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Group schemes and local densities of quadratic lattices in residue characteristic 2

Sungmun Cho

Dedicated to my parents

Abstract

The celebrated Smith–Minkowski–Siegel mass formula expresses the mass of a quadratic lattice (L, Q) as a product of local factors, called the local densities of (L, Q). This mass formula is an essential tool for the classification of integral quadratic lattices. In this paper, we will describe the local density formula explicitly by observing the existence of a smooth affine group scheme $\underline{G}$ over $\mathbb{Z}_2$ with generic fiber $\mathrm{Aut}_{\mathbb{Q}_2}(L, Q)$, which satisfies $\underline{G}(\mathbb{Z}_2) = \mathrm{Aut}_{\mathbb{Z}_2}(L, Q)$. Our method works for any unramified finite extension of $\mathbb{Q}_2$. Therefore, we give a long awaited proof for the local density formula of Conway and Sloane and discover its generalization to unramified finite extensions of $\mathbb{Q}_2$. As an example, we give the mass formula for the integral quadratic form $Q_n(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ associated to a number field k which is totally real and such that the ideal (2) is unramified over k.

1. Introduction

The problem of local densities has intrigued many great mathematicians, including Gauss and Eisenstein, Smith and Minkowski, and Siegel. If (L, Q) is a quadratic R-lattice, where R is the ring of integers of a number field, then the celebrated Smith–Minkowski–Siegel mass formula expresses the mass of a quadratic lattice (L, Q) as a product of local factors, called the local densities of (L, Q). The local density is defined as the limit of a certain sequence. Pall [Pal65] (for $p \neq 2$) and Watson [Wat76] (for $p = 2$) computed this limit for an arbitrary lattice over $\mathbb{Z}_p$, thereby deriving an explicit formula for its local density. For an expository sketch of their approach, see [Kit93]. There is another proof of Hironaka and Sato [SH00] computing the local density when $p \neq 2$. They treat an arbitrary pair of lattices, not just a single lattice, over $\mathbb{Z}_p$ (for $p \neq 2$). Conway and Sloane [CS88] further developed the formula for any p and gave a heuristic explanation for it. They mentioned in [CS88] regarding the local density formula for $p = 2$ that 'Watson's formula seems to us to be essentially correct' and in the footnote that 'in fact we could not quite reconcile Watson's version with ours; they appear to differ by a factor of 2^n for n-dimensional forms. This is almost certainly due to our misunderstanding of Watson's conventions, which differ considerably from ours.' In addition, they mentioned in the same paper that 'The reader may be confident that our version of the general mass formula is correct. . . . has enabled us to test the formula very stringently.' Their formula was computationally tested stringently and the proof of Conway–Sloane's local density formula, when $p = 2$, has not been published in literature.

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On the other hand, there is a simpler formulation of the local density as the integral of a certain volume form ω^{ld} over some open compact subgroup of an orthogonal group, due to Kneser, Tamagawa, and Weil. As explained in the introduction of [GY00], known methods unfortunately do not explain this formulation and involve complicated recursions.

Meanwhile, the work [GY00] by Gan and Yu is based on the existence of a smooth affine group scheme $\underline{G}$ over $\mathbb{Z}_p$ with generic fiber $\mathrm{Aut}_{\mathbb{Q}_p}(L,Q)$, which satisfies $\underline{G}(\mathbb{Z}_p)=\mathrm{Aut}_{\mathbb{Z}_p}(L,Q)$. By constructing $\underline{G}$ explicitly and determining its special fiber, they computed the integral and therefore obtained the formula for the local density when $p\neq 2$.

The main contribution of this paper is to construct $\underline{G}$ and investigate its special fiber in order to get an explicit formula for the local density when L is a quadratic A-lattice, where A is an unramified finite extension of $\mathbb{Z}_2$. Therefore, we give a long awaited proof for the local density formula of Conway and Sloane. Furthermore, we discover its generalization to unramified finite extensions of $\mathbb{Q}_2$. The special fiber of $\underline{G}$ has a large component group of the form $(\mathbb{Z}/2\mathbb{Z})^N$. That is, we discover a large number of independent homomorphisms from the special fiber of $\underline{G}$ to the constant group $\mathbb{Z}/2\mathbb{Z}$. Consequently, when replacing $\mathbb{Z}_2$ by an unramified finite extension with residue field F_q, where q is a power of 2, the 2-power factor 2^M in the formula of Conway and Sloane has to be replaced by $2^N\cdot q^{M-N}$. This fact is far from obvious from Conway–Sloane's explanation.

In conclusion, this paper, combined with [GY00], allows the computation of the mass formula for a quadratic R-lattice (L,Q) when the ideal (2) is unramified over R.

This paper is organized as follows. We first state the structural theorem for integral quadratic forms in § 2. We then give an explicit construction of $\underline{G}$ (in § 3) and its special fiber (in § 4). Finally, by comparing ω^{ld} and the canonical volume form ω^{can} of $\underline{G}$, we obtain an explicit formula for the local density in § 5. In § 6, as an example, we give the mass formula for the integral quadratic form $Q_n(x_1,\ldots,x_n)=x_1^2+\cdots+x_n^2$ associated to a number field k which is totally real and such that the ideal (2) is unramified over k. This formula is explicitly described using the Dedekind zeta function of k and a certain Hecke L-series.

As in [GY00], the smooth group schemes constructed in this paper should be of independent interest.

2. Structural theorem for quadratic lattices and notation

2.1 Notation

The notation and definitions in this subsection are taken from [O'Me55, O'Me00]. Let F be an unramified finite extension of $\mathbb{Q}_2$ with A its ring of integers and κ its residue field.

We consider an A-lattice L with a quadratic form $q:L\to A$. We denote by a pair (L,q) a quadratic lattice. Let $\langle -,-\rangle_q$ be the symmetric bilinear form on L such that

$$\langle x,y\rangle_q=\tfrac{1}{2}(q(x+y)-q(x)-q(y)).$$

We assume that $\langle x,y\rangle_q\in A$ and $V=L\otimes_A F$ is nondegenerate with respect to $\langle -,-\rangle_q$.

For any $\epsilon\in A$, we denote by (ϵ) the A-lattice of rank 1 equipped with the symmetric bilinear form having Gram matrix (ϵ). We use the symbol $\epsilon\cdot A(\alpha,\beta)$ to denote the A-lattice $A\cdot e_1+A\cdot e_2$ with the symmetric bilinear form having Gram matrix $\epsilon\cdot\left(\begin{smallmatrix}\alpha & 1\\ 1 & \beta\end{smallmatrix}\right)$.

A quadratic lattice L is the *orthogonal sum* of sublattices L_1 and L_2, written $L=L_1\oplus L_2$, if $L_1\cap L_2=0$, L_1 is orthogonal to L_2 with respect to the symmetric bilinear form $\langle -,-\rangle_q$, and L_1 and L_2 together span L.

The fractional ideal generated by $q(X)$ as X runs through L will be called the *norm* of L and written $N(L)$.

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By the *scale* $S(L)$ of L, we mean the fractional ideal generated by the subset $\langle L, L\rangle_q$ of F.

The *discriminant* of L, denoted by $d(L)$, is defined as the determinant of a Gram matrix defining the symmetric bilinear form $\langle -, -\rangle_q$, up to multiplication by the square of a unit.

We say (L, q) is *unimodular* if the discriminant $d(L)$ is a unit and a Gram matrix defining $\langle -, -\rangle_q$ has integral entries.

DEFINITION 2.1. For a given quadratic lattice L:

(a) a unimodular lattice L is *of parity type I* if $N(L) = A$; otherwise it is *of parity type II*;

(b) (L, q) is *modular* if $(L, a^{-1}q)$ is unimodular for some $a \in A\backslash\{0\}$, where a is unique up to a unit, and in this case the parity type of (L, q) is defined to be the parity type of $(L, a^{-1}q)$;

(c) the zero lattice is considered to be *modular of parity type II*.

DEFINITION 2.2. We define the dual lattice of L, denoted by $L^\perp$, as

$$L^\perp = \{x \in L \otimes_A F : \langle x, L\rangle_q \subset A\}.$$

Remark 2.3. (a) The scale $S(L)$ of a unimodular lattice L is always A. Based on the definition of a unimodular lattice, the discriminant $d(L)$ is a unit and a Gram matrix defining $\langle -, -\rangle_q$ has integral entries. Thus there is at least one unit entry of a Gram matrix and this implies our claim.

(b) [O'Me00, 93:15] It is well known that a lattice of parity type I is diagonalizable. In other words,

$$L = \bigoplus_i (u_i),$$

where the u_i are units.

(c) If a unimodular lattice L is *of parity type II*, then

$$L = \bigoplus_i A(a_i, b_i),$$

where a_i and b_i are elements of the prime ideal (2) of A. Thus the rank of L is even.

2.2 Structural theorem for quadratic lattices

We state the structural theorem for unimodular lattices as follows. Indeed, this theorem is a summary of some results from [O'Me00, § 93].

THEOREM 2.4. *Assume that L is unimodular. If L is of parity type I, then there is an orthogonal decomposition*

$$L \cong \bigoplus_i A_i(0, 0) \oplus K \oplus K'.$$

Here, $A_i(0, 0) = A(0, 0)$, K is empty or $A(2, \lambda)$ with $\lambda \in (2)$, and K' is (ϵ) or $A(1, 2\gamma)$ where $\epsilon \equiv 1 \bmod (2)$ and $\gamma \in A$.

If L is of parity type II, then we have an orthogonal decomposition

$$L \cong \bigoplus_i A_i(0, 0) \oplus A(a, b).$$

Here $A_i(0, 0) = A(0, 0)$ and $a, b \in (2)$.

Proof. We use notation and terminology from [O'Me00]. It is easily seen that the set $q(L) + 2(S(L))$ is an additive subgroup of F. We let $M(L)$ denote the largest fractional ideal contained in the group $q(L)+2(S(L))$. Then we define the *weight* by $2(M(L))+2(S(L))$. We call the scalar $\mathbf{a}$ a *norm generator* of L if $\mathbf{a}\in q(L)+2(S(L))$ and $\mathbf{a}A=N(L)$. We call the scalar $\mathbf{b}$ a *weight generator* of L if $\mathbf{b}A=2(M(L))+2(S(L))$. Let d be the discriminant of L. Regard d as an element of the unit group $\mathbf{u}$ of A.

We first state the following seven facts proved in [O'Me00, Example 93:10 and § 93:18].

(i) For every unit u in the unit group $\mathbf{u}$, there is a solution v in $\mathbf{u}$ such that $v^2\equiv u(\bmod 2)$ because the residue field κ is perfect and of characteristic 2. Thus an A-lattice (u) of rank 1 is isometric to an A-lattice (ϵ) of rank 1, where $\epsilon\equiv 1 \bmod (2)$.

(ii) If $\dim L\geqq 5$, then

$$L\cong A(0,0)\oplus\cdots.$$

(iii) We assume that $\dim L=4$ with $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ odd. Here, $\mathrm{ord}_2\mathbf{a}$ (respectively $\mathrm{ord}_2\mathbf{b}$) is the exponential order of $\mathbf{a}$ (respectively $\mathbf{b}$) at the prime ideal (2) in A. We suppose that d has been expressed in the form $d=1+\alpha$ with $\alpha\in\mathbf{ab}A$. For a given $\varrho\in A$ such that $4\varrho\in\mathbf{ab}A$, consider the lattices

$$\begin{aligned}J&=A(\mathbf{b},0)\oplus A(\mathbf{a},-\alpha\mathbf{a}^{-1}),\\ J'_\varrho&=A(\mathbf{b},4\varrho\mathbf{b}^{-1})\oplus A(\mathbf{a},-(\alpha-4\varrho)\mathbf{a}^{-1}).\end{aligned}$$

Then L is isomorphic to J or J'_ϱ.

(iv) If $\dim L=2$ with $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ odd, we have the following:

$$L\cong A(\mathbf{a},\mathbf{b}\varrho)\quad\text{for some } \varrho\in A.$$

(v) Let $\dim L=3$ with $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ odd. Then we have

$$L\cong A(\mathbf{b},0)\oplus(-d)$$

or

$$L\cong A(\mathbf{b},4\varrho\mathbf{b}^{-1})\oplus(-d(1-4\varrho))\quad\text{for some } \varrho\in A \text{ such that } 4\varrho\in\mathbf{b}A.$$

(vi) If $\dim L=2$ with $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ even, we have

$$L\cong A(0,0)\quad\text{or}\quad L\cong A(2,2\varrho)\quad\text{for some } \varrho\in A.$$

(vii) If $\dim L\geqq 3$ and $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ even, then

$$L\cong A(0,0)\oplus\cdots.$$

From (ii), we may and do assume that the rank of L is at most 4.

