# DIFFEOMORPHISMS ON $S^1$, PROJECTIVE STRUCTURES AND INTEGRABLE SYSTEMS

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## Abstract

In this paper we consider a projective connection as defined by the $n$th-order Adler-Gelfand-Dikii (AGD) operator on the circle. It is well-known that the Korteweg-de Vries (KdV) equation is the archetypal example of a scalar Lax equation defined by a Lax pair of scalar $n$th-order differential (AGD) operators. In this paper we derive (formally) the KdV equation as an evolution equation of the AGD operator (at least for $n \leq 4$) under the action of Vect($S^1$). The solutions of the AGD operator define an immersion $\mathbf{R} \to \mathbb{R}P^{n-1}$ in homogeneous coordinates. In this paper we derive the Schwarzian KdV equation as an evolution of the solution curve associated with $\Delta^{(n)}$, for $n \leq 4$.

## 1. Introduction

The space of linear differential operators on a manifold $M$ considered as a module over the group of diffeomorphisms is a well-known classical text. This space has various algebraic structures, for example, the structure of an associative algebra and of a Lie algebra [8]. In the one-dimensional case, this has been studied by E. Wilczynski [28] and E. Cartan [7].

The space of $n$th-order differential operators or Adler-Gelfand-Dikii (AGD) space [1, 9] is connected to the $gl(n, \mathbf{R})$ current algebra. Gelfand and Dikii [9] established the relation between the dual spaces of Kac-Moody algebras on the circle and the AGD space. The latter is a Poisson subspace of the former [6]. This space has an interesting structure, and this has been studied from a number of different angles. In this paper we will focus mainly from the point of view of projective connections on the circle. Projective connections on the circle have been classified from a geometric

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point of view by Kuiper [18]. Lazutkin and Pankratova [19] were the first to formulate this analytically.

The connection between the geodesic flow on the Virasoro-Bott group and the periodic KdV equation follows from the work of Arnold [4], Kirillov [15, 16, 17], Segal [26, 27], Ovsienko and Khesin [24] and Witten [31]. The connection between the C. Neumann system and the Bott-Virasoro group has been studied in [11, 12].

It is well-known that the KdV equation is the canonical example of a scalar Lax equation, which is an equation defined by a Lax pair of scalar differential operators

$$\frac{d\Delta}{dt} = [P, \Delta]$$

where

$$\Delta = \frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-1}}{dx^{n-2}} + \cdots + u_0.$$

Here $P$ is a differential operator whose coefficients are differential polynomials in the variables, essentially determined by the requirement that $[P, \Delta]$ be an operator of order less than $n$.

Physicists have studied the AGD operators and their connection to extended classical conformal algebras [5, 21]. Pierre Mathieu [21] has listed several extended conformal operators. These are denoted by $\Delta^{(n)}$, and some of the members of this family are:

$$\Delta^{(0)} = 1, \quad \Delta^{(1)} = \partial_x, \quad \Delta^{(2)} = \partial_x^2 + u, \quad \Delta^{(3)} = \partial_x^3 + 4u\partial_x + 2u',$$
$$\Delta^{(4)} = \partial_x^4 + 9u^2 + 3u'' + 10u'\partial_x + 10u\partial_x^2,$$

and so forth.

These operators define the Poisson (Gelfand-Dikii) bracket of the spin-$k$ field [5]. All these scalar differential operators are covariant operators, or in other words, they transform covariantly under the coadjoint action of $\mathrm{Diff}(S^1)$. As shown by Mathieu at least for $n \leq 4$, these operators $\Delta^{(n)}$ depend only on $u$ and its derivatives.

We will take a slightly different route [11]. We will analyse the evolution equation defined by the action of a vector field $\mathrm{Vect}(S^1)$ on a scalar differential operator. This gives rise to a KdV flow again. The solution of $\Delta^{(n)}$ defines an immersion in homogeneous coordinates [10, 25]. In this paper we establish a connection between this KdV flow and an evolution of a solution curve associated with the immersion. We know that there exists a projective action of $SL(n, \mathbf{R})$ on $\mathbb{R}P^{n-1}$, induced by the action of $SL(n, \mathbf{R})$ on $\mathbf{R}^n$. We will see that the evolution of the solution curve of $\Delta$ is invariant under the $SL(n, \mathbf{R})$ action. Earlier, Mari Beffa *et al.* [10] found an explicit formula for the most general vector evolution of curves on $\mathbb{R}P^{n-1}$. Our approach is different; it gives an elegant presentation of the Schwarzian KdV [23] ($SL(2, \mathbf{R})$ invariant KdV) equation, also called Ur-KdV [22, 29, 30], as an evolution equation of the space curve connected to the operator $\Delta^{(n)}$.

