A PROPERTY OF META-ABELIAN EXTENSIONS

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Let k be an algebraic number field of finite degree, $\mathbf{A}$ the maximal abelian extension over k, and M a meta-abelian field over k of finite degree, that is, M/k be a normal extension over k of finite degree with an abelian group as commutator group of its Galois group. Then $\mathbf{A}M$ is a kummerian extension over $\mathbf{A}$. If its kummerian generators are obtained from a subfield K of $\mathbf{A}$, namely if there exist elements $a_1, \ldots, a_t$ of K such that $\mathbf{A}M = \mathbf{A}(\sqrt[m_1]{a_1}, \ldots, \sqrt[m_t]{a_t})$, then we shall call M a *meta-abelian field over k attached to K*. If furthermore there exist $b_1, \ldots, b_s$ of K such that $\mathbf{A}M = \mathbf{A}(\sqrt[n_1]{b_1}, \ldots \sqrt[n_s]{b_s})$ and M contains all n_i-th roots of unity $(i = 1, \ldots, s)$, then we shall call M a *K-meta-abelian field over k* and $b_1, \ldots, b_s$ *M-reduced elements* of K. For k-meta-abelian fields over k, we have in [2] the decomposition law of primes of k in M.[1] The purpose of the present paper is to show that this decomposition law is effective also for meta-abelian fields over k attached to k, or more exactly these fields are already k-meta-abelian fields over k. We shall have a little more generally the following

THEOREM. *If M is a meta-abelian field over k attached to K, then MK is a K-meta-abelian field over k.*

In order to prove the theorem it is sufficient to observe the case where K is equal to k. Now let M be a meta-abelian field over k attached to k, $\mathbf{A}_0$ the largest abelian subfield of M, and M_i a cyclic subfield of M over $\mathbf{A}_0$ whose degree is a power of a prime l. Then there exists an element a_i of k such

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1) The symbol $\left[\frac{a}{\mathfrak{p}}\right]_n$ is not defined in [2] for the case $r=0$. Therefore to state the decomposion law it is necessry that $r \geqq 1$, namely k containes all l-th roots of unity. But if we define $\left[\frac{a}{\mathfrak{p}}\right]_n = 1$ or $=0$ according as $a^{(N\mathfrak{p}^{\rho}-1)} \equiv 1$ or $\not\equiv 1$ (mod. $\mathfrak{p}$), then, remarking that lemma 4 in [2] is also true for $r=0$, we have the decomposition law in M/k by means of this symbol also for the case $r=0$. Here $\left[\frac{a}{\mathfrak{p}}\right]_n \left[\frac{b}{\mathfrak{p}}\right]_n = \left[\frac{ab}{\mathfrak{p}}\right]_n$ does not hold when $\left[\frac{a}{\mathfrak{p}}\right]_n = 0$.

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that $\mathsf{A}M_i = \mathsf{A}(\sqrt[l^{n_i}]{a_i})$,[2] and M_i is necessarily normal over k. Since M/A_0 is abelian, M is the composite of such M_i. Therefore in order to prove the theorem we may assume that $\mathsf{A}M = \mathsf{A}(\sqrt[l^n]{a})$ with a single element a of k. Thus the essential point of the proof is the existence of a M-reduced element of k.

First in § **3**, we shall show that a can be so chosen that $\mathsf{A}_0(\sqrt[l^n]{a})/\mathsf{A}_0$ is a kummerian extension (weakly M-reduced element of k). To do this we study in § **1** extensions of the type $F(\sqrt[l^n]{a})/F$ with $a \in F$, which are not necessarily kummerian over any algebraic number field F of finite degree. Here the treatment of the case where $l=2$ and $\sqrt{-1} \notin F$ (radical case) is especially complicated, because the extension obtained by adjoining a 2^n-th root of unity is in general not cyclic. Next in § **4**, we shall study the characters of $\mathsf{A}M/\mathsf{A}_0$, substantially using the normality of M/k. Finally in § **5**, applying the above results, we shall conclude the proof. The results in § **2** are not necessary for the proof of the theorem, but we add them to complete the statement in § **1**.

Throughout this paper the following notations will be used.

P	the rational number field.
k	the ground field which is an algebraic number field of finite degree.
A	the maximal abelian extension over k.
F	any algebraic number field of finite degree.
l	a fixed positive rational prime
$\zeta_{l^n} = \zeta_{(n)}$	a fixed primitive l^n-th root of unity such that $\zeta_{l^n}^l = \zeta_{l^{n-1}}$.
$\zeta'_{l^n} = \zeta'_{(n)}, \zeta''_{(n)}, \ldots$	indefinite primitive l^n-th roots of unity.
$\zeta^*_{l^n} = \zeta^*_{(n)}, \zeta^{**}_{(n)}, \ldots$	indefinite l^n-th roots of unity.
$\sqrt[l^n]{a}$	an l^n-th root of a such that $\sqrt[l^m]{(\sqrt[l^n]{a})} = \sqrt[l^{n+m}]{a}$.
$F_{l^n} = F_{(n)} = F(\zeta_{l^n})$	
F_{l^∞}	the composite of all F_{l^n}, $n = 1, 2, \ldots$.

[2] Because, there exists an element b of k such that $\mathsf{A}M_i = \mathsf{A}(\sqrt[m]{b})$. Let $m = l^u x$, $(x, l) = 1$. Then $\mathsf{A}M_i = \mathsf{A}(\sqrt[m]{b}) \supset \mathsf{A}(\sqrt[l^u]{b}) \supset \mathsf{A}$, and $(\mathsf{A}M_i;\ \mathsf{A}(\sqrt[l^u]{b})) = (\mathsf{A}(\sqrt[l^u \cdot x]{b});\mathsf{A}(\sqrt[l^u]{b}))$ is prime to l. But since $(\mathsf{A}M_i;\ \mathsf{A})$ is equal to a power of l, we have necessarily $\mathsf{A}M_i = \mathsf{A}(\sqrt[m]{b}) = \mathsf{A}(\sqrt[l^u]{b})$.

F^{l^n}	the multiplicative group consisting of all l^n-th powers of non-zero elements of F.
$\mathfrak{G}(K/F)$	the Galois group of a normal extension K/F.
$(K;\ F)$	the relative degree of an extension K/F.

§1. Extensions $F(\sqrt[l^n]{a})/F$ which are not necessarily kummerian

LEMMA 1.1. *Suppose that $a \in F$, $\sqrt[l]{a} \notin F$. Then $F(\sqrt[l^n]{a})$ is a normal extension over F if and only if $F(\sqrt[l^n]{a})$ contains ζ_{l^n}.*

proof. Every conjugate of $\sqrt[l^n]{a}$ over F is of form $\zeta_{(n)}^x \cdot \sqrt[l^n]{a}$. Therefore, if $F(\sqrt[l^n]{a})$ contains $\zeta_{(n)}$, then $F(\sqrt[l^n]{a})$ is obviously normal over F. Now suppose conversely that $F(\sqrt[l^n]{a})$ is normal over F, and that l^m is the greatest common divisor (x, l^n), $\zeta_{(n)}^x \cdot \sqrt[l^n]{a}$ running through all conjugates of $\sqrt[l^n]{a}$ over F. If $m=0$, there exists an integer x such that $(x, l^n)=1$ and consequently $\zeta_{(n)}^x \cdot \sqrt[l^n]{a}/\sqrt[l^n]{a} = \zeta_{(n)}^x \in F(\sqrt[l^n]{a})$. Hence $F(\sqrt[l^n]{a})$ contains $\zeta_{(n)}$. But $m=0$. In fact, $(\sqrt[l^n]{a})^\sigma = \zeta_{(n)}^{l^m y} \cdot \sqrt[l^n]{a}$ for any automorphism σ of $F(\sqrt[l^n]{a})$ over F, and hence $(\sqrt[l^m]{a})^\sigma = ((\sqrt[l^n]{a})^\sigma)^{l^{n-m}} = \sqrt[l^m]{a}$, whence $\sqrt[l^m]{a} \in F$. Hence $m=0$ by the assumption of the lemma.

LEMMA 1.2. *Let σ be an automorphism of $F(\zeta_{(n)})/F$ such that $\zeta_{(r)}^\sigma = \zeta_{(r)}$ and $\zeta_{(r+1)}^\sigma \neq \zeta_{(r+1)}$ for $r<n$. Let further $\bar{\sigma}$ be an integer such that $\zeta_{(n)}^\sigma = \zeta_{(n)}^{\bar{\sigma}}$. Then $l^r \| \bar{\sigma}-1$.*[3]

Proof. From $(\zeta_{(n)}^\sigma)^{l^{n-r}} = (\zeta_{(n)}^{\bar{\sigma}})^{l^{n-r}} = \zeta_{(r)}^{\bar{\sigma}}$ and $(\zeta_{(n)}^\sigma)^{l^{n-r}} = \zeta_{(r)}^\sigma = \zeta_{(r)}$ follows $l^r | \bar{\sigma}-1$. If $l^{r+1} | \bar{\sigma}-1$, $\bar{\sigma}$ can be written $\bar{\sigma} = 1 + xl^{r+1}$ so that $\zeta_{(r+1)}^\sigma = (\zeta_{(n)}^{l^{n-(r+1)}})^\sigma = \zeta_{(n)}^{\bar{\sigma} l^{n-(r+1)}} = \zeta_{(n)}^{l^{n-(r+1)}} \zeta_{(n)}^{xl^n} = \zeta_{(r+1)}$, contradicting the assumption of the lemma.