Assume that L is *of parity type I* so that $N(L)=A$. Thus we may assume that $\mathbf{a}=1$ and $\mathrm{ord}_2\mathbf{a}=0$. Since $2(S(L))\subsetneqq 2(M(L))+2(S(L))\subseteqq(2)$ and $2(S(L))=(2)$, we have the following equality:

$$(\mathbf{b})=2(M(L))+2(S(L))=(2).$$

Hence $\mathrm{ord}_2\mathbf{b}=1$ and $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ is odd. We choose $\mathbf{b}=2$. If the rank of L is odd (respectively even), the theorem follows from (i) and (v) (respectively from (iii) and (iv)).

If L is *of parity type II* so that $N(L)=(2)$, $\mathrm{ord}_2\mathbf{a}=\mathrm{ord}_2\mathbf{b}=1$ and $\mathrm{ord}_2\mathbf{a}+\mathrm{ord}_2\mathbf{b}$ is even. Choose $\mathbf{a}=\mathbf{b}=2$. Then the theorem follows from (vi) and (vii). □

For a general lattice L, we have a Jordan splitting, namely $L=\bigoplus_i L_i$ such that L_i is *modular* and the sequence $\{s(i)\}_i$ increases, where $(2^{s(i)})=S(L_i)$. Unfortunately, a Jordan splitting of L

is not unique. Nevertheless, we will attach certain well-defined quantities to L at the end of this section. We recall the following theorem from [O'Me00].

THEOREM 2.5 [O'Me00, § 91:9]. *Let*

$$L = L_1 \oplus \cdots \oplus L_t, \quad L = K_1 \oplus \cdots \oplus K_T$$

be two Jordan splittings of L with L_i, K_j non-zero for all i, j. Then $t = T$. Furthermore, for $1 \leqq i \leqq t$, the scale, rank and parity type of L_i are the same as those of K_i.

If we allow L_i to be the zero lattice, then we may assume $S(L_i) = (2^i)$ without loss of generality. We can rephrase this theorem as follows. Let $L = \bigoplus_i L_i$ be a Jordan splitting with $s(i) = i$ for all $i \geqslant 0$. Then the scale, rank and parity type of L_i depend only on L. We will deal exclusively with a Jordan splitting satisfying $s(i) = i$ from now on.

2.3 Lattices

In this subsection, we will define several lattices and corresponding notation. Assume that a quadratic lattice (L, q) is given and $S(L) = (2^l)$. The following lattices will play a significant role in our construction of the smooth integral model:

(1) $A_i = \{x \in L \mid \langle x, L\rangle_q \in 2^i A\}$;
(2) $X(L)$, the sublattice of L such that $X(L)/2L$ is the kernel of the symmetric bilinear form $(1/2^l)\langle -, -\rangle_q$ mod 2 on $L/2L$;
(3) $B(L)$, the sublattice of L such that $B(L)/2L$ is the kernel of the linear form $(1/2^l)q$ mod 2 on $L/2L$.

To define our integral structure a few more lattices will be needed, but we need some preparation to define them.

Assume $B(L) \subsetneq L$. Hence the bilinear form $(1/2^l)\langle -, -\rangle_q$ mod 2 on the κ-vector space $L/X(L)$ is nonsingular symmetric and non-alternating. It is well known [KMRT98, Exercise 16, ch. 6] that there is a unique vector $e \in L/X(L)$ such that $((1/2^l)\langle v, e\rangle_q)^2 = (1/2^l)\langle v, v\rangle_q$ mod 2 for every vector $v \in L/X(L)$. Let $\langle e\rangle$ denote the 1-dimensional vector space spanned by the vector e and denote by $e^\perp$ the 1-codimensional subspace of $L/X(L)$ orthogonal to the vector e with respect to $(1/2^l)\langle -, -\rangle_q$ mod 2. Then

$$B(L)/X(L) = e^\perp.$$

If $B(L) = L$, then the bilinear form $(1/2^l)\langle -, -\rangle_q$ mod 2 on the κ-vector space $L/X(L)$ is nonsingular symmetric and alternating. In this case, we put $e = 0 \in L/X(L)$ and note that it is characterized by the same identity.

The remaining lattices we need for our definition are:

(4) $W(L)$, the sublattice of L such that $W(L)/X(L) = \langle e\rangle$;
(5) $Y(L)$, the sublattice of L such that $Y(L)/2L$ is the kernel of the symmetric bilinear form $(1/2^l)\langle -, -\rangle_q$ mod 2 on $B(L)/2L$;
(6) $Z(L)$, the sublattice of L such that $Z(L)/2L$ is the kernel of the quadratic form $(1/2^{l+1})q$ mod 2 on $B(L)/2L$.

Remark 2.6. (a) We can associate the five lattices above $(X(L), B(L), W(L), Y(L), Z(L))$ to $(A_i, (1/2^i)q)$. Denote the resulting lattices by X_i, B_i, W_i, Y_i, Z_i.

(b) As κ-vector spaces, the dimensions of $A_i/B_i, W_i/X_i, Y_i/Z_i$ are at most 1.

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Let $L=\bigoplus_i L_i$ be a Jordan splitting. We assign a type to each L_i as follows:

$$\begin{cases} I & \text{if } L_i \text{ is of parity } \textit{type } I,\\ I^o & \text{if } L_i \text{ is of parity } \textit{type } I \text{ and the rank of } L_i \text{ is odd},\\ I^e & \text{if } L_i \text{ is of parity } \textit{type } I \text{ and the rank of } L_i \text{ is even},\\ II & \text{if } L_i \text{ is of parity } \textit{type } II. \end{cases}$$

In addition, we say that L_i is

$$\begin{cases} \text{bound} & \text{if at least one of } L_{i-1} \text{ or } L_{i+1} \text{ is of parity } \textit{type } I,\\ \text{free} & \text{if both } L_{i-1} \text{ and } L_{i+1} \text{ are of parity } \textit{type } II. \end{cases}$$

Assume that a lattice L_i is *free of type* I^e. We denote by $\bar{V}_i$ the κ-vector space B_i/Z_i. Then we say that L_i is

$$\begin{cases} \text{of } \textit{type } I^e_1 & \text{if the dimension of } \bar{V}_i \text{ is odd, equivalently, } X_i=Z_i,\\ \text{of } \textit{type } I^e_2 & \text{otherwise.} \end{cases}$$

We stress that we do not assign a type I^e_1 or I^e_2 to *bound* lattices. Notice that each type of L_i is independent of the choice of a Jordan splitting.

Example 2.7. When L is *unimodular*, we assign a type to L according to the shape of K' described in Theorem 2.4 as follows.

K'	Type of L
$\emptyset$	II
(ϵ)	I^o
$A(1,2\gamma)$ with γ unit	I^e_1
$A(1,2\gamma)$ with $\gamma\in(2)$	I^e_2

Remark 2.8. We describe all these six lattices explicitly. We use the following conventions: $\mathcal{L}_i$ denotes $\bigoplus_{j\neq i}2^{\max\{0,i-j\}}L_j$. We denote by $\mathcal{M}_i$ the $\bigoplus_i A_i(0,0)\oplus K$ of Theorem 2.4, for any i, when L_i is *of type I*. When L_i is *of type II*, let $\mathcal{M}_i=L_i$. Theorem 2.4 involves a basis for a lattice K', which we write as $\{e_1^{(i)}\}$ or $\{e_1^{(i)},e_2^{(i)}\}$ if L_i is *of type* I^o or *of type* I^e, respectively.

For all cases, we have $A_i=\mathcal{L}_i\oplus L_i$ and $X_i=\mathcal{L}_i\oplus 2L_i$. If L_i is *of type I*, then the vector $e\in A_i/X_i$ is $(0,\dots,0,1)$. Based on this, the lattices B_i,W_i,Y_i are as follows.

Type	B_i	W_i	Y_i
I^o	$\mathcal{L}_i\oplus\mathcal{M}_i\oplus 2Ae_1^{(i)}$	$\mathcal{L}_i\oplus\cdot 2\mathcal{M}_i\oplus Ae_1^{(i)}$	X_i
I^e	$\mathcal{L}_i\oplus\mathcal{M}_i\oplus 2Ae_1^{(i)}\oplus Ae_2^{(i)}$	$\mathcal{L}_i\oplus 2\mathcal{M}_i\oplus 2Ae_1^{(i)}\oplus Ae_2^{(i)}$	W_i
II	A_i	X_i	X_i

The description of Z_i is a little complicated. Note that when L_i is *free of type* I^e_1 or *bound*, the dimension of Y_i/Z_i as a κ-vector space is *1*. We describe it case by case below. The main idea is to start from Y_i then to find a co-rank *1* sublattice as the kernel of the associated linear form $(1/2^{i+1})q$ mod 2 on $Y_i/2Y_i$.

(i) If L_i is *free of type* I^o*,* I^e_2*, or II*, then $Z_i=Y_i$.

(ii) If L_i is *free of type* I^e_1, then $Z_i=X_i$.

(iii) Let $\mathcal{J}=\{j\in\{i-1,i+1\}\mid L_j \textit{ is of type } I\}$. Let L_i be *bound of type* I^e and γ_i be an element in A such that $L_i=2^i\cdot(\mathcal{M}_i\oplus A(1,2\gamma_i))$.

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If γ_i is a unit in A, then

$$Z_i = \bigoplus_{j\notin\{i,i\pm1\}} 2^{\max\{0,i-j\}}L_j \oplus 2^{\max\{0,i-j\}}\Big(\bigoplus_{j\in\{i\pm1\}} \mathcal{M}_j \oplus Ae_2^{(j)}\Big) \oplus (2\mathcal{M}_i \oplus 2Ae_1^{(i)})$$
$$\oplus \Big\{\Big(\sum_{j\in\mathcal{J}} 2^{\max\{0,i-j\}}\cdot a_je_1^{(j)}\Big) + (\sqrt{\gamma_i}\cdot a_ie_2^{(i)}) \;\Big|\; \text{each } a_j, a_i \in A, \Big(\sum_{j\in\mathcal{J}} a_j\Big) + \sqrt{\gamma_i}\cdot a_i \in (2)\Big\},$$

where the $e_2^{(j)}$ factor should be ignored for those $j \in \{i \pm 1\}$ such that L_j is not of type I^e, and $\sqrt{\gamma_i}$ is an element of A such that $\sqrt{\gamma_i} \bmod 2 = \sqrt{\bar{\gamma}_i}(\in \kappa)$ where $\sqrt{\bar{\gamma}_i}$ is as explained at the beginning of Appendix A. If γ_i is not a unit in A, then

$$Z_i = \bigoplus_{j\notin\{i,i\pm1\}} 2^{\max\{0,i-j\}}L_j \oplus 2^{\max\{0,i-j\}}\Big(\bigoplus_{j\in\{i\pm1\}} \mathcal{M}_j \oplus Ae_2^{(j)}\Big) \oplus (2\mathcal{M}_i \oplus 2Ae_1^{(i)} \oplus Ae_2^{(i)})$$
$$\oplus \Big\{\sum_{j\in\mathcal{J}} 2^{\max\{0,i-j\}}\cdot a_je_1^{(j)} \;\Big|\; \text{each } a_j \in A, \sum_{j\in\mathcal{J}} a_j \in (2)\Big\},$$

where the $e_2^{(j)}$ factor should be ignored for those $j \in \{i \pm 1\}$ such that L_j is not of type I^e.

(iv) If L_i is *bound of type I^o or II*, then

$$Z_i = \bigoplus_{j\notin\{i,i\pm1\}} 2^{\max\{0,i-j\}}L_j \oplus 2^{\max\{0,i-j\}}\Big(\bigoplus_{j\in\{i\pm1\}} \mathcal{M}_j \oplus Ae_2^{(j)}\Big) \oplus (2\mathcal{M}_i \oplus 2Ae_1^{(i)})$$
$$\oplus \Big\{\sum_{j\in\mathcal{J}} 2^{\max\{0,i-j\}}\cdot a_je_1^{(j)} \;\Big|\; \text{each } a_j \in A, \sum_{j\in\mathcal{J}} a_j \in (2)\Big\},$$

where the $e_2^{(j)}$ factor should be ignored for those $j \in \{i \pm 1\}$ such that L_j is not of type I^e, and $e_1^{(i)}$ should be ignored if L_i is *of type II*.