**Organization of the paper**   This paper is organized as follows. In the next section, we present a brief description of the AGD space and the smooth orientation preserving immersion given by the solution of the equation involved. Section 3 is devoted to projective connections on the circle. We will describe the action of $\mathrm{Diff}(S^1)$ on the AGD manifold in Section 4. In Section 5 we will derive the KdV equation, which is an evolution equation defined by the action of $\mathrm{Vect}(S^1)$ on the space of $n$th-order linear differential operators. We derive the evolution of the solution curve in Section 6.

**Results of the paper**   In this paper we argue that the action of $\mathrm{Vect}(S^1)$ on the AGD manifold, $[\mathscr{L}_v, \Delta^{(n)}] = \mathscr{L}_v \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathscr{L}_v$, (this notation will be explained later) can be considered as a coadjoint action.

The action of $\mathrm{Vect}(S^1) = T_e \mathrm{Diff}^+(S^1)$ on the AGD operator $\Delta^{(n)}$ (for $n \leq 4$) yields an evolution equation

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathscr{L}_v, \Delta^{(n)}] = 0.$$

This generates a nonlinear evolution equation for periodic functions $u$

$$\frac{\partial u}{\partial t} = \frac{1}{2} f''' + 2f'u + f u'.$$

THEOREM 1.1. *The Euler-Arnold equation gives the KdV equation; it is a Hamiltonian flow on the "coadjoint orbits" in $\Delta^{(n)}$ for the Hamiltonian function $H(u) = u^2/2$.*

THEOREM 1.2. *The solutions of an operator $\Delta^{(n)}$ define an immersion $\mathbf{R} \to \mathbb{R}P^{n-1}$ in homogeneous coordinates. The evolution equation of the solution curve is a Hamiltonian flow, and it is given by the Schwarzian KdV equation.*

## 2. AGD space

In the late seventies, Adler [1] defined a family of second Hamiltonian structures with respect to which the generalized KdV equation can be written as a Hamiltonian system. The AGD brackets are defined on the space of $n$th-order scalar differential operators of the form

$$\mathscr{S} = \left\{ \frac{d^n}{dx^n} + u_{n-1}\frac{d^{n-1}}{dx^{n-1}} + u_{n-2}\frac{d^{n-2}}{dx^{n-2}} + \cdots + u_1\frac{d}{dx} + u_0 \right\}, \qquad (1)$$

where the coefficients $u_i \in C^\infty(S^1)$ are smooth and periodic. This space of differential operators is called the Adler-Gelfand-Dikii (AGD) manifold. It is known that the extended classical conformal algebras, that is, $n > 2$ spin algebras, may be obtained

from the second Hamiltonian structure of Lax equations for a Lax operator of type (1).

We realize (the regular part of) the dual space to $\mathscr{S}$ as follows:

$$\mathscr{S}^* = \left\{ l = \sum_{m=1}^{n} l_m \partial^{-m} \,\middle|\, l_m \in C^\infty(S^1) \right\},$$

and the pairing $\mathscr{S} \otimes \mathscr{S}^* \to \mathbf{R}$ is given by the formula

$$\langle \Delta^{(n)}, l \rangle = \int_{S^1} \mathrm{Res}(\Delta^{(n)} l)\, dx, \tag{2}$$

where $Res(X)$ is the coefficient of $\partial^{-1}$ in a pseudodifferential operator $X$.

In the AGD formalism, we assign a vector field $V_X$ on $\mathscr{S}$ to every regular linear functional $X$. Its value at a point $\Delta^{(n)} \in \mathscr{S}$ is

$$V_{X_f}(\Delta^{(n)}) = \Delta^{(n)} (X_f \Delta^{(n)})_+ - \Delta^{(n)} (X_f)_+ \Delta^{(n)}, \tag{3}$$

the positive sign denoting the projection to the differential part.

Thus we define a Lie-Poisson bracket on the space of smooth functions on $\mathscr{S}$:

$$\{f, g\}_{\Delta^{(n)}} = \langle dg|_{\Delta^{(n)}}, V_{df|_{\Delta^{(n)}}}(\Delta^{(n)}) \rangle. \tag{4}$$

This bracket is called the Gelfand-Dikii bracket.