PROPOSITION 1.1. *Suppose that $\zeta_{(r)} \in F$, $\zeta_{(r+1)} \notin F$ and $F(\sqrt[l^n]{a})/F$ is an abelian extension, a being an element of F. Then $(F_{(n)}(\sqrt[l^n]{a});\ F_{(n)}) \leqq l^r$.*

Proof. Put $\omega = \sqrt[l^n]{a}$, $K = F_{(n)}(\omega)$, $\mathfrak{G} = \mathfrak{G}(K/F)$, $\mathfrak{A} = \mathfrak{G}(K/F_{(n)})$ and $\mathfrak{g} = \mathfrak{G}(F_{(n)}/F)$. Define χ by $\chi(A) = \omega^A/\omega$ for any $A \in \mathfrak{A}$. Then χ generates the character group of $\mathfrak{A}$. For any $\sigma \in \mathfrak{g}$ let U_σ be a prolongation of σ to $\mathfrak{G}$. Then, since $(\omega^{l^n})^{U_\sigma}/\omega^{l^n} = a^\sigma/a = 1$, there exists an l^n-th root of unity $b_\sigma \in F_{(n)}$ such that $\omega^{U_\sigma} = \omega b_\sigma$, and we have

[3] $l^r \| b$ means $l^r | b$ and $l^{r+1} \nmid b$.

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$$\omega^{AU_\sigma} = (\chi(A)\omega)^{U_\sigma} = \chi(A)^\sigma \omega b_\sigma,$$
$$\omega^{U_\sigma A} = (\omega b_\sigma)^A = \chi(A)\omega b_\sigma.$$

Now suppose that K/F is abelian. Then for every $\sigma \in \mathfrak{g}$ and every $A \in \mathfrak{A}$ the above two equations should be equivalent. Hence $\chi(A)^\sigma = \chi(A)$, which means that $\chi(A) \in k$ and $\chi(A)$ is an l^r-th root of unity for every $A \in \mathfrak{A}$. Therefore the order of χ is at most equal to l^r. This implies our assertion.

LEMMA 1.3. *Suppose that* $\zeta_{l^2} \in F$ *and* $a \in F$, $\sqrt[l]{a} \notin F$. *Then* $(F(\sqrt[l^n]{a}); F) = l^n$.

Proof. Obviously $(F(\sqrt[l^n]{a}); F) \leqq l^n$. Now if the inequality occurs, then there is an integer $\nu \geqq 1$ such that $F(\sqrt[l^{\nu-1}]{a}) \subsetneqq F(\sqrt[l^\nu]{a}) = F(\sqrt[l^{\nu+1}]{a})$. So, putting $\sqrt[l^{\nu-1}]{a} = \alpha$ and $K = F(\alpha)$, we see $\sqrt[l]{\alpha} \notin K$ and $K(\sqrt[l^2]{\alpha}) = K(\sqrt[l^{\nu+1}]{a}) = K(\sqrt[l^\nu]{a}) = K(\sqrt[l]{\alpha})$. But since $\zeta_{l^2} \in K$, this equality contradicts a property of kummerian extentions.

LEMMA 1.4. *Let* $a \in F$, $a \notin F^l$. *Then it is neceasary and sufficient for* $F(\sqrt[l]{a})/F$ *to be normal that we have* $\zeta_l \in F$.

Proof. The sufficiency is clear. So we assume that $F(\sqrt[l]{a})/F$ is normal. Then we have $(F(\zeta_l); F) \leqq l-1$, $(F(\sqrt[l]{a}); F) = l$ and by lemma 1.1 $F(\sqrt[l]{a}) \supset F(\zeta_l) \supset F$. Hence $F(\zeta_l) = F$, namely $F \ni \zeta_l$.

Hereafter frequent special treatments about the case $l = 2$ are needed. So we give the following definitions. Denote by $P^0_{2^\nu}$ the largest real subfied of P_{2^ν}. Then P_{2^ν} is the composite of the quadratic field $P_{2^2} = P(\sqrt{-1})$ and the cyclic field $P^0_{2^\nu}$ of degree $2^{\nu-2}$ over P. Furthermore we denote by P'_{2^ν} the intermediate field of $P_{2^\nu}/P^0_{2^{\nu-1}}$ different from both $P^0_{2^\nu}$ and $P_{2^{\nu-1}}$. F is called, a *radical field* if F does not contain $\sqrt{-1}$, and otherwise a *non-radical field.* The *radical case with respect to* F is by definition the case where $l = 2$ and F is a radical field, and the *non-radical case with respect to* F is the case either $l \neq 2$ or F is a non-radical field (even if $l = 2$). If F is a radical field, then there is an integer $T \geqq 2$ such that $F \frown P_{2^\infty} = P'_{2^T}$ or $= P^0_{2^T}$ according as $F \frown P_{2^\infty}$ is imaginary or real. If $F \frown P_{2^\infty}$ is imaginary, i.e., if $F \frown P_{2^\infty} = P'_{2^T}$, then F is called a radical field *of the first kind*, otherwise a radical field *of the second kind.* Putting

$$\lambda_\nu = (\zeta_{2^\nu} + \zeta_{2^\nu}^{-1}) + 2 = (\zeta_{2^{\nu+1}} + \zeta_{2^{\nu+1}}^{-1})^2, \tag{1.1}$$

we call $\lambda_T = \lambda$ the *radical number* of F. Then

(1.2) $$P^0_{2^{\nu+1}} = P^0_{2^\nu}(\sqrt{\lambda_\nu}),\ P'_{2^\nu} = P^0_{2^\nu}(\sqrt{-\lambda_\nu}),$$

(1.3) $$\zeta_{2^\nu}\lambda_\nu = (1+\zeta_{2^\nu})^2.$$

In the case of F being a radical field of the first kind

(1.4) $$F(\zeta_{2^2}) \equiv F(\zeta_{2^3}) = \cdots = F(\zeta_{2^{T+1}}),\quad F(\zeta_{2^\nu}) \neq F(\zeta_{2^{\nu+1}})\quad \text{for } \nu \geqq T+1;$$

in the case of F being of the second kind

(1.5) $$F(\zeta_{2^2}) = F(\zeta_{2^3}) = \cdots = F(\zeta_{2^T}),\quad F(\zeta_{2^\nu}) \neq F(\zeta_{2^{\nu+1}})\quad \text{for } \nu \geqq T;$$

and in both cases

(1.6) $$F_\frown P_{2^T} = P^0_{2^T}.$$

Now let l be again an odd or even fixed rational prime, and $F^{(\nu)}$ the multiplicative group of all elements a of F such that $\sqrt[l^\nu]{a} \in F_{(\nu)}$. Then from Hasse [3] we have

LEMMA 1.5. *In either one of the following two cases* $F^{(\nu)} = F^{l^\nu}$;
i) *the non-radical case,*
ii) *the radical case with* $\nu \geqq T+1$.
In other cases the facter group $F^{(\nu)}/F^{2^\nu}$ *is of order* 2 *and its non-trivial coset is represented by* $\lambda_T^{2^{\nu-1}}$ *or* $-\lambda_\nu^{2^{\nu-1}}$ *according as*
iii) $\nu \geqq T+1$ *in the radical case of the second kind, or*
iv) $2 \leqq \nu \leqq T$ *in the radical case of the firet or second kind.*

Let us proceed our consideration about $F(\sqrt[l^n]{a})/F$.

PROPOSITION 1.2. *In the non-radical case with respect to* F, *if* $a \in F$ *and* $\notin F^l$, *then* $(F(\sqrt[l^n]{a});\ F) = l^n$.

Proof. First assume that $\sqrt[l]{a} \notin F_{(2)}$. Then $(F_{(2)}(\sqrt[l^n]{a});\ F_{(2)}) = l^n$ by lemma 1.3. Since $(F_{(2)}(\sqrt[l^n]{a});\ F_{(2)}) \leqq (F(\sqrt[l^n]{a});\ F) \leqq l^n$, we have $(F(\sqrt[l^n]{a});\ F) = l^n$. Nexst assume that $\sqrt[l]{a} \in F_{(2)}$. Then $F(\sqrt[l]{a}) \subset F_{(2)}$ and hence $F(\sqrt[l]{a})/F$ is a normal extension. Therefore $F \ni \zeta_{(1)}$ by lemma 1.4. This implies that $(F_{(2)};\ F) = l$ and further $F(\sqrt[l]{a}) = F_{(2)}$. Hence $\sqrt[l^2]{a} \notin F_{(2)}$. For, if $\sqrt[l^2]{a} \in F_{(2)}$ then $a \in F^{l^2}$ by lemma 1.5, contradicting the assumption. Now $(F(\sqrt[l^n]{a});\ F_{(2)}) = (F_{(2)}(\sqrt[l^n]{a});\ F_{(2)}) = l^{n-1}$ by lemma 1.3, and hence $(F(\sqrt[l^n]{a});\ F) = l^n$, owing to $(F_{(2)};\ F) = l$ as was seen above.

LEMMA 1.6. *If* $n \geqq 1$, $a \in F$ *and* $F(\sqrt[2^n]{a}) \neq F(\sqrt{-1})$, *then* $(F(\sqrt[2^n]{a});$

$(\sqrt[2^{n-1}]{a}))=2.$

Proof. Put $K=F(\sqrt{-1})$ and suppose that $\sqrt[2^\nu]{a}\in$, $\sqrt[2^{\nu+1}]{a}\notin K$ $(0\leqq\nu<n)$. Then, since K is a non-radical field, $(K(\sqrt[2^n]{a})\,;\,K)=2^{n-\nu}$ and $(K(\sqrt[2^{n-1}]{a})\,;\,K)=2^{n-\nu-1}$ by prop. 1.2. Therefore $2=(K(\sqrt[2^n]{a})\,;\,K(\sqrt[2^{n-1}]{a}))\leqq(F(\sqrt[2^n]{a})\,;\,F(\sqrt[2^{n-1}]{a}))\leqq 2$, which implies our assertion.

The followung proposition holds in both cases, radical or non-radical.