Remark 2.9. These six lattices have the following containment:

(i) when L_i is *free of type I^o*, $A_i \supsetneq B_i \supsetneq Y_i = X_i = Z_i$, $A_i \supsetneq W_i \supsetneq Y_i$;
(ii) when L_i is *free of type I_2^e*, $A_i \supsetneq B_i \supsetneq W_i = Y_i = Z_i \supsetneq X_i$;
(iii) when L_i is *free of type I_1^e*, $A_i \supsetneq B_i \supsetneq W_i = Y_i \supsetneq X_i = Z_i$;
(iv) when L_i is *free of type II*, $A_i = B_i \supsetneq W_i = Y_i = X_i = Z_i$;
(v) when L_i is *bound of type I^o*, $A_i \supsetneq B_i \supsetneq Y_i = X_i \supsetneq Z_i$, $A_i \supsetneq W_i \supsetneq Y_i$;
(vi) when L_i is *bound of type I^e*, $A_i \supsetneq B_i \supsetneq W_i = Y_i \supsetneq X_i$, $Y_i \supsetneq Z_i$;
(vii) when L_i is *bound of type II*, $A_i = B_i \supsetneq W_i = Y_i = X_i \supsetneq Z_i$.

From now on, the pair (L, q) is fixed throughout this paper and $\langle -, -\rangle$ denotes $\langle -, -\rangle_q$.

3. The smooth integral model $\underline{G}$

Let $\underline{G}'$ be a naive integral model of the orthogonal group $\mathrm{O}(V, q)$, where $V = L \otimes_A F$, such that for any commutative A-algebra R,

$$\underline{G}'(R) = \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R).$$

Let $\underline{G}$ be the smooth group scheme model of $\mathrm{O}(V, q)$ such that

$$\underline{G}(R) = \underline{G}'(R)$$

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for any étale A-algebra R. Notice that $\underline{G}$ is uniquely determined with these properties by [GY00, Proposition 3.7]. For a detailed exposition of the relation between the local density of (L,q) and $\underline{G}$, see [GY00, § 3].

In this section, we give an explicit construction of the smooth integral model $\underline{G}$. The construction of $\underline{G}$ is based on that of [GY00, § 5]. Let $K = \mathrm{Aut}_A(L,q) \subset \mathrm{GL}_F(V)$, and $\bar{K} = \mathrm{Aut}_{A^{\mathrm{sh}}}(L\otimes_A A^{\mathrm{sh}}, q\otimes_A A^{\mathrm{sh}})$, where A^{sh} is the strict henselization of A. To ease the notation, we say $g \in \bar{K}$ stabilizes a lattice $M \subseteq V$ if $g(M\otimes_A A^{\mathrm{sh}}) = M\otimes_A A^{\mathrm{sh}}$.

3.1 Main construction

In this subsection, we observe properties of elements of $\bar{K}$ and their matrix interpretation. We choose a Jordan splitting $L = \bigoplus_i L_i$ and, for each i, fix a basis of L_i according to Theorem 2.4. Let g be an element of $\bar{K}$.

(1) First of all, as explained in [GY00, § 5.1], g stabilizes the dual lattice $L^{\perp}$ of L. It is equivalent to saying that g stabilizes A_i for every integer i. We interpret this fact in terms of matrices.

Let $n_i = \mathrm{rank}_A L_i$, and $n = \mathrm{rank}_A L = \sum n_i$. Assume that $n_i = 0$ unless $0 \leqslant i < N$. We always divide a matrix g of size $n\times n$ into N^2 blocks such that the block in position (i,j) is of size $n_i \times n_j$. For simplicity, the row and column numbering starts at 0 rather than 1. The fact that g stabilizes $L^{\perp}$ means that the (i,j)-block has entries in $2^{\max\{0,j-i\}}A^{\mathrm{sh}}$.

From now on, we write

$$g = \left(2^{\max\{0,j-i\}} g_{i,j}\right).$$

(2) An element g of $\bar{K}$ stabilizes A_i, B_i, W_i, X_i and induces the identity on A_i/B_i and W_i/X_i. We also interpret these facts in terms of matrices as described below:

(a) if L_i is *of type* I^o, the diagonal (i,i)-block $g_{i,i}$ is of the form

$$\begin{pmatrix} s_i & 2y_i \\ 2v_i & 1+2z_i \end{pmatrix} \in \mathrm{GL}_{n_i}(A^{\mathrm{sh}}),$$

where s_i is an $(n_i-1)\times(n_i-1)$-matrix, etc;

(b) if L_i is *of type* I^e, the diagonal (i,i)-block $g_{i,i}$ is of the form

$$\begin{pmatrix} s_i & r_i & 2t_i \\ 2y_i & 1+2x_i & 2z_i \\ v_i & u_i & 1+2w_i \end{pmatrix} \in \mathrm{GL}_{n_i}(A^{\mathrm{sh}}),$$

where s_i is an $(n_i-2)\times(n_i-2)$-matrix, etc.

(3) An element g of $\bar{K}$ stabilizes Z_i and induces the identity on $W_i/(X_i\cap Z_i)$. To prove the latter, we choose an element w in W_i. It suffices to show that $gw-w \in X_i\cap Z_i$. By (2), it suffices to show that $gw-w\in Z_i$. This follows from the computation

$$\begin{aligned} \tfrac{1}{2}\cdot(1/2^i)q(gw-w) &= \tfrac{1}{2}(2\cdot(1/2^i)q(w) - 2\cdot(1/2^i)\langle gw,w\rangle) \\ &= (1/2^i)(q(w) - \langle w+x, w\rangle) = (1/2^i)\langle x,w\rangle \\ &= 0 \bmod 2, \end{aligned}$$

where $gw = w+x$ for some $x\in X_i$.

In terms of matrices, we have the following:

(a) if L_i is *of type II* or *free*, there is no new constraint. Thus we assume that L_i is *bound of type I*;

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(b) if L_{i-1} is *of type* I^o and L_{i+1} is *of type II*, then the (n_{i-1}, n_i)th-entry of $g_{i-1,i}$ lies in the prime ideal (2);

(c) if L_{i-1} is *of type* I^e and L_{i+1} is *of type II*, then the $(n_{i-1}-1, n_i)$th-entry of $g_{i-1,i}$ lies in the prime ideal (2);

(d) if L_{i-1} is *of type II* and L_{i+1} is *of type* I^o, then the (n_{i+1}, n_i)th-entry of $g_{i+1,i}$ lies in the prime ideal (2);

(e) if L_{i-1} is *of type II* and L_{i+1} is *of type* I^e, then the $(n_{i+1}-1, n_i)$th-entry of $g_{i+1,i}$ lies in the prime ideal (2);

(f) if L_{i-1} and L_{i+1} are *of type* I^o, then the sum of the (n_{i-1}, n_i)th-entry of $g_{i-1,i}$ and the (n_{i+1}, n_i)th-entry of $g_{i+1,i}$ lies in the prime ideal (2);

(g) if L_{i-1} is *of type* I^o and L_{i+1} is *of type* I^e, then the sum of the (n_{i-1}, n_i)th-entry of $g_{i-1,i}$ and the $(n_{i+1}-1, n_i)$th-entry of $g_{i+1,i}$ lies in the prime ideal (2);

(h) if L_{i-1} is *of type* I^e and L_{i+1} is *of type* I^o, then the sum of the $(n_{i-1}-1, n_i)$th-entry of $g_{i-1,i}$ and the (n_{i+1}, n_i)th-entry of $g_{i+1,i}$ lies in the prime ideal (2);

(i) if L_{i-1} and L_{i+1} are *of type* I^e, then the sum of the $(n_{i-1}-1, n_i)$th-entry of $g_{i-1,i}$ and the $(n_{i+1}-1, n_i)$th-entry of $g_{i+1,i}$ lies in the prime ideal (2).

(4) The fact that g induces the identity on $W_i/(X_i \cap Z_i)$ for all i is equivalent to the fact that g induces the identity on $(X_i \cap Z_i)^{\perp}/W_i^{\perp}$ for all i.

We give another description of this condition. Since the space V has a nondegenerate bilinear form $\langle -, - \rangle$, V can be identified with its own dual. We define the adjoint g^* characterized by $\langle gv, w \rangle = \langle v, g^* w \rangle$. Then the fact that g induces the identity on $(X_i \cap Z_i)^{\perp}/W_i^{\perp}$ is the same as the fact that g^* induces the identity on $W_i/(X_i \cap Z_i)$.

To interpret the above in terms of matrices, we notice that g^* is an element of $\bar{K}$ as well. Thus if we apply (3) to g^*, we have the following:

(a) if L_i is *of type II* or *free*, then there is no new constraint. Thus we assume that L_i is *bound of type I*;

(b) if L_{i-1} is *of type I*, L_i is *of type* I^o and L_{i+1} is *of type II*, then the (n_i, n_{i-1})th-entry of $g_{i,i-1}$ lies in the prime ideal (2);

(c) if L_{i-1} is *of type I*, L_i is *of type* I^e and L_{i+1} is *of type II*, then the $(n_i - 1, n_{i-1})$th-entry of $g_{i,i-1}$ lies in the prime ideal (2);

(d) if L_{i-1} is *of type II*, L_i is *of type* I^o and L_{i+1} is *of type I*, then the (n_i, n_{i+1})th-entry of $g_{i,i+1}$ lies in the prime ideal (2);

(e) if L_{i-1} is *of type II*, L_i is *of type* I^e and L_{i+1} is *of type I*, then the $(n_i - 1, n_{i+1})$th-entry of $g_{i,i+1}$ lies in the prime ideal (2);

(f) if L_{i-1} and L_{i+1} are *of type I* and L_i is *of type* I^o, then the sum of the (n_i, n_{i-1})th-entry of $g_{i,i-1}$ and the (n_i, n_{i+1})th-entry of $g_{i,i+1}$ lies in the prime ideal (2);

(g) if L_{i-1} and L_{i+1} are *of type I* and L_i is *of type* I^e, then the sum of the (n_i-1, n_{i-1})th-entry of $g_{i,i-1}$ and the $(n_i - 1, n_{i+1})$th-entry of $g_{i,i+1}$ lies in the prime ideal (2).

(5) By combining (3) and (4), we obtain the following:

(a) if L_i and L_{i+1} are *of type* I^o, the (n_i, n_{i+1})th- (respectively (n_{i+1}, n_i)th)-entry of $g_{i,i+1}$ (respectively $g_{i+1,i}$) lies in the prime ideal (2);

(b) if L_i and L_{i+1} are *of type* I^e, the $(n_i - 1, n_{i+1})$th- (respectively $(n_{i+1} - 1, n_i)$th)-entry of $g_{i,i+1}$ (respectively $g_{i+1,i}$) lies in the prime ideal (2);

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(c) if L_i is *of type* I^o and L_{i+1} is *of type* I^e, the (n_i, n_{i+1})th- (respectively $(n_{i+1}-1, n_i)$th)-entry of $g_{i,i+1}$ (respectively $g_{i+1,i}$) lies in the prime ideal (2);

(d) if L_i is *of type* I^e and L_{i+1} is *of type* I^o, the (n_i-1, n_{i+1})th- (respectively (n_{i+1}, n_i)th)-entry of $g_{i,i+1}$ (respectively $g_{i+1,i}$) lies in the prime ideal (2).

(6) Consequently, we have the following matrix form for g:

$$g = \left(2^{\max\{0,j-i\}} g_{i,j}\right),$$

where $g_{i,i}$ is as described in (2), and $g_{i,i+1}$ and $g_{i+1,i}$ are as described in (5).

3.2 Construction of $\underline{M}^*$

We first state the following lemma.

Lemma 3.1. *Let V be an F-vector space. Let $\{\phi_i\} \subset \mathrm{Hom}_F(V,F)$ be a finite set F-spanning $\mathrm{Hom}_F(V,F)$. Define a functor from the category of commutative flat A-algebras to the category of sets as follows:*

$$X : R \mapsto \{x \in V \otimes_A R \mid (\phi_i \otimes_A R)(x) \in R \text{ for all } i\}.$$

Then X is representable by the affine space $\mathbf{A}^{\dim_F V}$ over A of dimension $\dim_F V$.

Proof. Let $\{\psi_1, \ldots, \psi_m\}$ be a basis of an A-span of $\{\phi_i\}$. Let $v_1, \ldots, v_m$ be the dual basis. Then we have the following:

$$X(R) = \bigoplus_i R v_i.$$

This completes the proof. □

We define a functor from the category of commutative flat A-algebras to the category of monoids as follows. For any commutative flat A-algebra R, set

$$m \in \underline{M}(R) \quad \text{if and only if } m \in \mathrm{End}_R(L \otimes_A R)$$

with the following conditions:

(1) m stabilizes $A_i \otimes_A R, B_i \otimes_A R, W_i \otimes_A R, X_i \otimes_A R, Y_i \otimes_A R, Z_i \otimes_A R$ for all i;

(2) m induces the identity on $A_i \otimes_A R / B_i \otimes_A R, W_i \otimes_A R/(X_i \cap Z_i) \otimes_A R, (X_i \cap Z_i)^\perp \otimes_A R / W_i^\perp \otimes_A R$ for all i.