**2.1. Immersion and solution curve**   At first we will consider the $n = 2$ case. In this case, $\mathscr{S}$ is the space of Hill's operators of the form $\frac{\partial^2}{\partial x^2} + u(x)$.

LEMMA 2.1. *There is a one to one correspondence between*

(1)  *the Hill's equation on $S^1$*

$$\Delta\psi = \psi'' + u\psi = 0,$$

*where $u \in C^\infty(S^1)$ and $\psi$ is the unknown function; and*
(2)  *smooth orientation preserving immersions $g : S^1 \to \mathbb{R}P^1$, modulo the equivalence up to $PSL(2, \mathbf{R})$.*

This proof is very easy, it says that if we choose two independent solutions $\psi_1$ and $\psi_2$, then

$$x \mapsto (\psi_1(x), \psi_2(x)) \tag{5}$$

defines an immersion $\mathbf{R} \to \mathbb{R}P^1$ in homogeneous coordinates. This defines a curve in the projective line $\mathbb{R}P^1$. Since the Wronskian of the solution curve is constant up

to multiplication by a matrix in $SL(2, \mathbf{R})$, then the Wronskian $\psi_1'\psi_2 - \psi_1\psi_2'$ of any immersion can be written in the form of (5) and is equal to one.

This picture can be easily extended to the case of an $n$th-order scalar differential operator. The AGD manifold is an infinite-dimensional Fréchet manifold of scalar differential operators with smooth and periodic coefficients. Associating with the equation $\Delta\psi = 0$ we define $n$ independent solutions $(\psi_1, \psi_2, \ldots, \psi_n)$. The map

$$x \longmapsto (\psi_1(x), \psi_2(x), \ldots, \psi_n(x)) \tag{6}$$

defines an immersion $g : \mathbf{R} \to \mathbb{R}P^{n-1}$ in homogeneous coordinates. Thus we obtain a solution curve associated to $L$; once again the Wronskian of the components equals one. Since coefficients are periodic, if $\psi(x)$ is a solution then $\psi(x + 2\pi)$ is also a solution. This implies

$$\psi(x + 2\pi) = M_\psi\psi(x),$$

where $M_\psi = \psi(2\pi)\psi(0)^{-1}$ is a monodromy matrix. This matrix preserves the skew form given by the Wronskian, so $\det(M_\psi) = 1$, that is, $M_\psi \in SL(n, \mathbf{R})$. If one chooses a different solution curve then the new monodromy matrix will appear; this will be the conjugate of $M_\psi$ by an element of $SL(n, \mathbf{R})$. This means that for each Lax operator we can associate with it a projective curve whose monodromy will be an element of the conjugacy class $[M_\psi]$. This curve is unique up to the projective action of $SL(n, \mathbf{R})$.

## 3. Projective structures

Let $\Omega$ denote the cotangent bundle of the circle; this is a trivial real line bundle on $S^1$. Let $\Omega^{-1}$ and $\Omega^m$ be the tangent bundle and the $m$-fold tensor product of $\Omega$ respectively. The section of $\Omega^m$ is locally given by $s = g(x)\, dx^m$, where $g(x) = g(x + 2\pi)$. There is a natural action of $\mathrm{Diff}(S^1)$ on the sections of $\Omega^m$

$$\mathscr{L}_v s = (f g' + m f' g)\, dx^m, \tag{7}$$

where $\mathscr{L}_v$ is the Lie derivative of the vector field $v = f \frac{d}{dx}$. Suppose $w = g\frac{d}{dx} \in \Omega^{-1}$. Hence we get

$$\mathscr{L}_v w = [v, w] = (gf' - g'f)\, \frac{d}{dx}. \tag{8}$$

**3.1. Projective connection on the circle**    Let us denote by $\Omega^{\pm 1/2}$ the square root of the tangent and cotangent bundle respectively.

DEFINITION 3.1 ([12, 14]). A projective connection on the circle is a linear second-order differential operator $\Delta : \Gamma(\Omega^{-1/2}) \to \Gamma(\Omega^{3/2})$ such that

(1)  the symbol of $\Delta$ is the identity;
(2)  $\int_{S^1} (\Delta s_1) s_2 = \int_{S^1} s_1 (\Delta s_2)$ for all $s_i \in \Gamma(\Omega^{-1/2})$.