Proposition 1.3. *If $F(\sqrt[l^n]{a})\supset F_{(s)}$ with $a\in F$, then there exists an integer u such that $F(\sqrt[l^u]{a})=F_{(s)}$. If moreover $\sqrt[l]{a}\notin F$, $\zeta_{(r)}\in F$, $\zeta_{(r+1)}\notin F$ and $s>r$, then $F(\sqrt[l]{a})=F_{(r+1)}$.*

Proof. First we prove the latter. Suppose that $F(\sqrt[l]{a})\neq F_{(r+1)}$. Then $\sqrt[l]{a}\notin F_{(r+1)}$. Because, if $r\geqq 1$ then crearly $\sqrt[l]{a}\notin F_{(r+1)}$; and if $r=0$, $\sqrt[l]{a}\in F_{r+1}$, then $\zeta_{(1)}\in F$ by lemma 1.4, which is a contradiction. Now using $F_{(r+1)}$ instead of F in Prop. 1.2 we have $(F_{(r+1)}(\sqrt[l^n]{a})\,;\,F_{(r+1)})=l^n$. Then from the assumption of the proposition follows $(F(\sqrt[l^n]{a})\,;\,F)=(F_{(r+1)}(\sqrt[l^n]{a})\,;\,F_{(r+1)})(F_{(r+1)}\,;\,F)>l^n$. On the other hand $(F(\sqrt[l^n]{a})\,;\,F)\leqq l^n$. This is a contradiation. Hence $F(\sqrt[l]{a})=F_{(r+1)}$, which is the latter assertion of the proposition. Nèxt we prove the former. i) The non-radical case with respect to F: Let $\sqrt[l^u]{a}\in F_{(s)}$, $\sqrt[l^{u+1}]{a}\notin F_{(s)}$, then $F(\sqrt[l^n]{a})\supset F_{(s)}\supset F(\sqrt[l^u]{a})$ and by prop. 1.2 $(F(\sqrt[l^n]{a})\,;\,F(\sqrt[l^u]{a}))=(F(\sqrt[l^n]{a})\,;\,F_{(s)})=l^{n-u}$, Hence $F(\sqrt[l^u]{a})=F_{(s)}$, ii) The radical case with respect to F (i.e., $l=2$ and $r=1$): If $s=1$, i.e., $\zeta_{2^s}=-1$, then our assertion is trivially true, and so assume that $s\geqq 2$. Then by the latter assertion of the proposition, already proved above, we have $F(\sqrt{a})=F(\sqrt{-1})$. Therefore if we take $F(\sqrt{-1})$ resp. $\sqrt{a}$ instead of F resp. a, this case is reduced to the non-radical case i).

Now in the non-radical case we have the following two propositions.

Proposition 1.4. *In the non-radical case with respect to F, suppose that $\zeta_{(r)}\in F$, $\zeta_{(r+1)}\notin F$; and $a\in F$, $\sqrt[l]{a}\notin F$. Then $F(\sqrt[l^n]{a})=F(\sqrt[l^m]{\zeta_{(r)}})$ implies $m=n$.*

Proof. If $n=0$ our assertion is trivial. If $n\geqq 1$ then $r\geqq 1$ by lemma 1.4, and $\sqrt[l]{\zeta_{(r)}}\notin F$ by the assumption, from which our assertion follows immediately by prop. 1.2.

Proposition 1.5. *In the non-radical case with respect to F, suppose that $\zeta_{(r)}\in F$, $\zeta_{(r+1)}\notin F$; $a\in F$ and $a\notin F^l$. Then in order that $F(\sqrt[l^n]{a})=F(\sqrt[l^n]{\zeta_{(r)}})$ it* *is necessary and sufficient that* $a=\zeta'_{(r)}b^{l^n}$ *with some* $b\in F$.

Proof. The sufficiency is clear. In order to prove the necessity we may assume that $n\geqq 1$ and further $r\geqq 1$ by lemma 1.4. When $n=1$ the assertion is clear, because $r\geqq 1$ and $F(\sqrt[l]{a})=F(\sqrt[l]{\zeta_{(r)}})$ is a kummerian extension. Assume that $n\geqq 2$, the assertion is true for 1, 2, . . . , $n-1$, and further that $F(\sqrt[l^n]{a})=F(\sqrt[l^n]{\zeta_{(r)}})$. Then prop. 1.3 and prop. 1.4 imply $F(\sqrt[l^{n-1}]{a})=F(\sqrt[l^{n-1}]{\zeta_{(r)}})$, hence $a=\zeta'_{(r)}b^{l^{n-1}}$ with some $b\in F$ by the assumtion of the induction. Now if $\sqrt[l]{b}\in F$, then our assertion hase been already proved. So we may assume that $\sqrt[l]{b}\notin F$. Then $\sqrt[l]{b}\in F(\sqrt[l^n]{\zeta_{(r)}})$, since $F(\sqrt[l^n]{\zeta_{(r)}})=F(\sqrt[l^n]{a})=F(\sqrt[l^n]{\zeta'_{(r)}b^{l^{n-1}}})=F(\sqrt[l^n]{\zeta'_{(r)}}\cdot\zeta^*_{(n)}\cdot\sqrt[l]{b})$, and further $F(\sqrt[l]{b})=F(\sqrt[l]{\zeta_{(r)}})$, bcause in the non-radical case $F(\sqrt[l^n]{\zeta_{(r)}})/F$ is a cyclic extension. Therefore by the case $n=1$ we have $b=\zeta^*_{(r)}c^l$ with $c\in F$, and so $a=\zeta'_{(r)}(\zeta^*_{(r)}c^l)^{l^{n-1}}=\zeta''_{(r)}c^{l^n}$ with some $c\in F$. This proves the proposition.

In the radical case the following proposition holds, corresponding to prop. 1.5 in the non-radical case.

PROPOSITION 1.6. *In the radical case with respect to F, suppose that* $a\in F$, $\sqrt{a}\notin F$, *and denote by* λ *the radical number* λ_T *of F. Then a necessary and sufficient condition for* $F(\sqrt[2^n]{a})=F(\sqrt[2^n]{-1})$ *is* $a=-b^{2^n}$ *or* $a=-\lambda^{2^{n-1}}c^{2^n}$ *with* $b,\ c\in F$, *the latter occurring only when* $n>T$.[4]

Proof. Necessity. When $n=1$, $a=-b^2$ with $b\in F$, since $F(\sqrt{a})=F(\sqrt{-1})$ is a kummerian extension. Assume that $n\geqq 2$ and the assertion is true by 1,2, . . . , $n-1$, and further that $F(\sqrt[2^n]{a})=F(\sqrt[2^n]{-1})$. First we shall show that $F(\sqrt[2^{n-1}]{a})=F(\sqrt[2^{n-1}]{-1})$. If $F(\sqrt[2^n]{-1})=F(\sqrt{-1})$, then $F(\sqrt[2^{n-1}]{a})=F(\sqrt{a})=F(\sqrt{-1})=F(\sqrt[2^{n-1}]{-1})$, owing to $n\geqq 2$. Next suppose that $F(\sqrt[2^n]{-1})\neq F(\sqrt{-1})$, and put $K=F(\sqrt{-1})$. Then by the latter half of prop. 1.3 we have

$$(1.7)\qquad K=F(\sqrt{a})=K(\sqrt{a}),\ \text{and hence}\ K(\sqrt[2^n]{a})=F(\sqrt[2^n]{a})=K(\sqrt[2^n]{-1}).$$

Furthermore $(K(\sqrt[2^n]{a});\ K(\sqrt[2^{n-1}]{a}))=2$ and $(K(\sqrt[2^n]{-1});\ K(\sqrt[2^{n-1}]{-1}))=2$ by lemma 1.6. On the other hand, since $K(\sqrt[2^{n-1}]{-1})/K$ is cyclic, $K(\sqrt[2^{n-1}]{-1})=K(\sqrt[2^{n-1}]{a})=F(\sqrt[2^{n-1}]{a})$ and $K(\sqrt[2^{n-1}]{-1})=F(\sqrt[2^{n-1}]{-1})$ by the construction of K. This implies that $F(\sqrt[2^{n-1}]{a})=F(\sqrt[2^{n-1}]{-1})$. Therefore by the assumption of

[4] In the case where F is a radical field of the first kind, we have only $a=-b^{2^n}$, since $\sqrt{-\lambda}\in F$ for $n>T$ $(\geqq 2)$.

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the induction we have $a=-b^{2^{n-1}}$ or $a=-\lambda^{2^{n-1}}c^{2^{n-1}}$ with some $b,\ c\in F$. But the latter does not occur. For if F is a radical field of the first kind, we have $\sqrt{-\lambda}\in F$ by (1.2), and so $a=-\lambda^{2^{n-1}}c^{2^{n-1}}=-(\sqrt{-\lambda}\cdot c)^{2^{n-1}}$ for $n\geqq 3$, where $\sqrt{-\lambda}\cdot c\in F$. Hence we may write $a=-b^{2^{n-1}}$ for $n\geqq 3$. For $n=2$ we have $a=-\lambda c^2=(\sqrt{-\lambda}\cdot c)^2$, which contradicts the assumption of a. Next if F is of the second kind and $a=-\lambda^{2^{n-1}}c^{2^{n-1}}$, then $F(\sqrt[2^n]{-1})=F(\sqrt[2^n]{a})=F(\sqrt[2^n]{-1}\cdot\zeta^*_{(n)}\cdot\sqrt[4]{\lambda c^2})$. This implies that $\sqrt[4]{\lambda c^2}\in F(\sqrt[2^n]{-1})$, hence $F(\sqrt[4]{\lambda c^2})/F$ is a normal extension. Since $F(\sqrt{\lambda c^2})=F(\sqrt{\lambda})\neq F$, we have $F(\sqrt[4]{\lambda c^2})\supset F(\sqrt{-1})$ by lemma 1.1. Then prop. 1.3 implies that $F(\sqrt{\lambda})=F(\sqrt{-1})$. But this is a contradiction, for F was a radical field of the second kind. Thus in both cases we have $a=-b^{2^{n-1}}$. Now if $\sqrt{b}\in F$, then our proposition has been already proved. So we assume $\sqrt{b}\notin F$. Since $F(\sqrt[2^n]{-1})=F(\sqrt[2^n]{a})=F(\sqrt[2^n]{-1}\cdot\zeta^*_{(n)}\sqrt{b})$, we have $\sqrt{b}\in F(\sqrt[2^n]{-1})$. Hence $F(\sqrt{b})=F(\sqrt{-1}),=F(\sqrt{\lambda})$ or $=F(\sqrt{-\lambda})$, where the latter two equations may occur only when F is a radical field of the second kind. If $F(\sqrt{b})=F(\sqrt{-1})$, then $b=-c^2$, hence $a=-c^{2^n}$ with $c\in F$. If $F(\sqrt{b})=F(\sqrt{\lambda})$ or $=F(\sqrt{-\lambda})$ then $b=\lambda c^2$ or $=-\lambda c^2$, hence $a=-\lambda^{2^{n-1}}c^{2^n}$ with $c\in F$. This concludes the proof of the necessity of the proposition.

