$$

Finally, from the equation (3c) we obtain the following relations:

$$-X_{\gamma,k}^{k+2n-\beta} - X_{\beta,k}^{k+2n-\gamma} + X_{\gamma,2k}^{k+\beta} + X_{\beta,2k}^{k+\gamma} = 0 \quad \text{for } 1 \leqslant \beta \leqslant \gamma < n;$$

$$-X_{n,k}^{k+2n-\beta} - X_{\beta,k}^{2k+2n} + X_{n,2k}^{k+\beta} + X_{\beta,2k}^{k+n} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$-X_{n,k}^{2k+2n} + X_{n,2k}^{k+n} = 0;$$

$$Y_{\gamma,k}^{n-\beta} + Y_{\beta,k}^{n-\gamma} + Y_{\gamma,2k}^{2k+n+\beta} + X_{\beta,2k}^{2k+n+\gamma} = 0 \quad \text{for } 1 \leqslant \beta \leqslant \gamma < n;$$

$$Y_{n,k}^{n-\beta} + Y_{\beta,k}^{k+n} + Y_{n,2k}^{2k+n+\beta} + Y_{\beta,2k}^{2k+2n} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$Y_{n,k}^{k+n} + Y_{n,2k}^{2k+2n} = 0;$$

$$X_{\beta,k}^{n-\gamma} + X_{\beta,2k}^{2k+n+\gamma} - Y_{\gamma,k}^{k+2n-\beta} + Y_{\gamma,2k}^{k+\beta} = 0 \quad \text{for } 1 \leqslant \beta, \gamma < n;$$

$$X_{\beta,k}^{n+k} + X_{\beta,2k}^{2k+2n} - Y_{n,k}^{k+2n-\beta} + Y_{n,2k}^{k+\beta} = 0 \quad \text{for } 1 \leqslant \beta < n;$$

$$X_{n,k}^{n-\gamma} + X_{n,2k}^{2k+n+\gamma} - Y_{\gamma,k}^{2k+2n} + Y_{\gamma,2k}^{k+n} = 0 \quad \text{for } 1 \leqslant \gamma < n;$$

$$X_{n,k}^{k+n} + X_{n,2k}^{2k+2n} - Y_{n,k}^{2k+2n} + Y_{n,2k}^{k+n} = 0.$$

Putting altogether, we get, after an intricate computation, that $\dim \operatorname{Ker} \Psi_k < 12k + 20n - 10$, which proves our claim.

Using the **k**-linear map Ψ_k we obtain:

$$\begin{aligned}
syz_1(M_{u=1}(k)) &= \dim S_1(M_{u=1}(k)) = \dim(Ker\Psi_k) + \dim \operatorname{Im}(\Psi_k) \\
&\leqslant \dim(\operatorname{Ker} \Psi_k) + syz_1(M_{u=0}(k-1)) < (\text{Claim}) \\
&< 12k + 20n - 10 + syz_1(M_{u=0}(k-1)) = (\text{Lemma 3.2(1)}) \\
&= \binom{2n-1}{2}(k-2)^2 - k(2n^2k - 3nk - 5k - 8n^2 - 8n) \\
&= syz_1(M_{u=0}(k))
\end{aligned}$$

which gives what we want.

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Remark 3.2.1. For our purpose it is enough to see that $syz_1(M_{u=1}(k)) < syz_1(M_{u=0}(k))$ and we do not need to know the precise value of $syz_1(M_{u=1}(k))$. After computing the first cases using the computer program 'Macaulay' [BS], we guess:

$$syz_1(M_{u=1}(k)) = syz_1(M_{u=0}(k)) - (4n-5)(k-2).$$

References

[AO] V. Ancona and G. Ottaviani, *Stability of special symplectic instanton bundles on* $\mathbf{P}^{2n+1}$ TAMS 341 (1994), 677–693.

[AO1] V. Ancona and G. Ottaviani, *On moduli of instanton bundles on* $\mathbf{P}^{2n+1}$ Pac. J. Math. To appear (1995).

[B1] W. Barth, *Some properties of stable rank 2 vector bundles on* $\mathbf{P}^n$ Math. Ann. 226 (1977), 125–150.

[B2] W. Barth, *Irreducibility of the space of mathematical instanton bundles with rank 2 and* $c_2 = 4$, Math. Ann. 258 (1981), 81–106.

[BS] D. Bayer and M. Stillman, *Macaulay, a computer program for Commutative Algebra and Algebraic Geometry*.

[ES] G. Ellingsrud and S. A. Stromme, *Stable rank 2 bundles on* $\mathbf{P}^3$ *with* $c_1 = 0$ *and* $c_2 = 3.$, Math. Ann. 255 (1991), 123–135.

[G] R. Godement, *Théorie des faisceaux*, Ed. Hermann, (1958).

[H1] R. Hartshorne, *Algebraic Geometry*, GTM 52 (1977).

[H2] R. Hartshorne, *Stable vector bundles of rank 2 on* $\mathbf{P}^3$, Math. Ann. 238 (1978), 229–280.

[HN] A. Hirschowitz and M.S. Narasimhan, *Fibrés de 't Hooft speciaux et applications*, Proc. Nice Conf., Birkhauser (1981).

[LP] J. Le Potier, *Sur l'espace de modules des fibrés de Yang et Mills*, Prog. Math. 37 (1983), 65–137.

[NT] T. Nüssler and G. Trautmann, *Multiple Koszul structures on lines and instanton bundles*, Int. J. Math. 5 (1994), 373–388

[OSS] C. Okonek, M. Schneider and H. Spindler, *Vector bundles over complex projective spaces*, Progr. Math. 3 (1980).

[OS] C. Okonek and H. Spindler, *Mathematical Instanton Bundles on* $\mathbf{P}^{2n+1}$, Crelle J. 364 (1986), 35–50.

[OT] G. Ottaviani and G. Trautmann, *The tangent Space at a Special Symplectic Instanton Bundle on* $\mathbf{P}^{2n+1}$, Manusc. Math. 85 (1994), 97–107.

[ST] H. Spindler and G. Trautmann, *Special Instanton Bundles on* $\mathbf{P}^{2n+1}$*, their geometry and their moduli*, Math. Ann. 286 (1990), 35–50.