Let us take $s = \psi(x) \, dx^{-1/2} \in \Gamma(\Omega^{-1/2})$, then $\Delta s \in \Gamma(\Omega^{3/2})$ is locally described by

$$\Delta s = (a\psi'' + b\psi' + c\psi) \, dx^{3/2}.$$

As discussed in [7], any differential equation of the form

$$\frac{d^2 y}{dx^2} = p_3 \left( \frac{dy}{dx} \right)^3 + p_2 \left( \frac{dy}{dx} \right)^2 + p_1 \left( \frac{dy}{dx} \right) + p_0$$

defines a projective structure.

From the definition of the projective connection, condition (1) implies $a = 1$ and condition (2) implies $b = 0$, hence projective connection can be identified with the Hill operator $\Delta^{(2)} \equiv \Delta = d^2/dx^2 + u(x)$.

Using the equation $\mathscr{L}_v \Delta^{(2)} = \Delta^{(2)} \mathscr{L}_v$, we obtain the following proposition.

PROPOSITION 3.2. *A projective vector field $v = f \frac{d}{dx} \in \Gamma(\Omega^{-1})$ satisfies*

$$f''' + 4f'u + 2f u' = 0.$$

**3.2. Extended projective connections**  In this section we will explicitly compute the equations satisfied by projective vector fields for different values of $n$.

DEFINITION 3.3 (Extended Projective Connection).  An extended projective connection on the circle is a class of differential (conformal) operators

$$\Delta^{(n)} : \Gamma(\Omega^{-(n-1)/2}) \to \Gamma(\Omega^{(n+1)/2})$$

such that

(1)  the symbol of $\Delta^{(n)}$ is the identity;
(2)  $\int_{S^1} (\Delta^{(n)} s_1) s_2 = \int_{S^1} s_1 (\Delta^{(n)} s_2)$ for all $s_i \in \Gamma(\Omega^{-(n-1)/2})$.

It is known that the symbol of a $n$th-order operator from a vector bundle $U$ to $V$ is a section of $\mathrm{Hom}(U, V \otimes \mathrm{Sym}^n T)$, where $U = \Omega^{-(n-1)/2}$ and $V = \Omega^{(n+1)/2}$. Since $T = \Omega^{-1}$, we get $V \otimes \mathrm{Sym}^n T \cong U$, giving an invariant meaning to the first condition.

If $s_2 \in \Gamma(\Omega^{-(n-1)/2})$ then $s_1 \Delta^{(n)} s_2 \in \Gamma(\Omega)$ is a one form to integrate.

The consequence of the *first condition* is that all the differential operators are monic, that is, the coefficient of the highest derivative is always one, and the *second condition* says that the term $u_{n-1} = 0$.

The weights $-(n-1)/2$ and $(n+1)/2$ related to the space of operator $\Delta^{(n)}$ are known to physicists and mathematicians [13], but not in relation to extended projective connections.

DEFINITION 3.4. If $\Delta$ is a projective connection then a vector field is called a projective vector field (relative to $\Delta$) if

$$\mathscr{L}_v \Delta^{(n)} s = \Delta^{(n)}(\mathscr{L}_v s), \tag{9}$$

for all $s \in \Gamma(\Omega^{-(n-1)/2})$.

**3.3. Invariant property of the Wronskian**    In Section 2.1, we saw that the Wronskian of the solutions of a $n$th-order differential equation is constant, and that the solution defines an immersion $\mathbf{R} \to \mathbb{R}P^{n-1}$ in homogeneous coordinates.

The invariant meaning of the Wronskian can also be easily realized in terms of projective connections.

**Case when $n = 2$:**

If $a, b \in \Gamma(\Omega^{-1/2})$, then we find $ab' - a'b \in C^\infty(S^1)$, where $a', b' \in \Gamma(\Omega^{1/2})$.

If $a, b$ satisfy $d^2/dx^2 + u(x) = 0$, then the Wronskian is constant, since $a = \psi_1 dx^{-1/2}$, $b = \psi_2 dx^{-1/2}$ satisfy

$$(\psi_1 \psi_2' - \psi_1' \psi_2)' = \psi_1 \psi_2'' - \psi_1'' \psi_2 = 0.$$

**Case when $n = 3$:**

If $a, b, c \in \Gamma(\Omega^{-1})$, then the Wronskian

$$\mathrm{Wr}(a, b, c) = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

is an element of $C^\infty(S^1)$. Again, if $a, b, c$ satisfy $d^3/dx^3 + 4u \, d/dx + 2u' = 0$, then the Wronskian is constant, since $a = \psi_1 dx^{-1}$, $b = \psi_2 dx^{-1}$ $c = \psi_3 dx^{-1}$ satisfy

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}' = 0.$$

This can be generalized to higher values of $n$.