Sufficiency. The sufficiency of $a=-b^{2^n}$ is clear. Let $a=-\lambda^{2^{n-1}}c^{2^n}$ with $c\in F$ and $n>T$. Then we may assume that F is a radical field of the second kind.[5] Hence $\lambda^{2^{n-1}}$ is by lemma 1.5 a 2^n-th power in $F(\zeta_{(n)})$ and we have $F(\sqrt[2^n]{a})=F(\sqrt[2^n]{a},\ \zeta_{(n)})=F(\sqrt[2^n]{-1},\ \zeta_{(n)})=F(\zeta_{(n+1)})$. Finally if $n\leqq T$ then $a=-\lambda^{2^{n-1}}c^{2^n}$ can not occur. For, if $n=T$ then $F(\sqrt[2^T]{a})=F(\zeta'_{(T+1)}\cdot\sqrt{\lambda})=F(\zeta_{(T)})\neq F(\sqrt[2^T]{-1})$ by (1.3) and (1.5); and if $n<T$ then $\sqrt{\lambda}\notin F(\zeta_{(n+1)})$, hence $\zeta_{(n+1)}\sqrt{\lambda}\notin F(\zeta_{(n+1)})$, which means that $F(\sqrt[2^n]{a})\neq F(\sqrt[2^n]{-1})$.

Thus the proposition is proved.

§ 2. Supplements to the radical case

In order to prove our theorem the following propositions of this § are unnecessary, which are assertions in the radical case, corresponding to prop. 1.4 and prop. 1.5. We state them for the sake of completeness.

First we treat the case of the first kind.

PROPOSITION 2.1. *Let F be a radical field of the first kind and assume*

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5) See footnote 4.

$a \in F$, $\sqrt{a} \notin F$. *Then* $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ *if and only if the following conditions are satisfied:*

i) *When* $1 \leqq m \leqq T$, *we have* $1 \leqq n \leqq T$ *and* $a = -b^{2^n}$ *for* $n < T$, $a = -\lambda^{2^{T-1}} b^{2^T}$ *for* $n = T$,

ii) *When* $m \geqq T+1$, *we have* $n = m$ *and* $a = -b^{2^n}$,

where λ *is the radical number* λ_T *of* F *and* b *is an element of* F.

Proof. i) Suppose that $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ for $1 \leqq m \leqq T$. Then $F(\sqrt[2^n]{a}) \supset F(\zeta_{(n)})$ by lemma 1.1, and hence $T+1 \geqq n \geqq 1$ by the assumption and (1.4). If $n = T+1$, then $F(\sqrt[2^{T+1}]{a}) = F(\sqrt[2^m]{-1}) = F(\zeta_{(T+1)})$, and hence $\sqrt[2^{T+1}]{a} \in F_{(T+1)}$. This contradicts lemma 1.5 ii), since $\sqrt{a} \notin F$. Therefore $T \geqq n \geqq 1$. Let $n = T$; then as above $F(\sqrt[2^T]{a}) = F(\sqrt[2^m]{-1}) = F(\zeta_{(T)})$, and hence $\sqrt[2^T]{a} \in F_{(T)}$. Since $\sqrt{a} \notin F$, lemma 1.5, iv) implies $a = -\lambda^{2^{T-1}} b^{2^T}$ with $b \in F$. Next let $1 \leqq n < T$, then since $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{-1}) = F(\sqrt[2^n]{a})$, prop. 1.6 implies $a = -b^{2^n}$ with $b \in F$. Thus the necessity is proved. The sufficiency follows from lemma 1.5, iv) immediately. ii) First suppose that $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ for $m = T+1$. Similarly to the proof of case i), we have $F(\sqrt[2^n]{a}) \supset F(\zeta_{(n)})$, and hence $T+2 \geqq n$. If $T+2 = n$, then $F(\sqrt[2^{T+2}]{a}) = F(\zeta_{(T+2)})$, hence $a = b^{2^{T+2}}$ with $b \in F$ by lemma 1.5 ii). This contradicts $\sqrt{a} \notin F$. Hence we have $T+1 \geqq n$, and furthermore $T+1 \geqq n \geqq 2$, since $(F(\sqrt[2^{T+1}]{-1}) ; F) = 4$. Since $F(\sqrt[2^n]{a}) = F(\sqrt[2^{T+1}]{-1}) \neq F(\sqrt{-1})$, we have $(F(\sqrt[2^n]{a}) ; F(\sqrt[2^{n-1}]{a})) = 2$ by lemma 1.6, and hence $F(\sqrt[2^{n-1}]{a}) = F(\sqrt{-1}) = F(\sqrt[2^{n-1}]{-1})$, which is the subfield of $F(\sqrt[2^{T+1}]{-1})$ of index 2. Then $a = -b^{2^{n-1}}$ with $b \in F$ by prop. 1.6 and hence $F(\sqrt[2^{T+1}]{-1}) = F(\sqrt[2^n]{a}) = F(\zeta'_{(n+1)}\sqrt{b})$, which implies $\sqrt{b} \in F(\sqrt[2^{T+1}]{-1})$ owing to $n \leqq T+1$ as was seen above. However $F(\sqrt[2^n]{a}) = F(\zeta_{(n+1)}) = F(\sqrt[2^{T+1}]{-1})$, hence $n = T+1$, that is, $m = n$. Next we consider the case of $m > T+1$. It follows from $F(\sqrt[2^n]{a}) = F(\sqrt[2^m]{-1}) \supset F(\sqrt[2^{T+1}]{-1})$ and prop. 1.3 that there exists an intger u such that $F(\sqrt[2^u]{a}) = F(\sqrt[2^{T+1}]{-1})$. Then from the result of the above case where $m = T+1$ we have $u = T+1$, namely $F(\sqrt[2^{T+1}]{a}) = F(\sqrt[2^{T+1}]{-1})$. Furthermore $F(\sqrt[2^{T+1}]{a}) \neq F(\sqrt[2^{T+2}]{a})$ by lemma 1.6. Put $\alpha = \sqrt[2^{T+1}]{a}$ and $K = F(\alpha)$ which is equal to $F(\zeta_{(T+2)})$. Then K is a non-radical field, and contains neither $\zeta_{(T+3)}$ nor $\sqrt{\alpha}$, furthermore $K(\sqrt[2^{m-(T+1)}]{\zeta_{(T+2)}}) = K(\sqrt[2^{n-(T+1)}]{\alpha})$. Then prop. 1.4 implies $m-(T+1) = n-(T+1)$, hence $m = n$. After all $m = n$ when $m \geqq T+1$. Thus our assertion, including the sufficitency, follows from prop. 1.6.

Next we treat the case of the second kind, which is done by the same

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way as in the case of the first kind.

Proposition 2.2. *Let F be a radical field of the second kind and assume $a \in F$, $\sqrt{a} \notin F$. Then $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ if and only if the following conditions are satisfied:*

i) *when $1 \leqq m < T$, we have $1 \leqq n \leqq T$, and $a = -b^{2^n}$ for $1 \leqq n < T$, $a = -\lambda^{2^{T-1}} b^{2^T}$ for $n = T$,*

ii) *when $m = T$, we have $2 \leqq n \leqq T$, and $a = -b^{2^T}$ for $n = T$, $a = -\lambda^{2^{n-1}} b^{2^n}$ for $2 \leqq n < T$,*

iii) *when $m > T$, we have $n = m$, and $a = -b^{2^n}$ or $a = -\lambda^{2^{n-1}} b^{2^n}$,*

where λ is the radical number λ_T of F and b is an element of F.