Then by the above lemma, the functor $\underline{M}$ is representable by a unique flat A-algebra $A(\underline{M})$ which is a polynomial ring over A in n^2 variables. Moreover, it is easy to see that $\underline{M}$ has the structure of a scheme in monoids by showing that $\underline{M}(R)$ is closed under multiplication.

We stress that the above description of $\underline{M}(R)$, the set of R-points on the scheme $\underline{M}$, is no longer true when R is a κ-algebra. Now suppose that R is a κ-algebra. By choosing a basis for L as in § 3.1, we describe each element of $\underline{M}(R)$ formally as a matrix $(2^{\max\{0,j-i\}} m_{i,j})$, where $m_{i,j}$ is an $(n_i \times n_j)$-matrix with entries in R as described in § 3.1. To multiply $(m_{i,j})$ and $(m'_{i,j})$, we refer to the description of [GY00, § 5.3].

Let d be the determinant of the matrix $(2^{\max\{0,j-i\}} m_{i,j})$. Then $\mathrm{Spec}(A(\underline{M})_d)$ is an open subscheme of $\underline{M}$.

We define a functor from the category of commutative A-algebras to the category of groups as follows. For $m \in \underline{M}(R)$ with R a commutative A-algebra, set

$$\underline{M}^*(R) = \{m \in \underline{M}(R) : \text{there exists } m^{-1} \in \underline{M}(R) \text{ such that } m \cdot m^{-1} = m^{-1} \cdot m = 1\}.$$

We claim that $\underline{M}^*$ is representable by $\mathrm{Spec}(A(\underline{M})_d)$. Notice that $\mathrm{Spec}(A(\underline{M})_d)(R)$, the set of R-points of $\mathrm{Spec}(A(\underline{M})_d)$ for a flat A-algebra R, is characterized by

$$\{m \in \underline{M}(R) : \text{there exists } \widetilde{m}^{-1} \in \mathrm{End}_R(L \otimes_A R) \text{ such that } m \cdot \widetilde{m}^{-1} = \widetilde{m}^{-1} \cdot m = 1\}.$$

Note that $\underline{M}(R) \subset \mathrm{End}_R(L \otimes_A R)$ for a flat A-algebra R.

We first show that $\widetilde{m}^{-1}$ ($\in \mathrm{End}_R(L \otimes_A R)$) with $m \in \underline{M}(R)$ is an element of $\underline{M}(R)$ for every flat A-algebra R. To verify this statement, it suffices to show that $\widetilde{m}^{-1}$ satisfies conditions (1) and (2) defining $\underline{M}$ and it follows from Lemma 3.2 which will be stated below. For any flat A-algebra R, we consider the following well-defined map:

$$\mathrm{Spec}\,(A(\underline{M})_d)(R) \longrightarrow \mathrm{Spec}\,(A(\underline{M})_d)(R), \quad m \mapsto \widetilde{m}^{-1}.$$

Since $\mathrm{Spec}(A(\underline{M})_d)$ is affine and flat, this map is represented by a morphism as schemes. This implies that if $m \in \mathrm{Spec}(A(\underline{M})_d)(R)$ then $\widetilde{m}^{-1} \in \underline{M}(R)$ for any commutative A-algebra R. Therefore,

$$\mathrm{Spec}\,(A(\underline{M})_d)(R) = \underline{M}^*(R)$$

for any commutative A-algebra R. We now state Lemma 3.2.

LEMMA 3.2. *Let L' be a sublattice of L and m be an element of* $\mathrm{Spec}\,(A(\underline{M})_d)(R)$, *where R is a flat A-algebra. Assume that m stabilizes $L' \otimes_A R$. Then this lattice $L' \otimes_A R$ is stabilized by $\widetilde{m}^{-1}$ as well.*

Proof. Since m stabilizes $L' \otimes_A R$, we have that $m \cdot L' \otimes_A R \subset L' \otimes_A R$. In addition, m is an element of $\mathrm{Spec}(A(\underline{M})_d)(R)$ and so the determinant of m is a unit in R. Therefore, $m \cdot L' \otimes_A R = L' \otimes_A R$. This implies that $\widetilde{m}^{-1} \cdot L' \otimes_A R = L' \otimes_A R$. □

Therefore, we conclude that $\underline{M}^*$ is an open subscheme of $\underline{M}$, with generic fiber $M^* = \mathrm{GL}_F(V)$, and that $\underline{M}^*$ is smooth over A. Moreover, $\underline{M}^*$ is a group scheme since $\underline{M}$ is a scheme in monoids.

Remark 3.3. We give another description for the functor $\underline{M}$. Let us define a functor from the category of commutative flat A-algebras to the category of rings as follows.

For any commutative flat A-algebra R, set

$$\underline{M}'(R) \subset \{m \in \mathrm{End}_R(L \otimes_A R)\}$$

with the following conditions:

(1) m stabilizes $A_i \otimes_A R, B_i \otimes_A R, W_i \otimes_A R, X_i \otimes_A R, Y_i \otimes_A R, Z_i \otimes_A R$ for all i;
(2) m maps $A_i \otimes_A R, W_i \otimes_A R, (X_i \cap Z_i)^\perp \otimes_A R$ into $B_i \otimes_A R, (X_i \cap Z_i) \otimes_A R, W_i^\perp \otimes_A R$, respectively.

Then the functor $\underline{M}$ is the same as the functor $1 + \underline{M}'$, where $(1 + \underline{M}')(R) = \{1 + m : m \in \underline{M}'(R)\}$.

3.3 Construction of $\underline{Q}$

Recall that the pair (L, q) is fixed throughout this paper and the lattices A_i, B_i, W_i, X_i, Y_i, Z_i only depend on the quadratic pair (L, q). For any flat A-algebra R, let $\underline{Q}(R)$ be the set of quadratic forms f on $L \otimes_A R$ such that $S(L, f) \subseteq S(L, q)$ and f satisfies the following conditions:

(a) $\langle L \otimes_A R, A_i \otimes_A R\rangle_f \subset 2^i R$ for all i;
(b) $X_i \otimes_A R/2A_i \otimes_A R$ is contained in the kernel of the symmetric bilinear form $\langle -, -\rangle_{f,i}$ mod 2 on $A_i \otimes_A R/2A_i \otimes_A R$. Here, $\langle -, -\rangle_{f,i} = (1/2^i)\langle -, -\rangle_f$;

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(c) $B_i \otimes_A R/2A_i \otimes_A R$ is contained in the kernel of the linear form $(1/2^i)f$ mod 2 on $A_i \otimes_A R/2A_i \otimes_A R$;

(d) assume $B_i \subsetneq A_i$. We have seen the existence of the unique vector $e \in A_i/X_i$ such that $\langle v, e\rangle_{q,i}^2 = \langle v, v\rangle_{q,i}$ mod 2 for every vector $v \in A_i/X_i$ in § 2.3. Then $e\otimes 1 \in A_i\otimes_A R/X_i\otimes_A R$ also satisfies the condition that $\langle v, e\otimes 1\rangle_{f,i}^2 = \langle v, v\rangle_{f,i}$ mod 2 for every vector $v \in A_i\otimes_A R/X_i\otimes_A R$;

(e) $Y_i \otimes_A R/2A_i \otimes_A R$ is contained in the kernel of the symmetric bilinear form $\langle -, -\rangle_{f,i}$ mod 2 on $B_i \otimes_A R/2A_i \otimes_A R$;

(f) $Z_i \otimes_A R/2A_i \otimes_A R$ is contained in the kernel of the quadratic form $\frac{1}{2} \cdot (1/2^i)f$ mod 2 on $B_i \otimes_A R/2A_i \otimes_A R$;

(g) $(1/2^i)f$ mod $2 = (1/2^i)q$ mod 2 on $A_i \otimes_A R/2A_i \otimes_A R$;

(h) $(1/2^i)f(w_i) - (1/2^i)q(w_i) \in (4)$, where $w_i \in W_i \otimes_A R$;

(i) $\langle a_i, w_i\rangle_{f,i} \equiv \langle a_i, w_i\rangle_{q,i}$ mod 2, where $a_i \in A_i \otimes_A R$ and $w_i \in W_i \otimes_A R$;

(j) $\langle w_i', w_i\rangle_f \in R$ and $\langle z_i', z_i\rangle_f \in R$. Here, $w_i' \in W_i^{\perp}\otimes_A R$, $w_i \in W_i\otimes_A R$ and $z_i' \in (X_i\cap Z_i)^{\perp}\otimes_A R$, $z_i \in (X_i \cap Z_i) \otimes_A R$. In addition, $\langle w_i, z_i'\rangle_f - \langle w_i, z_i'\rangle_q \in R$.

We interpret the above conditions in terms of matrices. For a flat A-algebra R, $\underline{Q}(R)$ is the set of symmetric matrices

$$\left(2^{\max\{i,j\}} f_{i,j}\right)$$

of size $n \times n$ satisfying the following.

(1) The size of $f_{i,j}$ is $n_i \times n_j$.

(2) If L_i is *of type* I^o with respect to q, then $f_{i,i}$ is of the form

$$\begin{pmatrix} a_i & 2b_i \\ 2\cdot {}^t b_i & \epsilon + 4c_i \end{pmatrix}.$$

Here, the diagonal entries of a_i are $\equiv 0$ mod 2, where a_i is an $(n_i - 1) \times (n_i - 1)$-matrix, etc.

(3) If L_i is *of type* I^e, then $f_{i,i}$ is of the form

$$\begin{pmatrix} d_i & {}^t b_i & 2e_i \\ b_i & 1 + 2a_i & 1 + 2c_i \\ 2\cdot {}^t e_i & 1 + 2c_i & 2\gamma_i + 4f_i \end{pmatrix}.$$

Here, the diagonal entries of d_i are $\equiv 0$ mod 2, where d_i is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

(4) If L_i is *of type II*, then the diagonal entries of $f_{i,i}$ are $\equiv 0$ mod 2.

(5) If L_i and L_{i+1} are *of type I*, then the (n_i, n_{i+1})th-entry of $f_{i,i+1}$ lies in the ideal (2).

It is easy to see, by Lemma 3.1, that $\underline{Q}$ is represented by a flat A-scheme which is isomorphic to an affine space of dimension $(n^2 + n)/2$. Note that our fixed quadratic form q is an element of $\underline{Q}(A)$.

3.4 Smooth affine group scheme $\underline{G}$

Theorem 3.4. *For any flat A-algebra R, the group $\underline{M}^*(R)$ acts on the right of $\underline{Q}(R)$ by $f \circ m = {}^t m \cdot f \cdot m$. Then this action is represented by an action morphism of schemes*

$$\underline{Q} \times \underline{M}^* \longrightarrow \underline{Q}.$$

Proof. We start with any $m \in \underline{M}^*(R)$ and $f \in \underline{Q}(R)$. It suffices to show that $f \circ m$ satisfies conditions (a) to (j) given in § 3.3.

From the construction of $\underline{M}^*$, it is obvious that $f \circ m$ satisfies conditions (a) to (f).

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Condition (g) is obvious from the fact that m stabilizes A_i and B_i and induces the identity on A_i/B_i.

The fact that m induces the identity on W_i/X_i and W_i/Z_i implies that $f \circ m$ satisfies condition (h).

For condition (i), it suffices to show that $\langle ma_i, mw_i\rangle_{f,i} \equiv \langle a_i, w_i\rangle_{q,i} \bmod 2$. We denote $ma_i = a_i + b_i$ and $mw_i = w_i + x_i$, where $b_i \in B_i \otimes_A R, x_i \in X_i \otimes_A R$. Now it suffices to show $\langle a_i + b_i, x_i\rangle_{f,i} + \langle b_i, w_i\rangle_{f,i} \equiv 0 \bmod 2$. Firstly, $\langle a_i + b_i, x_i\rangle_{f,i} \equiv 0 \bmod 2$ by the definition of the lattice X_i. Secondly, we have $\langle b_i, w_i\rangle_{f,i} \bmod 2 \equiv 0$ by observing that $\langle b_i, e\rangle^2_{f,i} \equiv \langle b_i, b_i\rangle_{f,i} \equiv 0 \bmod 2$ when $B_i \subsetneq A_i$, where e is the unique vector chosen earlier. If $B_i = A_i$, it is obvious because $W_i = X_i$.