## 4. Diff($S^1$) action on the AGD manifold

In this section, we describe the transformations of the AGD operator under the action of Diff($S^1$). This transformation of scalar differential operators has been

known since last century. The action of $\mathrm{Diff}(S^1)$ induces a change of variable in the independent parameter $x$.

DEFINITION 4.1. The $\mathrm{Vect}(S^1)$ action on $\Delta^n$ is defined by

$$[\mathscr{L}_v, \Delta^{(n)}] := \mathscr{L}_v^{-(n+1)/2} \circ \Delta^{(n)} - \Delta^{(n)} \circ \mathscr{L}_v^{(n-1)/2}. \tag{10}$$

Hence (10) can be considered as some coadjoint action of $\mathrm{Vect}(S^1)$ on some "dual" space [20].

Suppose $\Delta^{(n)}$ to be a scalar differential operator. The action of $\mathrm{Diff}(S^1)$ transforms the solutions of $\Delta^{(n)}\psi = 0$ as densities of degree $(n-1)/2$.

Let us consider the action of $\mathrm{Diff}(S^1)$:

$$\mathrm{Diff}(S^1) \times AGD \to AGD$$
$$(x, \Delta^{(n)}) \longmapsto (\sigma(x), \Delta^{(n)}),$$

where $\sigma(x)\Delta^{(n)}$ is the unique scalar operator of the form $\Delta^{(n)}$ having $u_{n-1} = 0$. If $\mu$ and $\xi$ are the solutions of $\sigma(x)\Delta^{(n)}$ and $\Delta^{(n)}$, then their solutions are connected by

$$\mu = \left(x'^{(n-1)/2}\right) \circ \xi.$$

In the case of $n = 2$, this coincides with the action of the Virasoro group on the space of Hill operators, the dual space of Virasoro algebra.

It should be noted that the operators $\Delta^{(n)}$ do not preserve their form under the action of $\mathrm{Diff}(S^1)$, $x \mapsto \sigma(x)$, due to the appearance of the $(n-1)$th term

$$-\frac{n(n-1)}{2}\left(\frac{\sigma''}{\sigma'^{n+1}}\right).$$

Hence we should think of the operators as acting on *densities of weight* $-1/2(n-1)$ rather than on scalar functions; in this case we can always find $u_{n-1} = 0$ as a reparametrization invariant. Therefore the action of $\mathrm{Diff}(S^1)$ on $\Delta^n$ is given by

$$\partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \cdots + u_0(x) \to \sigma'^{-(n+1)/2}(\partial_x^n + \tilde{u}_{n-2}\partial_x^{n-2} + \cdots + \tilde{u}_0)\sigma'^{-(n-1)/2},$$

where $\tilde{u}_{n-2} = \sigma' u_{n-2}(\sigma(x)) + (1/12)n(n-1)(n+1)S(x)$. Here $S(x)$ is the Schwarzian.

In particular, for $n = 3$ we find

$$\tilde{u}_1(x) = \sigma'^2 u_1(\sigma(x)) + 2S(x),$$
$$\tilde{u}_0(x) = \sigma'^3 u_0(\sigma(x)) + \sigma'\sigma'' u_1(\sigma(x)) + S'(x).$$

For $n = 4$ we obtain

$$\tilde{u}_2(x) = \sigma'^2 u_2(\sigma(x)) + 5S(x),$$

$$\tilde{u}_1(x) = \sigma'^3 u_1(\sigma(x)) + 2\sigma'\sigma'' u_2(\sigma(x)) + 5S'(x),$$

$$\tilde{u}_0(x) = \sigma'^4(\sigma(x)) + \frac{3}{2}\sigma'^2\sigma'' u_1(\sigma(x)) + \frac{3}{2}\sigma''^2 u_2(\sigma(x))$$

$$+ \frac{3}{2}\sigma'^2 u_2(\sigma(x))S(x) + \frac{3}{2}S''(x) + \frac{3}{2}S^2(x).$$

## 5.  The Euler-Arnold equation and the KdV equation

We want to investigate the analogous equation for the extended projective connection on the circle, that is,

$$\frac{\partial \Delta^{(n)}}{\partial t} + [\mathscr{L}_v, \Delta^{(n)}] = 0, \tag{11}$$

where $[\mathscr{L}_v, \Delta^{(n)}]$ is given in (10). It should be noted that one single equation appears here, this has been considered in various places in different contexts [2, 3].