Proof. i) Suppose that $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ for $1 \leqq m < T$. Then $F(\sqrt[2^n]{a}) \supset F(\zeta_{(n)})$ by lemma 1.1, and hence $T \geqq n \geqq 1$. Then our proof concludes by repeating the same argument as in the proof i) of Prop. 2.1 under the assertion $T \geqq n \geqq 1$. ii) Suppose that $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ for $m = T$. By taking T instead of $T+1$, and using lemma 1.5 iii) instead of lemma 1.5 ii) in the first part of the proof ii) of prop. 2.1, we obtain $T \geqq n \geqq 2$, $a = -b^{2^{n-1}}$ and $\sqrt{b} \in F(\sqrt[2^{T+1}]{-1})$ for some $b \in F$. Then $F(\sqrt{b}) = F$, $= F(\sqrt{-1})$, $= F(\sqrt{\lambda})$ or $= F(\sqrt{-\lambda})$ in the present case where F is of the second kind. If $F(\sqrt{b}) = F$ or $= F(\sqrt{-1})$, we have $n = T$ by fhe same way as in prop. 2.1, and hence by prop. 1.6 $a = -b^{2^T}$. If $F(\sqrt{b}) = F(\sqrt{\lambda})$ or $= F(\sqrt{-\lambda})$, then $b = \pm \lambda c^2$ with $c \in F$, hence $a = -\lambda^{2^{n-1}} c^{2^n}$ owing to $n \geqq 2$. Conversely let $a = -\lambda^{2^{n-1}} c^{2^n}$ with $c \in F$. Then $F(\sqrt[2^n]{a}) = F(\zeta_{(n+1)}\sqrt{\lambda})$. Now $F(\zeta_{(n+1)}\sqrt{\lambda}) = F(\zeta_{(T)})$ or $= F(\zeta_{(T+1)})$ according as $n = T$ or $T > n \geqq 2$. In fact, putting $\zeta_{(n+1)} = \zeta_{(T+1)}^{2^x}$ and $\kappa = \zeta_{(n+1)}\sqrt{\lambda}$, we have $F(\kappa^2) = F(\zeta_{(n)}) = F(\zeta_{(T)})$ owing to $T \geqq n \geqq 2$. Hence we can write $\zeta_{(T)} = f(\kappa^2)$ for some rational function f in F, and so by (1.3) $\pm \kappa = \pm \zeta_{(T+1)}^{2^x}\sqrt{\lambda}$ $= \zeta_{(T+1)}^{2}(\zeta_{(T+1)} + \zeta_{(T+1)}^{-1}) = \zeta_{(T+1)}^{2^x-1}(\zeta_{(T)} + 1) = \zeta_{(T+1)}^{2^x-1}(f(\kappa^2) + 1)$. This implies that $F(\kappa) = F(\zeta_{(T)})$ or $F(\kappa) = F(\zeta_{(T+1)})$ according as $x = 0$ or $x \geqq 1$. iii) Suppose that $F(\sqrt[2^m]{-1}) = F(\sqrt[2^n]{a})$ for $m > T$. It follows from $F(\sqrt[2^n]{a}) = F(\sqrt[2^m]{-1}) \supset F(\sqrt[2^T]{-1})$ and from prop. 1.3 that there exists an integer u such that $F(\sqrt[2^u]{a}) = F(\sqrt[2^T]{-1})$. Let u be the largest integer with such a property. Then $2 \leqq u \leqq T$ by ii). Assume $u < T$, then $a = -\lambda^{2^{u-1}} b^{2^u}$ with $b \in F$ by ii). This follows $F(\zeta_{(m+1)}) = F(\sqrt[2^n]{a}) \supset F(\sqrt[2^{u+1}]{a}) = F(\zeta_{(u+2)} \cdot \sqrt[4]{\lambda b^2})$. Then $F(\zeta_{(m+1)}) \ni \sqrt[4]{\lambda b^2}$, because $m > T > u$ and hence $m + 1 \geqq u + 2$. Therefore $F(\sqrt[4]{\lambda b^2})/F$ is an abelian extension of degree 4, hence $F(\sqrt[4]{\lambda b^2}) \supset F(\sqrt{-1})$ by lemma 1.1. Furthermore $F(\sqrt{\lambda}) = F(\sqrt{-1})$ by prop. 1.3. But this is a contradiction, since we treat the case of the second kind. Hence we have $u = T$, and so $F(\sqrt[2^T]{a}) = F(\sqrt[2^T]{-1})$. Then by repeating the same argument as in the latter part of the proof ii) of prop. 2.1 by using T inatead of $T+1$, we have $m = n$, which implies our assertion by prop. 1.6.

§ 3. First reduction of kummerian generators

LEMMA 3.1. *Let M be a meta-abelian extension over k attached to k such that $\mathbf{A}M = \mathbf{A}(\sqrt[l^n]{a})$ with $a \in k$, $\sqrt[l]{a} \notin k$, Put $\sqrt[l^m]{a} = \alpha$ and $n - m = v$. Suppose that $\zeta_{(t)} \in \mathbf{A}_0 = \mathbf{A} \frown M$, $\zeta_{(t+1)} \notin \mathbf{A}_0$, $\alpha \in \mathbf{A}_0$. If $\alpha = \zeta_{(t)}^{*} \beta^{l^{v-t}}$ with $\beta \in \mathbf{A}_0$, then there exists an elements b of k such that $b \notin k^l$, $\mathbf{A}M = \mathbf{A}(\sqrt[l^\nu]{b})$ and $\sqrt[l^{\nu-t}]{b} \in \mathbf{A}_0$ for some integer ν.*

Proof. It follows from $\sqrt[l^m]{a} \in \mathbf{A}_0$ that $k(\sqrt[l^m]{a})/k$ is an abelian extension, and hence $\mathbf{A}_0 \supset k(\sqrt[l^m]{a}) \supset k(\zeta_{(m)})$ by lemma 1.1. Therefore $t \geqq m$ by the assumption of t. Since $\beta^{l^{v-t}} = \zeta_{(t)}^{**} \alpha$, we have $\beta^{l^v} = a^{l^{t-m}} \in k$. Then there is a positive integer z such that $\beta^{l^{z-1}} \notin k$ and $\beta^{l^z} \in k$. Putting $\beta^{l^z} = b$, we see $\mathbf{A}M = \mathbf{A}(\sqrt[l^n]{a}) = \mathbf{A}(\sqrt[l^v]{\alpha}) = \mathbf{A}(\zeta_{(v+t)}^{*} \cdot \sqrt[l^t]{\beta}) = \mathbf{A}(\sqrt[l^t]{\beta}) = \mathbf{A}(\sqrt[l^{t+z}]{b})$ and $\sqrt[l^z]{b} \in \mathbf{A}_0$, which proves the lemma.

For a meta-abelian extension M over k attached to k such that $\mathbf{A} \frown M = \mathbf{A}_0 \ni \zeta_{(t)}$, we shall call an element a of k *weakly M-reduced*, whenever $a \notin k^l$, $\mathbf{A}M = \mathbf{A}(\sqrt[l^n]{a})$ and $\mathbf{A}_0 \ni \sqrt[l^{n-t}]{a}$.[6]

First, in the non-radical case with respect to $\mathbf{A}_0$, we prove the existence of a weakly M-reduced element.

PROPOSITION 3.1. *Let M be a meta-abelian extension over k attached to k and put $\mathbf{A} \frown M = \mathbf{A}_0$. Then in the non-radical case with respect to $\mathbf{A}_0$, there exists a weakly M-reduced element of k.*

Proof. Suppose that $\zeta_{(t)} \in$, $\zeta_{(t+1)} \notin \mathbf{A}_0$; $\mathbf{A}M = \mathbf{A}(\sqrt[l^n]{a})$ with $a \in k$; and $\sqrt[l^m]{a} \in \mathbf{A}_0$, $\sqrt[l^{m+1}]{a} \notin \mathbf{A}_0$. Put $\sqrt[l^m]{a} = \alpha$ and $n - m = v$. If $v \leqq t$, then a is already weakly M-reduced. So we assume $v > t$. Since $M/\mathbf{A}_0$ is abelian, $\mathbf{A}M/\mathbf{A}_0$ is abelian and hence $\mathbf{A}_0(\sqrt[l^n]{a}) = \mathbf{A}_0(\sqrt[l^v]{\alpha})$ is also abelian. $\mathbf{A}_0(\sqrt[l^v]{\alpha}) \supset \mathbf{A}_0(\zeta_{(v)})$ by lemma 1.1. Therefore by prop. 1.3 there exists an integer u such that

[6] Set $\sqrt[l^n]{a} = a^{l^{-n}}$ for a negative n.

$\mathsf{A}_0(\sqrt[l^u]{\alpha}) = \mathsf{A}_0(\zeta_{(v)}) = \mathsf{A}_0(\sqrt[l^{v-t}]{\zeta_{(t)}})$. Then $u = v - t$ by prop. 1.4, for we treat the non-radical case with respect to A_0. Furthermore $\alpha = \zeta'_{(t)}\beta^{l^{v-t}}$ with $\beta \in \mathsf{A}_0$ by prop. 1.5. From this and lemma 3.1 our proposition follows at once.

Next we consider the radical case with respect to A_0.

LEMMA 3.2. *Let M be a non-abelian meta-abelian extension over k attached to k such that $\mathsf{A}M = \mathsf{A}(\sqrt[2^n]{a})$ with $a \in k$, and put $\mathsf{A} \frown M = \mathsf{A}_0$. Suppose that $\sqrt{-1} \notin \mathsf{A}_0$ (i.e. A_0 is a radical field); $\sqrt[2^m]{a} \in \mathsf{A}_0$, $\sqrt[2^{m+1}]{a} \notin \mathsf{A}_0$; and $n - m > 2$. Then putting $\sqrt[2^m]{a} = \alpha$ and $n - m = v$, we have $\mathsf{A}_0(\sqrt[2^{v-1}]{-1}) = \mathsf{A}_0(\sqrt[2^{v-1}]{\alpha}) \neq \mathsf{A}_0(\sqrt[2^v]{\alpha})$.*

Proof. Since $\mathsf{A}_0(\sqrt[2^n]{a}) = \mathsf{A}_0(\sqrt[2^v]{\alpha})$ is an abelian extension over A_0, we have $\mathsf{A}_0(\sqrt[2^v]{\alpha}) \supset \mathsf{A}_0(\zeta_{2^v})$ by lemma 1.1. Furthermore by prop. 1.3 there exists an integer u such that $\mathsf{A}_0(\zeta_{2^v}) = \mathsf{A}_0(\sqrt[2^u]{\alpha})$. Suppose that u is the largest integer with such a property. If $v \leqq u$, then $\mathsf{A}_0(\sqrt[2^v]{\alpha}) \subset \mathsf{A}_0(\zeta_{2^v})$ and hence $\mathsf{A}M = \mathsf{A}\mathsf{A}_0(\sqrt[2^v]{\alpha}) \subset \mathsf{A}_0(\zeta_{2^v}) \subset \mathsf{A}$, contradicting the assumption that M is non-abelian. Hence $v > u$. Since $v \geqq 2$, $\mathsf{A}_0(\zeta_{2^v})$ is a non-radical field. Therefore, by the assumption that u is largest, prop. 1.2 implies that $\mathsf{A}_0(\sqrt[2^v]{\alpha})$ is an extension of degree 2^{v-u} over $\mathsf{A}_0(\zeta_{2^v}) = \mathsf{A}_0(\sqrt[2^u]{\alpha})$. On the other hand by prop. 1.1 the degree is at most equal to 2. Since $v > u$, we have $v - u = 1$ and $\mathsf{A}_0(\sqrt[2^v]{\alpha}) \neq \mathsf{A}_0(\sqrt[2^{v-1}]{\alpha})$, which proves the lemma.