For condition (j), it suffices to prove that $\langle mw_i, mz_i'\rangle_f - \langle w_i, z_i'\rangle_q \in R$. Notice that $mw_i = w_i + z_i$ and $mz_i' = z_i' + w_i'$, where $z_i \in (X_i \cap Z_i) \otimes_A R$ and $w_i' \in W_i^{\perp}$. Thus, this can be easily seen by the fact that $\langle w_i, w_i'\rangle_f + \langle z_i, z_i'\rangle_f + \langle z_i, w_i'\rangle_f \in R$. □

THEOREM 3.5. *Let ρ be the morphism $\underline{M}^* \to \underline{Q}$ defined by $\rho(m) = q \circ m$. Then ρ is smooth of relative dimension dim* $\mathrm{O}(V, q)$.

Proof. The theorem follows from [GY00, Theorem 5.5] and the following lemma. □

LEMMA 3.6. *The morphism $\rho \otimes \kappa : \underline{M}^* \otimes \kappa \to \underline{Q} \otimes \kappa$ is smooth of relative dimension* dim $\mathrm{O}(V, q)$.

Proof. The proof is based on [GY00, Lemma 5.5.2]. It is enough to check the statement over the algebraic closure $\bar{\kappa}$ of κ. By [Har77, III.10.4], it suffices to show that, for any $m \in \underline{M}^*(\bar{\kappa})$, the induced map on the Zariski tangent space $\rho_{*,m} : T_m \to T_{\rho(m)}$ is surjective.

We define two functors from the category of commutative flat A-algebras to the category of abelian groups as follows:

$$T_1(R) = \{m - 1 : m \in \underline{M}(R)\},$$
$$T_2(R) = \{f - q : f \in \underline{Q}(R)\}.$$

The functor T_1 (respectively T_2) is representable by a flat A-algebra which is a polynomial ring over A in n^2 (respectively $(n^2 + n)/2$) variables. Moreover, they have the structure of a commutative group scheme since they are closed under addition. In fact, T_1 is the same as the functor $\underline{M}'$ in Remark 3.3.

We still need to introduce another functor on flat A-algebras. Define $T_3(R)$ to be the set of all $(n \times n)$-matrices y over R with the following conditions.

(a) The (i, j)-block $y_{i,j}$ of y has entries in $2^{\max(i,j)}R$ so that

$$y = \left(2^{\max(i,j)} y_{i,j}\right).$$

Here, the size of $y_{i,j}$ is $n_i \times n_j$.

(b) If L_i is *of type* I^o, $y_{i,i}$ is of the form

$$\begin{pmatrix} s_i & 2y_i \\ 2v_i & 2z_i \end{pmatrix} \in \mathrm{M}_{n_i}(R),$$

where s_i is an $(n_i - 1) \times (n_i - 1)$-matrix, etc.

(c) If L_i is *of type* I^e, $y_{i,i}$ is of the form

$$\begin{pmatrix} s_i & r_i & 2t_i \\ y_i & x_i & 2z_i \\ 2v_i & 2u_i & 2w_i \end{pmatrix} \in \mathrm{M}_{n_i}(R),$$

where s_i is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

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(d) If L_i and L_{i+1} are *of type I*, then both the (n_i, n_{i+1})th-entry of $y_{i,i+1}$ and the (n_{i+1}, n_i)th-entry of $y_{i+1,i}$ lie in the ideal (2).

It is easy to see that the functor T_3 is represented by a flat A-scheme.

Then we identify T_m with $T_1(\bar{\kappa})$ and $T_{\rho(m)}$ with $T_2(\bar{\kappa})$. The map $\rho_{*,m} : T_m \to T_{\rho(m)}$ is then $X \mapsto m^t \cdot q \cdot X + X^t \cdot q \cdot m$, where the sum and the multiplication are to be interpreted as in [GY00, § 5.3].

To prove surjectivity, it suffices to show the following three statements:

(1) $X \mapsto q \cdot X$ is a bijection $T_1(\bar{\kappa}) \to T_3(\bar{\kappa})$;
(2) for any $m \in \underline{M}^*(\bar{\kappa})$, $Y \mapsto {}^t m \cdot Y$ is a bijection from $T_3(\bar{\kappa})$ to itself;
(3) $Y \mapsto {}^t Y + Y$ is a surjection $T_3(\bar{\kappa}) \to T_2(\bar{\kappa})$.

(3) is direct from the construction of $T_3(\bar{\kappa})$. Hence we provide the proof of (1) and (2).

For (1), we first observe that two functors T_1 and T_3 are representable by flat affine schemes. Therefore, it suffices to show that the map

$$T_1(R) \longrightarrow T_3(R), \quad X \mapsto q \cdot X$$

is bijective for a flat A-algebra R. To prove this, it suffices to show that the map $T_1(R) \to T_3(R)$, $X \mapsto q \cdot X$ and the map $T_3(R) \to T_1(R), Y \mapsto q^{-1} \cdot Y$ are well defined for all flat A-algebras R.

For the first map, it suffices to show that $q \cdot X$ satisfies the four conditions defining the functor T_3. We represent the given quadratic form q by a symmetric matrix $\left(2^i \cdot \delta_i\right)$ with $2^i \cdot \delta_i$ for the (i,i)-block and 0 for the remaining blocks. We express

$$X = \left(2^{\max\{0,j-i\}} x_{i,j}\right).$$

Then

$$q \cdot X = \left(2^{\max(i,j)} y_{i,j}\right).$$

Here, $y_{i,i} = \delta_i \cdot x_{i,i}$, $y_{i,i+1} = \delta_i \cdot x_{i,i+1}$ and $y_{i+1,i} = \delta_{i+1} \cdot x_{i+1,i}$. These matrix equations are easily computed and so we conclude $q \cdot X \in T_3(R)$.

For the second map, we express $Y = \left(2^{\max(i,j)} y_{i,j}\right)$ and $q^{-1} = \left(2^{-i} \cdot \delta_i^{-1}\right)$. Then we have the following:

$$q^{-1} \cdot Y = \left(2^{\max\{0,j-i\}} x_{i,j}\right).$$

Here, $x_{i,i} = \delta_i^{-1} \cdot y_{i,i}$, $x_{i,i+1} = \delta_i^{-1} \cdot y_{i,i+1}$ and $x_{i+1,i} = \delta_{i+1}^{-1} \cdot y_{i+1,i}$. From these, it is easily checked that $q^{-1} \cdot Y$ is an element of $T_1(R)$.

For (2), it suffices to show that the map

$$T_3(\bar{\kappa}) \to T_3(\bar{\kappa}), \quad Y \mapsto {}^t m \cdot Y \text{ for any } m \in \underline{M}^*(\bar{\kappa}),$$

is well defined so that its inverse map $Y \mapsto {}^t m^{-1} \cdot Y$ is well defined as well.

We again express $m = \left(2^{\max\{0,j-i\}} m_{i,j}\right)$ and $Y = \left(2^{\max(i,j)} y_{i,j}\right)$. We need to show that ${}^t m \cdot Y$ satisfies four conditions defining the functor T_3.

Condition (a) is explained in [GY00, Lemma 5.5.2]. For the second and the third, we observe that the diagonal (i,i)-block of ${}^t m \cdot Y$ is

$$2^i \cdot {}^t m_{i,i} \cdot y_{i,i} + \sum_{j \neq i} 2^{\max\{0,i-j\} + \max\{j,i\}} \cdot {}^t m_{j,i} \cdot y_{i,j}.$$

Notice that $\max\{0, i-j\} + \max\{j,i\}$ is greater than i if $j \neq i$. By observing the above equation, it is easily seen that ${}^t m \cdot Y$ satisfies conditions (b) and (c).

Finally we observe that the $(i, i+1)$-block of ${}^t m \cdot Y$ is

$$2^{i+1} \cdot {}^t m_{i,i} \cdot y_{i,i+1} + 2^{i+1} \cdot {}^t m_{i,i+1} \cdot y_{i+1,i+1} + 2^{i+2} \cdot \widetilde{y_{i,i+1}}$$

for some $\widetilde{y_{i,i+1}}$ and the $(i+1, i)$-block of ${}^t m \cdot Y$ is

$$2^{i+1} \cdot {}^t m_{i+1,i} \cdot y_{i,i} + 2^{i+1} \cdot {}^t m_{i+1,i+1} \cdot y_{i+1,i} + 2^{i+2} \cdot \widetilde{y_{i+1,i}}$$

for some $\widetilde{y_{i+1,i}}$. From these, one can easily check that ${}^t m \cdot Y$ satisfies condition (d). □

Let $\underline{G}$ be the stabilizer of q in $\underline{M}^*$. It is an affine group subscheme of $\underline{M}^*$, defined over A. Thus, we have the following theorem.

Theorem 3.7. *The group scheme $\underline{G}$ is smooth, and $\underline{G}(R) = \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R)$ for any étale A-algebra R.*

4. The special fiber

In this section, we will determine the structure of the special fiber $\tilde{G}$ of $\underline{G}$ by observing the maximal reductive quotient and the component group. From this section to the end, the identity matrix is denoted by id.

4.1 The reductive quotient of the special fiber

Recall that Z_i is the sublattice of B_i such that $Z_i/2A_i$ is the kernel of the quadratic form $(1/2^{i+1})q$ mod 2 on $B_i/2A_i$. Let $\bar{V}_i = B_i/Z_i$ and let $\bar{q}_i$ denote the nonsingular quadratic form $(1/2^{i+1})q$ mod 2 on $\bar{V}_i$. It is obvious that each element of $\underline{G}(R)$ fixes $\bar{q}_i$ for every flat A-algebra R. Based on this, we claim to have a morphism of algebraic groups

$$\varphi_i : \tilde{G} \to \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$$

defined over κ, where $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$ is the reduced subgroup scheme of $\mathrm{O}(\bar{V}_i, \bar{q}_i)$. Notice that if the dimension of $\bar{V}_i$ is even and positive, then $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}\,(= \mathrm{O}(\bar{V}_i, \bar{q}_i))$ is disconnected. If the dimension of $\bar{V}_i$ is odd, then $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}\,(= \mathrm{SO}(\bar{V}_i, \bar{q}_i))$ is connected.

To prove the above claim, let R be an étale local A-algebra with κ_R the residue field of R. Since $\underline{G}$ is smooth over A, the map $\underline{G}(R) \to \tilde{G}(\kappa_R)$ is surjective by Hensel's lemma.

Now, we choose an element $g \in \tilde{G}(\kappa_R)$ and its lifting $\tilde{g} \in \underline{G}(R)$. Since $\tilde{g}$ induces an element of $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}(\kappa_R)$, we have a map from $\tilde{G}(\kappa_R)$ to $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}(\kappa_R)$. It is easy to see that this map is well defined, i.e. independent of a lifting $\tilde{g}$ of g.

In order to show that this map is representable, we interpret it as matrices. Recall that a matrix form of elements of $\tilde{G}(\kappa_R)$ is

$$\left(2^{\max\{0, j-i\}} m_{i,j}\right),$$

where the diagonal block $m_{i,i}$ is

$$\begin{pmatrix} s_i & 2y_i \\ 2v_i & 1+2z_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_i & r_i & 2t_i \\ 2y_i & 1+2x_i & 2z_i \\ v_i & u_i & 1+2w_i \end{pmatrix}$$

if L_i is *of type* I^o or *of type* I^e, respectively.

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Let $g = (2^{\max\{0,j-i\}} m_{i,j})$. Then g maps to the following.

When L_i is *of type II*, g maps to $m_{i,i}$ (if L_i is free) or to

$$\begin{pmatrix} m_{i,i} & 0 \\ \delta_{i-1}e_{i-1}\cdot m_{i-1,i} + \delta_{i+1}e_{i+1}\cdot m_{i+1,i} & 1 \end{pmatrix}$$

(if L_i is bound). Here,

$$\delta_j = \begin{cases} 1 & \text{if } L_j \text{ is of } \textit{type } I, \\ 0 & \text{if } L_j \text{ is of } \textit{type } II, \end{cases}$$

and $e_j = (0, \ldots, 0, 1)$ (respectively $e_j = (0, \ldots, 0, 1, 0)$) of size $1 \times n_j$ if L_j is *of type* I^o (respectively *of type* I^e).