PROPOSITION 5.1. *Equation* (11) *with* $n < 5$ *generates only one single equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}f''' + 2f'u + f u'. \tag{12}$$

PROOF. We prove this by direct computation.

For a special choice of $f = u$ we obtain the KdV equation. Geometrically this follows from the argument below.

We have argued that the action $[\mathscr{L}_v, \Delta^{(n)}]$ can be considered as a coadjoint action of $\text{Vect}(S^1)$ on $\Delta^{(n)}$. Using (12) we know

$$\tilde{u} = \frac{1}{2}f''' + 2f'u + f u' = \left(\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x\right)f.$$

The operator $(\frac{1}{2}\partial_x^3 + 2u\partial_x + u_x)$ is called the symplectic operator. The Euler equation is the Hamiltonian flow on the coadjoint orbits in $\widehat{\Omega\mathscr{G}}^*$ generated by the Hamiltonian $H(u) = u^2(x)/2$, given by

$$\frac{d}{dt}u(t) = -ad_{u(t)}^* u(t).$$

Thus by applying the Euler-Arnold equation we prove Theorem 1.1.

## 6. Flows on the curve space

In Section 2.1, we saw how an immersion associated with $\Delta^{(n)}$ yields a curve $\psi : \mathbf{R} \to \mathbb{R}P^{n-1}$ in the projective space.

Let us write $\psi$ in terms of inhomogeneous coordinates. We lift $\psi$ to a curve on $\mathbf{R}^n$. This we may denote by $\bar{\psi} = \eta(x)(1, \psi)$. We choose the factor $\eta(x)$ so that the Wronskian of the components of the new curve equals 1.

It turns out that there is a unique choice of $\eta(x)$ with such a property, and this is given by $\eta(x) = \mathrm{Wr}(\psi_1', \dots, \psi_{n-1}')^{-1/n}$.

In particular, for the case when $n = 2$,

$$\psi \equiv (\psi_1, \psi_2) = (\psi'^{-1/2}, \psi'^{-1/2}\psi) \tag{13}$$

is the solution curve [10]. It retains the unitarity of the Wronskian.

LEMMA 6.1. *The equation $f''' + 2u'f + 4uf' = 0$ traces out a three-dimensional space of solutions.*

PROOF. If $\psi_1$ and $\psi_2$ are the solutions of

$$\Delta\psi = (d^2/dx^2 + u)\psi = 0, \tag{14}$$

then it is easy to see that $\psi_i\psi_j \in \Gamma(\Omega^{-1})$ satisfies the above equation. Hence the solution space is spanned by $\psi_1^2$, $\psi_2^2$ and $\psi_1\psi_2$.

Substituting (13) in to $(d^2/dx^2 + u)\psi_i = 0$ (for $i = 1, 2$), we obtain

$$u = \frac{1}{2}\left(\frac{\psi'''}{\psi'} - \frac{\psi''^2}{\psi'^2}\right).$$

The right-hand side is invariant under the $PSL(2, \mathbf{R})(= SL(2, \mathbf{R}/\pm 1)$ group. If we substitute this expression into the Euler-Arnold equation, we obtain the evolution equation of the solution curve on the projective space

$$\psi_t = \psi_{xxx} - 3\psi_{xx}^2\psi_x^{-1}/2.$$

This equation is called the Schwarzian KdV (or Ur KdV by the Wilson school [22, 29, 30]). One can check directly that this equation is $SL(2, \mathbf{R})$ invariant. Thus we obtain Theorem 1.2.

The Schwarzian KdV has a bihamiltonian structure [29]

$$H_1 = 4u, \qquad D_1 = -2\psi_x^{-1}\partial\psi_x^{-1},$$

$$H_2 = u^2, \qquad D_2 = -\frac{1}{2}\psi_x^{-2}\partial^3 - 3\psi_{xx}\psi_x^{-3}\partial^2 + (3\psi_{xx}^2\psi_x^{-4} - \psi_{xxx}\psi_x^{-3})\partial.$$

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