PROPOSITION 3.2. *Let M be a meta-abelian extension over k attached to k such that $\mathsf{A}M = \mathsf{A}(\sqrt[2^n]{a})$ with $a \in k$, and put $\mathsf{A} \frown M = \mathsf{A}_0$. Furthermore suppose that $\sqrt{-1} \notin \mathsf{A}_0$ (the radical case with respect to A_0). Then there exists a weakly M-reduced element of k.*

Proof. Let $\sqrt[2^m]{a} \in \mathsf{A}_0$, $\sqrt[2^{m+1}]{a} \notin \mathsf{A}_0$; and put $\alpha = \sqrt[2^m]{a}$, $n - m = v$. If $v = 0$ or $= 1$, then a is already weakly M-rduced. So assume $v \geqq 2$. Then $\mathsf{A}_0(\sqrt[2^{v-1}]{-1}) = \mathsf{A}_0(\sqrt[2^{v-1}]{\alpha})$ by lemma 3.2. Hence prop. 1.6 implies $\alpha = -\beta^{2^{v-1}}$ or $= -\lambda^{2^{v-2}}\beta^{2^{v-1}}$, where $\beta \in \mathsf{A}_0$ and λ is the radical number of A_0. However we need not cosider the latter case. For, if A_0 is a radical field of the first kind, then only the case $\alpha = -\beta^{2^{v-1}}$ occurs.[7] If A_0 is of the second kind and $\alpha = -\lambda^{2^{v-2}}\beta^{2^{v-1}}$, then $\mathsf{A}M = \mathsf{A}(\sqrt[2^v]{\alpha}) = \mathsf{A}(\sqrt[4]{\lambda\beta^2})$. Since $\mathsf{A}M/\mathsf{A}_0$ is abelian, $\mathsf{A}_0(\sqrt[4]{\lambda\beta^2})/\mathsf{A}_0$ is also abelian and $\mathsf{A}_0(\sqrt[4]{\lambda\beta^2}) \supset \mathsf{A}_0(\sqrt{-1})$ by lemma 1.1. Furthermore prop. 1.3 shows

[7] See footnote 4.

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$\mathbf{A}_0(\sqrt{\lambda\beta^2}) = \mathbf{A}_0(\sqrt{\lambda}) = \mathbf{A}_0(\sqrt{-1})$. But this is a cotradiction. Therefore we have also $\alpha = -\beta^{2^{v-1}}$ with $\beta \in \mathbf{A}_0$. Then our proposition follows at once from lemma 3.1.

§4. Characters of $M/\mathbf{A}_0$

Let M be as in §3 a meta-abelian extension over k attached to k and put $\mathbf{A}\frown M = \mathbf{A}_0$. Suppose that $\zeta_{(t)} \in \mathbf{A}_0$, $\zeta_{(t+1)} \notin \mathbf{A}_0$ and a is a weakly M-reduced element of k, whose existence has been proved in §3. Puf $\omega = \sqrt[l^n]{a}$, $\mathfrak{G} = \mathfrak{G}(\mathbf{A}M/k)$, $\mathfrak{A} = \mathfrak{G}(\mathbf{A}M/\mathbf{A}_0)$, $\mathfrak{g} = \mathfrak{G}(\mathbf{A}_0/k)$ and

$$\psi_0(A) = \omega^A/\omega \qquad \text{for any } A \in \mathfrak{A}. \tag{4.1}$$

Then ψ_0 generates the character group of $\mathfrak{G}(\mathbf{A}_0(\sqrt[l^n]{a})/\mathbf{A}_0)$, for we see $\psi_0(AB) = \omega^{AB}/\omega = (\omega^{AB}/\omega^B)(\omega^B/\omega) = \psi_0(A)^B\psi_0(B)$ and $\psi_0(A) \in \mathbf{A}_0$. For $\sigma \in \mathfrak{g}$ let U_σ be a prolongation of σ to $\mathbf{A}M$. Then since $(\omega^{l^n})^{U_\sigma} = a^\sigma = a = \omega^{l^n}$, we have

$$\omega^{U_\sigma} = \omega\zeta_{(\sigma)}, \qquad \text{where } \zeta_{(\sigma)}^{l^n} = 1. \tag{4.2}$$

If we choose another prolongation V_σ of σ and set $\omega^{V_\sigma} = \omega\eta_{(\sigma)}$, then $V_\sigma = BU_\sigma$ with some $B \in \mathfrak{A}$ and $\omega^{V_\sigma} = \omega^{BU_\sigma} = (\psi_0(B)\omega)^{U_\sigma} = \psi_0(B)^{U_\sigma}\omega\zeta_{(\sigma)}$. Hence

$$\eta_{(\sigma)} = \psi_0(B)^{U_\sigma}\zeta_{(\sigma)}. \tag{4.3}$$

For $A \in \mathfrak{A}$ and $\sigma \in \mathfrak{g}$ we write $A^\sigma = U_\sigma^{-1}AU_\sigma$. Then $\omega^{U_\sigma A^\sigma} = (\omega\zeta_{(\sigma)})^{A^\sigma} = \psi_0(A^\sigma)\omega\zeta_{(\sigma)}^{A^\sigma} = \psi_0(A^\sigma)\omega\zeta_{(\sigma)}^{A}$, $\omega^{AU_\sigma} = (\psi_0(A)\omega)^{U_\sigma} = \psi_0(A)^{U_\sigma}\omega\zeta_{(\sigma)}$. Hence

$$\psi_0(A^\sigma) = \psi_0(A)^{U_\sigma}\zeta_{(\sigma)}^{1-A}. \tag{4.4}$$

Define ψ_0^σ and ζ_σ by

$$\psi_0^\sigma(A) = \psi_0(A^\sigma) \tag{4.5}$$

and

$$\zeta_\sigma(A) = \zeta_{(\sigma)}^{1-A}. \tag{4.6}$$

Let further $\bar{\sigma}$ be an integer determined by $\zeta_{(t)}^\sigma = \zeta_{(t)}^{\bar{\sigma}}$, the order of ψ_0 being l^t. Then by (4.4) we have

$$\psi_0^\sigma = \psi_0^{\bar{\sigma}}\zeta_\sigma. \tag{4.7}$$

Through different choice of the prolongation of σ to $\mathbf{A}M$, $\zeta_{(\sigma)}$ is multiplied by an l^t-th root of unity. However the root of unity is contained in $\mathbf{A}_0$, hence ζ_σ is determined only by σ, does not depend on the choice of the prolongation,

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and is a character of $\mathfrak{A}$ by (4.7).

Now let Φ resp. Ψ be character groups of $\mathfrak{G}(\mathbf{A}/\mathbf{A}_0)$ resp. $\mathfrak{G}(\mathbf{A}_0(\sqrt[l^n]{a})/\mathbf{A}_0)$, the latter being generated by ψ_0 which is defined by (4.1). For $A \in \mathfrak{A}$ let A_1 and A_2 be restrictions of A to $\mathbf{A}$ and to $\mathbf{A}_0(\sqrt[l^n]{a})$ respectively, and set

$$\varphi\psi(A) = \varphi(A_1)\psi(A_2) \qquad \text{for } \varphi \in \Phi,\ \psi \in \Psi. \tag{4.8}$$

Then such products of φ and ψ form the character group $\mathbf{X}$ of $\mathfrak{A}$. Denote by $\mathbf{X}_M$ the subgroup of $\mathbf{X}$ corresponding to M; and for $\chi \in \mathbf{X}$, $\sigma \in \mathfrak{g}$ let χ^σ be the character of $\mathfrak{A}$ defined by $\chi^\sigma(A) = \chi(A^\sigma)$ for $A \in \mathfrak{A}$. Then we have

Lemma 4.1. *Let K_1 be an intermediate field of $\mathbf{A}M/\mathbf{A}_0$, and $\mathbf{X}_1$ be the subgroup of the character group $\mathbf{X}$ of $\mathfrak{A}$ corresponding to K_1. Then in order that K_1/k is a normal extension it is necessary and sufficient that $\varphi\psi \in \mathbf{X}_1$, $\varphi \in \Phi$, $\psi \in \Psi$ imply $\psi^{\sigma-1} \in \mathbf{X}_1$ for all $\sigma \in \mathfrak{g} = \mathfrak{G}(\mathbf{A}_0/k)$.*

Proof. Put $\mathfrak{A}_1 = \mathfrak{G}(\mathbf{A}M/K_1)$. Then, in order that K_1/k is normal it is necessary and sufficient that $A^\sigma \in \mathfrak{A}_1$ for all $A \in \mathfrak{A}_1$ and all $\sigma \in \mathfrak{g}$. Furthermore we have $A^\sigma \in \mathfrak{A}$ if and only if $\chi^\sigma(A) = \chi(A^\sigma) = 1$ for all $\chi \in \mathbf{X}_1$. Therefore K_1/k is normal if and only if $\chi^\sigma \in \mathbf{X}_1$ for all $\chi \in \mathbf{X}_1$ and all $\sigma \in \mathfrak{g}$. On the other hand if $\chi = \varphi\psi$, $\varphi \in \Phi$, $\psi \in \Psi$, then $\chi^{\sigma-1} = \psi^{\sigma-1}$ owing to $\varphi^\sigma = \varphi$. Hence the lemma is proved.