When L_i is *of type I*, g maps to s_i if L_i is *free of type* I^o or *free of type* I_2^e. For the other cases with L_i *of type I*, namely if L_i is *free of type* I_1^e, *bound of type* I^o, or *bound of type* I^e, g maps to

$$\begin{pmatrix} s_i & 0 \\ \delta_i'\sqrt{\tilde{\gamma}_i}\cdot v_i + (\delta_{i-1}e_{i-1}\cdot m_{i-1,i} + \delta_{i+1}e_{i+1}\cdot m_{i+1,i})\cdot \tilde{e}_i & 1 \end{pmatrix}.$$

Here, δ_j and e_j are as explained above and $\sqrt{\tilde{\gamma}_i}$ is as explained at the beginning of Appendix A. In addition,

$$\delta_i' = \begin{cases} 1 & \text{if } L_i \text{ is of } \textit{type } I^e, \\ 0 & \text{if } L_i \text{ is of } \textit{type } I^o, \end{cases}$$

and $\tilde{e}_i = \binom{\mathrm{id}}{0}$ of size $n_i \times (n_i - 1)$ (respectively $n_i \times (n_i - 2)$), where id is the identity matrix of size $(n_i - 1) \times (n_i - 1)$ (respectively $(n_i - 2) \times (n_i - 2)$), if L_i is *of type* I^o (respectively *of type* I^e).

This matrix interpretation induces the Hopf algebra morphism (polynomials of degree at most 1) from the coordinate ring of $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$ to the coordinate ring of $\tilde{G}$, which accordingly induces an algebraic group homomorphism $\varphi_i : \tilde{G} \to \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$ such that the group homomorphism induced by φ_i at the level of κ_R-points is as given above.

Since $\tilde{G}$ is smooth over κ, the set of κ_R-points of $\tilde{G}$ for all finite extensions κ_R/κ is dense in $\tilde{G}$ by [BLR90, Corollary 13 of § 2.2]. Therefore, φ_i is uniquely determined by the map constructed above at the level of κ_R-points.

Theorem 4.1. *The morphism φ defined by*

$$\varphi = \prod_i \varphi_i : \tilde{G} \longrightarrow \prod_i \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$$

is surjective.

Proof. Assume that dimension of $\tilde{G}$ = dimension of $\mathrm{Ker}\,\varphi + \sum_i$ (dimension of $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$). Thus we see that $\mathrm{Im}\,\varphi$ contains the identity component of $\prod_i \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$. Here $\mathrm{Ker}\,\varphi$ denotes the kernel of φ and $\mathrm{Im}\,\varphi$ denotes the image of φ.

Recall that a matrix form of elements of $\tilde{G}(R)$ for a κ-algebra R is

$$m = \left(2^{\max\{0,j-i\}} m_{i,j}\right),$$

where the diagonal block $m_{i,i}$ is

$$\begin{pmatrix} s_i & 2y_i \\ 2v_i & 1+2z_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_i & r_i & 2t_i \\ 2y_i & 1+2x_i & 2z_i \\ v_i & u_i & 1+2w_i \end{pmatrix}$$

if L_i is *of type* I^o or *of type* I^e, respectively.

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Let $\mathcal{H}$ be the set of i such that $\mathrm{O}(\bar{V}_i,\bar{q}_i)^{\mathrm{red}}$ is disconnected. Notice that $\mathrm{O}(\bar{V}_i,\bar{q}_i)^{\mathrm{red}}$ is disconnected exactly when L_i is *free of type II*, *free of type* I^o, or *free of type* I^e_2. For such a lattice L_i, we define the closed subgroup scheme H_i of $\tilde{G}$ as follows.

– If L_i is *free of type II*, then H_i is defined by the equations $m_{j,k}=0$ if $j\neq k$, and $m_{j,j}=\mathrm{id}$ if $j\neq i$.
– If L_i is *free of type* I^o, then H_i is defined by the equations $m_{j,k}=0$ if $j\neq k$, $m_{j,j}=\mathrm{id}$ if $j\neq i$, and $y_i=0, v_i=0, z_i=0$.
– If L_i is *free of type* I^e_2, then H_i is defined by the equations $m_{j,k}=0$ if $j\neq k$, $m_{j,j}=\mathrm{id}$ if $j\neq i$, and $r_i=0, t_i=0, y_i=0, x_i=0, z_i=0, v_i=0, u_i=0, w_i=0$.

Then φ_i induces an isomorphism between H_i and $\mathrm{O}(\bar{V}_i,\bar{q}_i)^{\mathrm{red}}$. We consider the morphism

$$\prod_{i\in\mathcal{H}} H_i \longrightarrow \tilde{G},$$

$(h_i)_{i\in\mathcal{H}}\mapsto\prod_{i\in\mathcal{H}}h_i$. Note that H_i and H_j commute with each other in the sense that $h_i\cdot h_j=h_j\cdot h_i$ for all $i\neq j$, where $h_i\in H_i(R)$ and $h_j\in H_j(R)$ for a κ-algebra R. Based on this, the above morphism becomes a group homomorphism. In addition, the fact that $H_i\cap H_j=0$ for all $i\neq j$ implies that this morphism is injective. Thus the product $\prod_{i\in\mathcal{H}}H_i$ is embedded into $\tilde{G}$ as a closed subgroup scheme. Since $\varphi_i|_{H_j}$ is trivial for $i\neq j$, the morphism

$$\prod_{i\in\mathcal{H}}\varphi_i : \prod_{i\in\mathcal{H}} H_i \rightarrow \prod_{i\in\mathcal{H}} \mathrm{O}(\bar{V}_i,\bar{q}_i)$$

is an isomorphism. Therefore, φ is surjective. Now it suffices to establish the assumption made at the beginning of the proof, which is the next lemma. □

Lemma 4.2. $\operatorname{Ker}\varphi$ *is isomorphic to* $\mathbf{A}^l\times(\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$ *as κ-varieties, where* $\mathbf{A}^l$ *is an affine space of the dimension l. Here:*

- *α is the number of i such that L_i is free of type I^e_1;*
- *β is the size of the set of j such that L_j is of type I and L_{j+2} is of type II;*
- *l is such that $l+\sum_i$ (dimension of $\mathrm{O}(\bar{V}_i,\bar{q}_i)^{\mathrm{red}}$) = dimension of $\tilde{G}$.*

The proof is postponed to Appendix A.

Remark 4.3. We describe $\mathrm{Im}\,\varphi_i$ as follows.

Type of lattice L_i	$\mathrm{Im}\,\varphi_i$
I^o, free	$\mathrm{O}(n_i-1,\bar{q}_i)$
I^e_1, free	$\mathrm{SO}(n_i-1,\bar{q}_i)$
I^e_2, free	$\mathrm{O}(n_i-2,\bar{q}_i)$
II, free	$\mathrm{O}(n_i,\bar{q}_i)$
I^o, bound	$\mathrm{SO}(n_i,\bar{q}_i)$
I^e, bound	$\mathrm{SO}(n_i-1,\bar{q}_i)$
II, bound	$\mathrm{SO}(n_i+1,\bar{q}_i)$

4.2 The first construction of component groups

The purpose of this subsection and the next subsection is to define the surjective morphism from $\tilde{G}$ to $(\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$, where α and β are defined in Lemma 4.2.

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We first define two constant group schemes E_i and F_i. We consider the closed subgroup scheme $\tilde{H}_i$ of $\tilde{G}$ defined by equations

$$m_{i,j} = 0 \quad \text{and} \quad m_{j,j} = \text{id} \quad \text{for all } j \neq i.$$

Then $\tilde{H}_i$ is isomorphic to the special fiber of the smooth affine group scheme associated to the lattice L_i.

If L_i is *of type* I^e, then a matrix form of elements of $\tilde{H}_i$ is

$$\begin{pmatrix} s & r & 2t \\ 2y & 1+2x & 2z \\ v & u & 1+2w \end{pmatrix}.$$

The subgroup scheme E_i is defined by the following equations:

$$s = \text{id}, \quad r = 0, \quad t = 0, \quad y = 0, \quad v = 0, \quad z = 0 \quad \text{and} \quad w = 0.$$

The equations defining the subgroup scheme F_i is as follows:

$$s = \text{id}, \quad r = 0, \quad t = 0, \quad y = 0, \quad v = 0, \quad u = 0 \quad \text{and} \quad w = 0.$$

Obviously, these are closed subgroup schemes of $\tilde{G}$. If L_i is *of type* I^e, then F_i is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In particular, if L_i is *free of type* I^e_1, then E_i is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as well.

If L_i is *of type* I^o, then a matrix form of elements of $\tilde{H}_i$ is

$$\begin{pmatrix} s_i & 2y_i \\ 2v_i & 1+2z_i \end{pmatrix}.$$

The closed subgroup scheme F_i of $\tilde{H}_i$ is defined by equations

$$s_i = \text{id}, \quad y_i = 0 \quad \text{and} \quad v_i = 0$$

and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

We now construct the morphism $\xi_i : \tilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}$. Assume that L_i is *free of type* I^e_1. Consider the linear form

$$\frac{1}{2^i}\langle -, e\rangle \mod 2 \text{ on } A_i/X_i,$$

where e is the vector in A_i/X_i as defined in § 2.3. This linear form is fixed by elements of $\underline{G}(R)$ for a flat A-algebra R, because the vector e is fixed by elements of $\underline{G}(R)$. We choose an arbitrary linear form l on A_i such that

$$l \bmod 2 = \frac{1}{2^i}\langle -, e\rangle \mod 2 \text{ on } A_i/X_i.$$

Define the quadratic form q' by l^2. Notice that the norm of the quadratic lattice $(A_i, (1/2^i)q + q')$ is the prime ideal (2). If we consider the quadratic form $\frac{1}{2}((1/2^i)q + q') \bmod 2$ defined over the κ-vector space A_i/X_i, then it is stabilized by elements of $\underline{G}(R)$ for a flat A-algebra R. It is obvious that this quadratic form defined over A_i/X_i is nonsingular and independent of the choice of l. Thus we have a morphism of algebraic groups

$$\tilde{G} \longrightarrow \mathrm{O}\left(A_i/X_i, \frac{1}{2}\left(\frac{1}{2^i}q + q'\right)\right)$$

defined over κ. As matrices, if we express $g = \left(2^{\max\{0,j-i\}} m_{i,j}\right) \in \tilde{G}(R)$ with $m_{i,i} = \begin{pmatrix} s_i & r_i & 2t_i \\ 2y_i & 1+2x_i & 2z_i \\ v_i & u_i & 1+2w_i \end{pmatrix}$ for a κ-algebra R, then g maps to $\begin{pmatrix} s_i & r_i & 0 \\ 0 & 1 & 0 \\ v_i & u_i & 1 \end{pmatrix}$.

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Note that the dimension of the κ-vector space A_i/X_i is the same as that of $L_i/2L_i$, which is the even integer n_i. On the other hand, there is a surjective morphism from this orthogonal group onto $\mathbb{Z}/2\mathbb{Z}$, namely the Dickson invariant, since the dimension of the κ-vector space A_i/X_i is even. We define

$$\xi_i : \tilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

to be the composition of the Dickson invariant and the preceding morphism.

Remark 4.4. We describe the Dickson invariant as the determinant morphism of a smooth affine group scheme. Let $(\bar{V}, \bar{q})$ be a nonsingular quadratic space, where $\bar{V}$ is a κ-vector space of even dimension. Then we can choose a unimodular lattice (L, q) *of type II* such that $(L/2L, \frac{1}{2}q \bmod 2) = (\bar{V}, \bar{q})$. If $\mu_{n,A}$ is the group scheme of nth roots of unity defined over A, then the determinant morphism gives a morphism of group schemes

$$\det : \underline{G}' \to \mu_{2,A}$$

defined over A. Here, $\underline{G}'$ is a naive integral model such that $\underline{G}'(R) = \mathrm{Aut}_R(L \otimes_A R, q \otimes_A R)$ for every commutative A-algebra R. Moreover, we can regard $\mu_{2,A}$ as a naive integral model of the orthogonal group associated to a quadratic lattice of rank 1. Then it is easily seen that the smooth affine group scheme associated to this quadratic lattice of rank 1 is $\mathbb{Z}/2\mathbb{Z}$, by observing the equation defining it.

Based on the above, the morphism det induces a morphism of group schemes from $\underline{G}$ to $\mathbb{Z}/2\mathbb{Z}$ defined over A by functoriality of smooth integral models and this morphism gives the morphism

$$\widetilde{\det} : \tilde{G} \to \mathbb{Z}/2\mathbb{Z}$$

defined over κ. Furthermore, it is easily seen that $\widetilde{\det}$ is surjective. In fact, the kernel of $\widetilde{\det}$ is the identity component. To see this, we observe that the morphism

$$\varphi : \tilde{G} \longrightarrow \mathrm{O}(L/2L, \tfrac{1}{2}q \bmod 2) \quad (= \mathrm{O}(\bar{V}, \bar{q}))$$

is an isomorphism by Theorem 4.1 and Lemma 4.2 and so $\tilde{G}$ has two connected components.

On the other hand, the Dickson invariant gives a surjective morphism from $\tilde{G}\,(\cong \mathrm{O}(\bar{V}, \bar{q}))$ to $\mathbb{Z}/2\mathbb{Z}$ and its kernel is the identity component as well. Therefore, the Dickson invariant is the same as the morphism $\widetilde{\det}$.

Lemma 4.5. *The restricted morphism*

$$\xi_i|_{E_i} : E_i \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism. Recall that E_i is defined at the beginning of this subsection.

Proof. If we consider the closed subgroup scheme of $\tilde{H}_i$ defined by equations

$$s = \mathrm{id}, \quad r = 0, \quad t = 0, \quad y = 0 \quad \text{and} \quad v = 0,$$

then this group scheme is isomorphic to the special fiber of the smooth affine group scheme associated to a lattice *free of type* I_1^e with rank 2. Since E_i is a subgroup scheme of this group scheme, we may and do assume that $n_i = 2$.

Based on Remark 4.4, we describe the morphism $\widetilde{\det}$ associated to the orthogonal group $\mathrm{O}(A_i/X_i, \frac{1}{2}((1/2^i)q + q'))$ explicitly. Choose a lattice of rank 2 with a Gram matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2\gamma_i \end{pmatrix}$. Here, γ_i is a unit in A such that $L_i = A(1, 2\gamma_i)$. Since this lattice is unimodular *of type II*, a matrix form of a flat A-algebra point of the associated smooth integral model is $\begin{pmatrix} x & y \\ u & z \end{pmatrix}$. In other words,

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there are no congruence conditions. This matrix satisfies three equations:

$$x^2+ux+\gamma_i\cdot u^2=1,\quad 2xy+xz+uy+2\gamma_i\cdot zu=1,\quad y^2+yz+\gamma_i\cdot z^2=\gamma_i.$$

The determinant of this matrix is $xz-uy=1-2(uy+xy+\gamma_i\cdot zu)$. We also express a κ-algebra point of the smooth integral model as the matrix $\left(\begin{smallmatrix} x & y\\ u & z\end{smallmatrix}\right)$. Then $\widetilde{\det}$ maps $\left(\begin{smallmatrix} x & y\\ u & z\end{smallmatrix}\right)$ to $uy+xy+\widetilde{\gamma_i}\cdot zu$. Here, $\widetilde{\gamma_i}(\neq 0)$ is the image of γ_i in the residue field κ.

For $g\in E_i(R)$, $\xi_i|_{E_i}(g)=\widetilde{\det}\left(\left(\begin{smallmatrix} 1 & 0\\ u_i & 1\end{smallmatrix}\right)\right)=\widetilde{\gamma_i}\cdot u_i$. Therefore, $\xi_i|_{E_i}$ is an isomorphism. □

Combining all morphism ξ_i, we have the following theorem.

Theorem 4.6. *The morphism $\xi=\prod_i\xi_i:\tilde{G}\to(\mathbb{Z}/2\mathbb{Z})^{\alpha}$ is surjective. Here, α is the number of i such that L_i is free of type I^e_1.*

Proof. Define the scheme E to be the product of E_i such that L_i is *free of type* I^e_1. Notice that E_i and E_j commute with each other in the sense that $e_i\cdot e_j=e_j\cdot e_i$ for all $i\neq j$, where $e_i\in E_i$ and $e_j\in E_j$ and L_i and L_j are *free of type* I^e_1, and that $E_i\cap E_j=0$. Thus E can be embedded into $\tilde{G}$ as a closed subgroup scheme. In addition, it is obvious that $\xi_i|_{E_j}$ is trivial for $i\neq j$. Therefore ξ induces an isomorphism of algebraic groups from E to $(\mathbb{Z}/2\mathbb{Z})^{\alpha}$ defined over κ. This completes the proof. □

4.3 The second construction of component groups

In this subsection, we will construct the morphism ψ from $\tilde{G}$ to $(\mathbb{Z}/2\mathbb{Z})^{\beta}$. We begin by defining several lattices.

Definition 4.7. We define the lattice L^1, which is the sublattice of L such that $L^1/2L$ is the kernel of the symmetric bilinear form $\langle -,-\rangle$ mod 2 on $L/2L$. Similarly we define the lattice L^i, which is the sublattice of L^{i-1} such that $L^i/2L^{i-1}$ is the kernel of the symmetric bilinear form $(1/2^{i-1})\langle -,-\rangle$ mod 2 on $L^{i-1}/2L^{i-1}$ for all $i\geqslant 1$. For simplicity, put

$$L^0=L=\bigoplus_{i\geqslant 0}L_i, 0\leqslant i<N.$$

The description of L^i is

$$L^{2m}=2^m(L_0\oplus L_1)\oplus 2^{m-1}(L_2\oplus L_3)\oplus\cdots\oplus 2(L_{2m-2}\oplus L_{2m-1})\oplus\bigoplus_{i\geqslant 2m}L_i$$

and

$$L^{2m-1}=2^mL_0\oplus 2^{m-1}(L_1\oplus L_2)\oplus\cdots\oplus 2(L_{2m-3}\oplus L_{2m-2})\oplus\bigoplus_{i\geqslant 2m-1}L_i.$$

We choose a Jordan splitting for the quadratic lattice $(L^{2m},(1/2^{2m})q)$ as follows:

$$L^{2m}=\bigoplus_{i\geqslant 0}M_i,$$

where

$$M_0=2^mL_0\oplus 2^{m-1}L_2\oplus\cdots\oplus 2L_{2m-2}\oplus L_{2m},$$
$$M_1=2^mL_1\oplus 2^{m-1}L_3\oplus\cdots\oplus 2L_{2m-1}\oplus L_{2m+1}$$
$$\text{and}\quad M_k=L_{2m+k}\quad\text{if } k\geqslant 2.$$

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Here, M_i is *modular* and $S(M_i) = (2^i)$. For the quadratic lattice $(L^{2m-1}, (1/2^{2m-1})q)$, a chosen Jordan splitting is as follows:

$$L^{2m-1} = \bigoplus_{i \geqslant 0} M_i,$$

where

$$M_0 = 2^{m-1}L_1 \oplus 2^{m-2}L_3 \oplus \cdots \oplus 2L_{2m-3} \oplus L_{2m-1},$$
$$M_1 = 2^m L_0 \oplus 2^{m-1}L_2 \oplus \cdots \oplus 2L_{2m-2} \oplus L_{2m}$$
$$\text{and} \quad M_k = L_{2m-1+k} \quad \text{if } k \geqslant 2.$$

Definition 4.8. We define $C(L)$ to be the sublattice of L such that

$$C(L) = \{x \in L \mid \langle x, y\rangle \in (2) \text{ for all } y \in B(L)\}.$$

We choose any integer j such that L_j is *of type I* and L_{j+2} is *of type II*. We stress that M_0 is *of type I* and $M_2 = L_{j+2}$ is *of type II*. We choose a basis $(\langle e_i\rangle, e)$ (respectively $(\langle e_i\rangle, a, e)$) for M_0 based on Theorem 2.4 when the rank of M_0 is odd (respectively even). Then $B(L^j)$ is spanned by

$$(\langle e_i\rangle, 2e) \text{ (respectively } (\langle e_i\rangle, 2a, e)) \quad \text{and} \quad M_1 \oplus \left(\bigoplus_{i \geqslant 2} M_i\right)$$

and $C(L^j)$ is spanned by

$$(\langle 2e_i\rangle, e) \text{ (respectively } (\langle 2e_i\rangle, 2a, e)) \quad \text{and} \quad M_1 \oplus \left(\bigoplus_{i \geqslant 2} M_i\right).$$

We now construct the morphism $\psi_j : \tilde{G} \to \mathbb{Z}/2\mathbb{Z}$ as follows. (There are two cases depending on whether M_0 is *of type* I^e or *of type* I^o.)

(1) Firstly, we assume that M_0 is *of type* I^e. We choose a Jordan splitting for the quadratic lattice $(C(L^j), (1/2^{j+1})q)$ as follows:

$$C(L^j) = \bigoplus_{i \geqslant 0} M_i'.$$

Notice that M_1' is *of type II* so that M_0' is *free*. Let G_j denote the special fiber of the smooth affine group scheme associated to the quadratic lattice $(C(L^j), (1/2^{j+1})q)$. We now have a morphism from $\tilde{G}$ to G_j.

If M_0' is *of type II*, *of type* I^o or *of type* I^e_2, then we have a morphism from G_j to the even orthogonal group associated to M_0' as explained in § 4.1. Thus, we see that the Dickson invariant of this orthogonal group induces the morphism

$$\psi_j : \tilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

If M_0' is *of type* I^e_1, then we have a morphism from G_j to $\mathbb{Z}/2\mathbb{Z}$ associated to M_0' as explained in § 4.2. It induces the morphism

$$\psi_j : \tilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

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(2) We next assume that M_0 is *of type* I^o. We choose a Jordan splitting for the quadratic lattice $(C(L^j),(1/2^j)q)$ as follows:

$$C(L^j)=\bigoplus_{i\geqslant 0} M_i'.$$

Notice that the rank of the unimodular lattice M_0' is 1 and the lattice M_2' is *of type II*. If G_j denotes the special fiber of the smooth affine group scheme associated to the quadratic lattice $(C(L^j),(1/2^j)q)$, we have a morphism from $\tilde{G}$ to G_j.

We now consider the new quadratic lattice $M_0'\oplus C(L^j)$. The smooth affine group scheme associated to the quadratic lattice $(C(L^j),(1/2^j)q)$ can be embedded into the smooth affine group scheme associated to the quadratic lattice $M_0'\oplus C(L^j)$ as a closed subgroup scheme. Thus the special fiber G_j of the former group scheme is embedded into the special fiber of the latter group scheme. Since the unimodular lattice $M_0'\oplus M_0'$ is *of type* I^e, where $(M_0'\oplus M_0')\oplus\bigoplus_{i\geqslant 1}M_i'$ is a Jordan splitting of the quadratic lattice $M_0'\oplus C(L^j)$, we have a morphism from the special fiber of the latter group scheme to $\mathbb{Z}/2\mathbb{Z}$ as constructed in the first case. It induces the morphism

$$\psi_j:\tilde{G}\longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

(3) Combining all cases, we have the morphism

$$\psi=\prod_j \psi_j:\tilde{G}\longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\beta},$$

where β is the size of the set of j such that L_j is *of type I* and L_{j+2} is *of type II*.

Remark 4.9. There is another description for β. We consider the type sequence $\{a_i\}$ such that a_i is I or II according to the parity type of L_i. Define two sequences $b_m=a_{2m+1}$ and $c_m=a_{2m}$. Then we take maximal consecutive terms consisting of I in each b_m or c_m. The set consisting of these terms is finite and its size is β. For example, if

$$\{a_n\}_{n\geqslant 0}=\{I\ I\ I\ II\ I\ I\ II\ I\ II\ I\ I\},$$

then

$$\{b_m\}_{m\geqslant 0}=\{I\ II\ I\ I\ I\}\quad\text{and}\quad \{c_m\}_{m\geqslant 0}=\{I\ I\ I\ II\ II\ I\}.$$

Hence β is $2+2=4$.

We now have the following result.

Theorem 4.10. *The morphism*

$$\psi=\prod_j \psi_j:\tilde{G}\longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\beta}$$

is surjective.

Moreover, the morphism

$$\varphi\times\xi\times\psi:\tilde{G}\rightarrow\prod_i \mathrm{O}(\bar{V}_i,\bar{q}_i)^{\mathrm{red}}\times(\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$$

is also surjective.

Proof. We first show that ψ_j is surjective. Note that for such a j, L_j is *of type I* and L_{j+2} is *of type II*. Recall that we have defined the closed subgroup scheme F_j of $\tilde{G}$ at the beginning of § 4.2 and it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Now it suffices to show that $\psi_j|_{F_j}$ is an isomorphism and its proof is similar to that of Lemma 4.5 so we may skip.

Surjectivity of ψ is similar to Theorem 4.6. Notice that F_i and F_j commute with each other for all $i \neq j$, where L_i and L_j (respectively L_{i+2} and L_{j+2}) are *of type I* (respectively *of type II*), and that $F_i \cap F_j = 0$. Thus the product $F = \prod_j F_j$ is embedded into $\tilde{G}$ as a closed subgroup scheme. In addition, it is obvious that $\psi_i|_{F_j}$ is trivial for all $i < j$. Hence the morphism ψ induces an isomorphism of algebraic groups from F to $(\mathbb{Z}/2\mathbb{Z})^\beta$ defined over κ. This shows surjectivity of the morphism ψ.