Lemma 4.2. *Let M be as above and non-abelian over k. Then for every $\sigma \in \mathfrak{g}$ and every $\psi \in \Psi$ we have $\psi^{\sigma-1} \in \mathbf{X}_M$.*

Proof. M is a normal extension over k. So, if we can show that for any $\psi \in \Psi$ there exists $\varphi \in \Phi$ such that $\varphi\psi \in \mathbf{X}_M$, then our assertion follows from lemma 4.1. Now let ψ be any element of Ψ. Then there exists an integer x and $\varphi \in \Phi$ such that $\varphi\psi^x \in \mathbf{X}_M$ and $\psi^x \neq 1$, because $\mathbf{A}M = \mathbf{A}\cdot\mathbf{A}_0(\sqrt[l^n]{a})$ and further M/k is non-abelian. Let $x = l^\nu x'$, $(x', l) = 1$. Since the order of ψ^{l^ν} is then at most equal to $l^{t-\nu}$, we have $\varphi^y\psi^{l^\nu} \in \mathbf{X}_M$, where $x'y \equiv 1 \bmod. l^{t-\nu}$. Let m be a smallest integer of ν such that $\varphi\psi^{l^\nu} \in \mathbf{X}_M$, φ running through all elements of Φ. Then the character group of $\mathbf{A}M/\mathbf{A}$ is generated by ψ^{l^m}, and hence $\mathbf{A}M = \mathbf{A}(\sqrt[l^{n-m}]{a})$. Since M/k is non-abelian, namely $\mathbf{A}M \neq \mathbf{A}$, we have $m = 0$ by prop. 1.2. This means that there exists $\varphi \in \Phi$ such that $\varphi\psi \in \mathbf{X}_M$, which is to be proved.

Lemma 4.3. *Let $\sigma_1, \ldots, \sigma_s$ be elements of $\mathfrak{g} = \mathfrak{G}(\mathbf{A}_0/k)$, and $x_1, \ldots, x_s$ be* *integers. Then* $\zeta_{\sigma_1}^{x_1}\cdots\zeta_{\sigma_s}^{x_s}\in \mathbf{X}_M$ *is equivalent to* $\zeta_{(\sigma_1)}^{x_1}\cdots\zeta_{(\sigma_s)}^{x_s}\in \mathbf{A}_0$, *where* $\zeta_{(\sigma_i)}$ *is obtained by* (4.2) *from any representatives* U_{σ_i} *of* σ_i *in* $\mathfrak{G}=\mathfrak{G}(\mathbf{A}M/k)$ $(i=1, \ldots, s)$.

Proof. Since $\zeta_{\sigma_1}^{x_1}\cdots\zeta_{\sigma_s}^{x_s}\in \mathbf{X}_M$ is equivalent to $\zeta_{\sigma_1}^{x_1}\cdots\zeta_{\sigma_s}^{x_s}(A)=1$ for all $A\in\mathfrak{G}(\mathbf{A}M/M)$, our assertion follows at once from $\zeta_\sigma^x(A)=(\zeta_{(\sigma)}^{1-A})^x=(\zeta_{(\sigma)})^{1-A}$ for any U_σ.

LEMMA 4.4. *If* $\zeta_{(t)}^\sigma=\zeta_{(t)}$ *for* $\sigma\in\mathfrak{g}$, *then* $\zeta_\sigma\in\mathbf{X}_M$.

Proof. If $\zeta_{(t)}^\sigma=\zeta_{(t)}$, then $\psi^{\sigma-1}=\zeta_\sigma$ by (4.7), and hence $\zeta_\sigma\in\mathbf{X}_M$ by lemma 4.2.

LEMMA 4.5. *Let* U_σ *be a prolongation of* $\sigma\in\mathfrak{g}$ *to* $\mathfrak{G}$ *and* $\zeta_{(\sigma)}$ *be as in* (4.2). *Suppose further* $(\omega^{l^\nu})^{U_\sigma}=\omega^{l^\nu}$ *and* $(\omega^{l^{\nu-1}})^{U_\sigma}\neq\omega^{l^{\nu-1}}$. *Then* $\zeta_{(\sigma)}$ *is a primitive* l^ν*-th root of unity.*

Proof. Since $(\omega^{l^\nu})^{U_\sigma}=\omega^{l^\nu}\zeta_{(\sigma)}^{l^\nu}$ and $(\omega^{l^\nu})^{U_\sigma}=\omega^{l^\nu}$, we have $\zeta_{(\sigma)}^{l^\nu}=1$ by (4.2). Analogously, $\zeta_{(\sigma)}^{l^\nu}\neq 1$ from $(\omega^{l^{\nu-1}})^{U_\sigma}\neq\omega^{l^{\nu-1}}$. Therefore $\zeta_{(\sigma)}$ is a primitive l^ν-th root of unity.

§ 5. Completion of the proof

Let M be a meta-abelian extension over k attached to k and put $\mathbf{A}\frown M=\mathbf{A}_0$. Furthermore suppose that $\zeta_{(r)}\in k$, $\zeta_{(r+1)}\notin k$; and $\zeta_{(t)}\in\mathbf{A}_0$, $\zeta_{(t+1)}\notin\mathbf{A}_0$. Then we have

LEMMA 5.1. *Let* M *be as above and* $\mathbf{A}M=\mathbf{A}(\sqrt[l^n]{a})$, *where* a *is a weakly* M*-reduced element of* k. *Then it follows from* $k(\sqrt[l]{a})\neq k(\zeta_{(r+1)})$ *that* $\zeta_{(n)}\in\mathbf{A}_0$; *namely* $n\leqq t$, a *is* M*-reduced and* M *is* k*-meta-abelian over* k.

Proof. a is weakly M-reduced. Hence if $\sqrt[l]{a}\notin\mathbf{A}_0$ then $n\leqq t$ and our assertion is already true. So assume $\sqrt[l]{a}\in\mathbf{A}_0$. Then prop. 1.4 implies $r\geqq 1$, and the proof is separated into two cases:

Case 1. $\sqrt[l]{a}\notin k(\zeta_{(t)})$: First we remark that this case contains all the non-radical cases and the radical cases of the first kind. Because, in these cases $k(\zeta_{(t)})/k$ is cyclic, and so $k(\sqrt[l]{a})\subset k(\zeta_{(t)})$ yields $k(\sqrt[l]{a})=k(\sqrt[l]{\zeta_{(r)}})$, contradicting the assumption of the proposition. Now let σ be a prolongation to $\mathbf{A}_0$ of a non-trivial automorphism of the normal extension $k(\sqrt[l]{a}, \zeta_{(t)})/k(\zeta_{(t)})$. Then $\zeta_{(\sigma)}$ is a primitive l^n-th root of unity by $(\sqrt[l]{a})^\sigma\neq\sqrt[l]{a}$ and lemma 4.5. On the other hand $\zeta_\sigma\in\mathbf{X}_M$ by $\zeta_{(t)}^\sigma=\zeta_{(t)}$ and by lemma 4.4, hence lemma 4.3 implies

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$\zeta_{(\sigma)} \in M$. Thus our assertion is proved in the case where $\sqrt[l]{a} \notin k(\zeta_{(t)})$.

Case 2. $\sqrt[l]{a} \in k(\zeta_{(t)})$: As remarked in case 1 this case occurs only in the radical case of the second kind. Therefore suppose that $l=2$, $\sqrt{-1} \notin k$, and let $\lambda = \lambda_T$ be the radical number of k. Then $k(\sqrt{a}) \subset k(\zeta_{(t)})$ and $k(\sqrt{a}) \neq k(\sqrt{-1})$. Hence $(k(\zeta_{(t)});\ k) \geqq 4$, and $t \geqq T+1$ by (1.5). Then a quadratic subfield of $k(\zeta_{(t)})$ over k is $k(\sqrt{-1})$, $k(\sqrt{\lambda})$ or $k(\sqrt{-\lambda})$; and we see $k(\sqrt{a}) = k(\sqrt{\lambda})$ or $= k(\sqrt{-\lambda})$. Now let σ resp. τ be prolongstions to $\mathbf{A}_0$ of non-trivial automorphisms of the quadratic extensions $k(\zeta_{(T+1)})/k(\sqrt{-1})$ resp. $k(\zeta_{(T+1)})/k(\sqrt{a})$; and $\bar{\sigma}$ resp. $\bar{\tau}$ integers such that $\zeta_{(t)}^{\sigma} = \zeta_{(t)}^{\bar{\sigma}}$ resp. $\zeta_{(t)}^{\tau} = \zeta_{(t)}^{\bar{\tau}}$. Then $\sqrt{a}^{\sigma} \neq \sqrt{a}$, and lemma 4.5 implies that $\zeta_{(\sigma)}$ is a primitive 2^n-th root of unity. Furthermore since $\zeta_{(T)}^{\sigma} = \zeta_{(T)}$ and $\zeta_{(T+1)}^{\sigma} \neq \zeta_{(T+1)}$, we may write by lemma 1.2 $\bar{\sigma} = 1 + 2^T y(\sigma)$, where $(y(\sigma), 2) = 1$. $\psi^{\sigma-1} = \psi^{2^T y(\sigma)} \zeta_\sigma \in \mathbf{X}_M$ by lemma 4.1 and (4.7). Now $y'(\sigma)$ being an integer such that $y'(\sigma) y(\sigma) \equiv 1 \bmod. 2^{t-T}$, we have

$$\psi^{2^T} \zeta_\sigma^{y'(\sigma)} \in \mathbf{X}_M, \qquad (y'(\sigma), 2) = 1. \tag{5.1}$$

As for $\sigma\tau$, proceed similarly as above. Namely, $\sqrt{a}^{\sigma\tau} = \sqrt{a}^{\sigma} \neq \sqrt{a}$ implies that $\zeta_{\sigma\tau}$ is a primitive 2^n-th root of unity, and $\sqrt{-1}^{\sigma\tau} = \sqrt{-1}^{\tau} \neq \sqrt{-1}$ implies $\overline{\sigma\tau} = 1 + 2y(\sigma\tau)$, where $(y(\sigma\tau), 2) = 1$. Hence $\psi^{\sigma\tau-1} = \psi^{2y(\sigma\tau)} \zeta_{\sigma\tau} \in \mathbf{X}_M$, and so

$$\psi^2 \zeta_{\sigma\tau}^{y'(\sigma\tau)} \in \mathbf{X}_M, \qquad (y'(\sigma\tau), 2) = 1, \tag{5.2}$$

where $y(\sigma\tau) y'(\sigma\tau) \equiv 1 \bmod. 2^{t-1}$. Therfore $\zeta_\sigma^{y'(\sigma)} \zeta_{\sigma\tau}^{-y'(\sigma\tau)2^{T-1}} \in \mathbf{X}_M$ by (5.1) and (5.2), and further $\zeta_{(\sigma)}^{y'(\sigma)} \zeta_{(\sigma\tau)}^{-y'(\sigma\tau)2^{T-1}} \in M$ by lemma 4.3. Since $T \geqq 2$, this is a primitive 2^n-th root of unity, which proves our assertion.

LEMMA 5.2. *Let M be as above a meta-abelian extension over k attached to k and $\mathbf{A}M = \mathbf{A}(\sqrt[l^n]{a})$, where a is a weakly M-reduced element of k. Then $\zeta_{l^{n-u+r}} \in \mathbf{A}_0$, provided that $\sqrt[l^{n-u}]{a} \in \mathbf{A}_0 = \mathbf{A} \frown M$ for $n > u \geqq r$.*

Proof. Let t, r be as in the beginning of this section, and σ be an automorphism of $\mathbf{A}_0/k$ such that $(\sqrt[l]{a})^\sigma \neq \sqrt[l]{a}$ and $\zeta_{(t)}^\sigma = \zeta_{(t)}^{1-l^r x}$, where $(x, l) = 1$. Such a σ really exists. In fact, if $t = r$, then σ is obtained as a prolongation to $\mathbf{A}_0$ of a non-trivial automorphism of $k(\sqrt[l]{a})/k$. If $t > r$ and $k(\sqrt[l]{a}) = k(\zeta_{(r+1)})$, then an automorphism σ with $\zeta_{(t)}^\sigma = \zeta_{(t)}^{1-l^r}$ also satisfies the condition $(\sqrt[l]{a})^\sigma \neq \sqrt[l]{a}$, for we have $\zeta_{(r+1)}^\sigma \neq \zeta_{(r+1)}$ by lemma 1.2. Finally if $t > r$ and $k(\sqrt[l]{a}) \neq k(\zeta_{(r+1)})$, then a prolongation σ to $\mathbf{A}_0$ of a non-trivial automorphism of $k(\sqrt[l]{a}, \zeta_{(r+1)})/k(\sqrt[l]{\zeta_{(r)} a})$ satisfies $(\sqrt[l]{a})^\sigma \neq \sqrt[l]{a}$ and $\zeta_{(r+1)}^\sigma \neq \zeta_{(r+1)}$, hence $\zeta_{(t)}^\sigma = \zeta_{(t)}^{1-l^r x}$

where $(x, l)=1$ by lemma 1.2 and 1.4. For such a σ we have $\psi^{\sigma}=\psi^{1-l^r x}\zeta_\sigma$ by (4.7). Now from $(\sqrt[l]{a})^{\sigma} \neq \sqrt[l]{a}$ and from lemma 4.5 follows that $\zeta_{(\sigma)}$ is a primitive l^n-th root of unity for any prolongation U_σ of σ to $\mathfrak{G}$. On the other hand, $\psi^{\sigma-1}=\psi^{-l^r x}\zeta_\sigma \in \mathbf{X}_M$ by (5.3) and lemma 4.2, hence $\psi^{-l^u x}\zeta_\sigma^{l^{u-r}} \in \mathbf{X}_M$. Since, however, the order of ψ is at most equal to l^u, owing to the assumption $\sqrt[l^{n-u}]{a} \in \mathbf{A}_0$, we have $\zeta_\sigma^{l^{u-r}} \in \mathbf{X}_M$. From this and from lemma 4.3 follows $\zeta_{(\sigma)}^{l^{u-r}} \in M$. Since $\zeta_{(\sigma)}$ was a primitive l^n-th root of unity as was seen above, we have $\zeta_{(n-u+r)} \in M$, and furthermore $\zeta_{(n-u+r)} \in \mathbf{A}_0$, which is to be proved.

Conclusion of the proof of the theorem

Hereafter the same notations as above will be used; $\zeta_{(r)} \in k$, $\zeta_{(r+1)} \notin k$; M be a meta-abelian extension over k attached to k; $\mathbf{A} \frown M=\mathbf{A}_0$; $\zeta_{(t)} \in \mathbf{A}_0$, $\zeta_{(t+1)} \notin \mathbf{A}_0$; $\mathbf{A}M=\mathbf{A}(\sqrt[l^n]{a})$, where a can be assumed a weakly M-reduced element of k, namely $\sqrt[l^{n-t}]{a} \in \mathbf{A}_0$, by §2.

Now if $n \leqq t$, then M is already k-meta-abelian over k. So suppose $n>t$. Since of couse $\sqrt[l]{a} \in \mathbf{A}_0$, we have $r \geqq 1$ by lemma 1.4. Furthermore

$$\zeta_{(n-t+r)} \in \mathbf{A}_0 \tag{5.4}$$

by putting $u=t$ in lemma 5.1.

Let

$$a=\zeta_{(r)}^{*} b^{l^{\mu}} \quad \text{with } b \in k, \text{ and } a \neq \zeta_{(r)}^{**} c^{l^{\mu+1}} \quad \text{with any } c \in k. \tag{5.5}$$

Then

$$\mathbf{A}M=\mathbf{A}(\sqrt[l^n]{a})=\mathbf{A}(\zeta_{(n+r)}^{*} \cdot \sqrt[l^{n-\mu}]{b})=\mathbf{A}(\sqrt[l^{n-\mu}]{b}). \tag{5.6}$$

Hence if $n-\mu \leqq t$, then M is already k-meta-abelian over k. So suppose $n-\mu > t$, and put $n-\mu=\nu$. Then $\mathbf{A}M=\mathbf{A}(\sqrt[l^\nu]{b})$ by (5.6). On the other hand from (5.5) and from the reducibility of a follows $\sqrt[l^{n-t}]{a}=\zeta_{(n-t+r)}^{*} \cdot \sqrt[l^{\nu-t}]{b} \in \mathbf{A}_0$, hence $\sqrt[l^{\nu-t}]{b} \in \mathbf{A}_0$ by (5.4). Moreover $b \notin k^l$ because of (5.5). Hence b is also a weakly M-reduced element of k. Furthermore $k(\sqrt[l]{b}) \neq k(\sqrt[l]{\zeta_{(r)}})$. Indeed, otherwise $b=\zeta_{(r)}^{*} c^l$ with $c \in k$ owing to $r \geqq 1$, and $a=\zeta_{(r)}^{**} c^{l^{\mu+1}}$, contradicting (5.5). Using b resp. ν instead of a resp. n in lemma 5.1, we see from the above results that M is a k-meta-abelian extension over k. This concludes the proof of the theorem.

§ 6. Remarks on P-meta-abelian fields of degree 8

We shall characterize P-meta-abelian fields of degree 8, P being the rational number field, by using our theorem. The following lemma is easily proved.

LEMMA 6.1. *Let F be any algebraic number field of finite degree and Ω be a quadratic extension of F. Then Ω is embeddable in a biquadratic cyclic extension over Ω if and only if there exists an element μ in Ω such that $N_{\Omega/F}\mu = -1$.*

Now let M be a non-abelian P-meta-abelian field over P of degree 8 and $\mathbf{A}$ be the maximal abelian field over P. Then $\mathbf{A}M = \mathbf{A}(\sqrt[4]{a})$ for some $a \in P$ and our theorem implies $\sqrt{-1} \in M$. Conversely, let M be a non-abelian normal field over P of degree 8 and assume $\sqrt{-1} \in M$. Then, since M/P is a quarternion or dihedral extension, there exists a positive element d of P such that quadratic subfields of M are $P(\sqrt{-1})$, $P(\sqrt{d})$ and $P(\sqrt{-d})$. Put $\Omega = P(\sqrt{-1}, \sqrt{d})$ and $k = P(\sqrt{d})$. Since Ω is imaginary and k is real, M/k is a non-cyclic biquadratic extension by lemma 6.1. Hence M/P is necessarily a dihedral extension. Let $k(\sqrt{\mu})$ for $\mu \in k$ be a subfield of M distinct to Ω, and $\bar{\mu}$ be a conjugate of μ over P. If $k(\sqrt{\bar{\mu}}) = k(\sqrt{\mu})$, then $k(\sqrt{\mu})/P$ is biquadratic normal, and so abelian. Hence $M = k(\sqrt{\mu})\Omega$ is also abelian over P, which contradict our assumption. Therefore $k(\sqrt{\mu}) \neq k(\sqrt{\bar{\mu}})$, and hence $M = k(\sqrt{\bar{\mu}}, \sqrt{\mu})$, moreover $k(\sqrt{\mu\bar{\mu}}) = k(\sqrt{-1})$. Then $\mu\bar{\mu} = -\gamma^2$ for some $\gamma \in k$. Put $\sqrt{d}\,\mu = \omega$ and $k(\sqrt{\omega}) = A$. Since $\bar{\omega} = -\sqrt{d}\,\bar{\mu} = \omega\left(-\frac{\mu\bar{\mu}}{\mu^2}\right) = \omega\left(\frac{\gamma}{\mu}\right)^2$, A/P is normal, and moreover abelian. Therefore M is a subfield of $A\Omega(\sqrt[4]{d})$, that is, M is a P-meta-abelian. Then we have proved.

PROPOSITION 6.1. *A necessary and sufficient condition for a non-abelian normal extension M over P of degree 8 to be a P-meta-abelian is that M contains $\sqrt{-1}$. And then M is a dihedral extension over P.*

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