For surjectivity of $\varphi \times \xi \times \psi$, it suffices to show that $\xi \times \psi|_{\operatorname{Ker}\varphi}$ is surjective onto $(\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$. Since the morphism φ vanishes on E and F, the two schemes E and F are subschemes of $\operatorname{Ker}\varphi$. Notice that the intersection of E and F as subgroup schemes of $\operatorname{Ker}\varphi$ is trivial. This fact implies that the product $E \times F$ is embedded into $\operatorname{Ker}\varphi$ as κ-schemes. Notice that E and F may not commute with each other so $E \times F$ may not inherit subgroup scheme structure of $\operatorname{Ker}\varphi$. It is easily seen from the construction of E and F that the restricted morphisms $\xi|_F$ and $\psi|_E$ are trivial. Therefore, $\xi \times \psi$ induces an isomorphism from $E \times F$ to $(\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$ as κ-schemes. This completes the proof. □

4.4 The maximal reductive quotient of $\tilde{G}$

Let $\tilde{M}$ be the special fiber of $\underline{M}^*$. Let

$$\tilde{M}_i = \mathrm{GL}_\kappa(B_i/Y_i).$$

For any κ-algebra R, let $m = (2^{\max\{0,j-i\}} m_{i,j}) \in \tilde{M}(R)$. Recall that s_i is a block of $m_{i,i}$ if L_i is *of type I*, as explained in § 3.1. Then $s_i \in \tilde{M}_i(R)$. If L_i is *of type II*, then $m_{i,i} \in \tilde{M}_i(R)$. Therefore, we have a surjective morphism of algebraic groups

$$r : \tilde{M} \longrightarrow \prod \tilde{M}_i,$$

defined over κ. We now have the following lemma.

LEMMA 4.11. *The kernel of r is the unipotent radical $\tilde{M}^+$ of $\tilde{M}$, and $\prod \tilde{M}_i$ is the maximal reductive quotient of $\tilde{M}$.*

We finally have the structural theorem for the algebraic group $\tilde{G}$.

THEOREM 4.12. *The morphism*

$$\varphi \times \xi \times \psi : \tilde{G} \longrightarrow \prod_i \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$$

is surjective and the kernel is unipotent and connected. Consequently, $\prod_i \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$ is the maximal reductive quotient. Here, $\mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}}$ is explained in § 4.1 (especially, Remark 4.3), and α and β are defined in Lemma 4.2.

Proof. We only need to prove that the kernel is unipotent and connected. Since the kernel of φ is a closed subgroup scheme of the unipotent group $\tilde{M}^+$, it suffices to show that the kernel of $\varphi \times \xi \times \psi$ is connected. Equivalently, it suffices to show that the kernel of the restricted morphism $\xi \times \psi|_{\operatorname{Ker}\varphi}$ is connected. From Lemma 4.2, $\operatorname{Ker}\varphi \cong \mathbf{A}^l \times (\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$ as κ-varieties. Since the restricted morphism $\xi \times \psi|_{\operatorname{Ker}\varphi}$ is surjective onto $(\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$, we complete the proof by counting the number of connected components. □

Remark 4.13. Recall that α is the number of i such that L_i is *free of type I_1^e*. For such a lattice, Im φ_i is $\mathrm{SO}(n_i - 1)$, which is connected. If a lattice is *free* with nontrivial $\bar{V}_i$ but not *of type I_1^e*, then Im φ_i is disconnected with two connected components. Therefore the maximal reductive

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quotient
$$\prod \mathrm{O}(\bar{V}_i, \bar{q}_i)^{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^{\alpha+\beta}$$
is isomorphic to
$$\prod \mathrm{SO}(\bar{V}_i, \bar{q}_i) \times (\mathbb{Z}/2\mathbb{Z})^{\alpha'+\beta},$$
as κ-varieties, where α' is the number of i such that L_i is *free* with nontrivial $\bar{V}_i$.

5. Comparison of volume forms and final formulas

This section is based on [GY00, § 7]. In the construction of [GY00, § 3.2], pick ω'_M and ω'_Q to be such that
$$\int_{\underline{M}(A)} |\omega'_M| = 1 \quad \text{and} \quad \int_{\underline{Q}(A)} |\omega'_Q| = 1.$$
Put $\omega^{\mathrm{can}} = \omega'_M/\rho^*\omega'_Q$. By Theorem 3.5, we have an exact sequence of locally free sheaves on $\underline{M}^*$
$$0 \longrightarrow \rho^*\Omega_{\underline{Q}/A} \longrightarrow \Omega_{\underline{M}^*/A} \longrightarrow \Omega_{\underline{M}^*/\underline{Q}} \longrightarrow 0.$$
It follows that ω^{can} is of the type discussed in [GY00, § 3].

Lemma 5.1. *Let π be a uniformizer of A. Then*
$$\omega_M = \pi^{N_M}\omega'_M, \quad N_M = \sum_{L_i:\mathit{type}\ I} (2n_i - 1) + \sum_{i<j} (j-i) \cdot n_i \cdot n_j + 2b,$$
$$\omega_Q = \pi^{N_Q}\omega'_Q, \quad N_Q = \sum_{L_i:\mathit{type}\ I} 2n_i + \sum_{i<j} j \cdot n_i \cdot n_j + \sum_i d_i + b + c,$$
$$\omega^{\mathrm{ld}} = \pi^{N_M - N_Q}\omega^{\mathrm{can}}.$$

Here:

- *b is the total number of pairs of adjacent constituents L_i and L_{i+1} that are both of type I (b is denoted by $n(I,I)$ in [CS88]);*
- *c is the sum of dimensions of all nonempty Jordan constituents L_i that are of type II (c is denoted by $n(II)$ in [CS88]);*
- *$d_i = i \cdot n_i \cdot (n_i + 1)/2$.*

Theorem 5.2. *Let f be the cardinality of κ. The local density of (L,q) is*
$$\beta_L = \frac{1}{[\mathrm{O}(V,q) : \mathrm{SO}(V,q)]} f^N \cdot f^{-\dim \mathrm{O}(V,q)} \sharp\tilde{G}(\kappa),$$
where $N = N_Q - N_M = t + \sum_{i<j} i \cdot n_i \cdot n_j + \sum_i d_i - b + c$, t = the total number of L_i that are of type I. Here, $\sharp\tilde{G}(\kappa)$ can be computed based on Remark 5.3(1) and Theorem 4.12.

Remark 5.3. (i) In the above local density formula, $\sharp\tilde{G}(\kappa)$ is computed as follows. We denote by $R_u\tilde{G}$ the unipotent radical of $\tilde{G}$ so that the maximal reductive quotient of $\tilde{G}$ is $\tilde{G}/R_u\tilde{G}$. That is, there is the following exact sequence of group schemes over κ:
$$1 \longrightarrow R_u\tilde{G} \longrightarrow \tilde{G} \longrightarrow \tilde{G}/R_u\tilde{G} \longrightarrow 1.$$
Furthermore, the sequence of groups
$$1 \longrightarrow R_u\tilde{G}(\kappa) \longrightarrow \tilde{G}(\kappa) \longrightarrow (\tilde{G}/R_u\tilde{G})(\kappa) \longrightarrow 1$$
is also exact by [GY00, Lemma 6.3.3]. [GY00, Lemma 6.3.3] also induces that $\sharp R_u\tilde{G}(\kappa)$ is f^m, where m is the dimension of $R_u\tilde{G}$. Notice that the dimension of $R_u\tilde{G}$ can be computed explicitly

based on Theorem 4.12 or Remark 4.13, since the dimension of $\tilde{G}$ is $n(n-1)/2$ with $n = \mathrm{rank}_A L$. In addition, the order of an orthogonal group defined over a finite field is well known. Thus, one can compute $\sharp(\tilde{G}/R_u\tilde{G})(\kappa)$ explicitly based on Theorem 4.12 or Remark 4.13. Finally, the order of the group $\tilde{G}(\kappa)$ is identified as follows:

$$\sharp\tilde{G}(\kappa) = \sharp R_u\tilde{G}(\kappa) \cdot \sharp(\tilde{G}/R_u\tilde{G})(\kappa).$$

(ii) As in [GY00, Remark 7.4], although we have assumed that $n_i = 0$ for $i < 0$, it is easy to check that the formula in the preceding theorem remains true without this assumption.

6. The mass formula for $Q_n(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$

Let us apply the local density formula to obtain the mass formula for the integral quadratic form

$$Q_n(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2.$$

As we are working globally, we differ from our previous notation at times. Let k be a totally real number field of degree d over $\mathbb{Q}$ and R be its ring of integers. Assume that the ideal (2) is unramified over R. For a place v of k, we let k_v be the corresponding completion of k. For a finite place v of k, let κ_v denote the residue field of the completion of k at v, and q_v be the cardinality of κ_v. We consider the quadratic R-lattice (L, Q) such that $Q_n(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. The paper [GY00] of Gan and Yu includes a complete discussion of the Smith–Minkowski–Siegel mass formula. Applying it to the quadratic lattice (L, Q), we have the following formula.

PROPOSITION 6.1 [GY00, Theorem 10.20]. *We have*

$$\mathrm{Mass}(L, Q_n) = c(L) \cdot \frac{d_k^{n(n-1)/4}}{\prod_{v \text{ finite}} \beta_{L_v}}.$$

Here, $c(L) = (\lambda^{-1}\mu)^d$, d_k is the discriminant of k over $\mathbb{Q}$, and β_{L_v} is the local density associated to the quadratic lattice L_v:

$\lambda = \prod_i ((2\pi)^{d_i}/(d_i - 1)!)$, the d_i run over the degrees of G;

$\mu = 2^n$(respectively $2^{(n+1)/2}$) if n is even (respectively n is odd).

We define $\mathcal{I}_1 = \{v : 2|q_v \text{ and } [\kappa_v : F_2] \text{ is odd}\}$ and $\mathcal{I}_2 = \{v : 2|q_v \text{ and } [\kappa_v : F_2] \text{ is even}\}$, where F_q is the finite field with q elements. Let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$.

Based on [GY00, Theorem 7.3] and Theorem 5.2 of this paper, we have the following formula for the local density:

$$\begin{cases} \beta_{L_v} = 1/2 q_v^{-n(n-1)/2} \cdot \sharp\mathrm{O}(n, Q_n) & \text{if } v \nmid (2), \\ \beta_{L_v} = 1/2 q_v^{1-n(n-1)/2} \cdot \sharp\tilde{G}(\kappa_v) & \text{if } v \mid (2), \end{cases}$$

where

$$\sharp\tilde{G}(\kappa_v) = \begin{cases} 4q_v^{n-1} \cdot \sharp\mathrm{SO}(n-1) & \text{if } n \equiv \pm 1 \bmod 8, \text{ or } n \equiv \pm 3 \bmod 8 \text{ and } v \in \mathcal{I}_2, \\ 4q_v^{n-1} \cdot \sharp{}^2\mathrm{SO}(n-1) & \text{if } n \equiv \pm 3 \bmod 8 \text{ and } v \in \mathcal{I}_1, \\ 4q_v^{n-1} \cdot \sharp\mathrm{SO}(n-1) & \text{if } n \equiv \pm 2 \bmod 8, \\ 4q_v^{2n-3} \cdot \sharp\mathrm{SO}(n-2) & \text{if } n \equiv \pm 0 \bmod 8, \text{ or } n \equiv 4 \bmod 8 \text{ and } v \in \mathcal{I}_2, \\ 4q_v^{2n-3} \cdot \sharp{}^2\mathrm{SO}(n-2) & \text{if } n \equiv 4 \bmod 8 \text{ and } v \in \mathcal{I}_1. \end{cases}$$

Here, $\mathrm{SO}(n)$ (respectively, ${}^2\mathrm{SO}(n)$) denotes the split (respectively, nonsplit) connected orthogonal group over κ_v.

The order of an orthogonal group defined over a finite field is well known. We state it below according to the characteristic of a finite field and the dimension n.

If the characteristic of the finite field κ_v is greater than 2, then the order of an orthogonal group is as follows:

$$\begin{cases} \sharp \mathrm{O}(2m+1, Q_{2m+1}) = 2q_v^{m^2}\prod_{i=1}^{m}(q_v^{2i}-1), & \\ \sharp \mathrm{O}(2m, Q_{2m}) = 2q_v^{m(m-1)}(q_v^m-1)\prod_{i=1}^{m-1}(q_v^{2i}-1) & \text{if } -1 \text{ is a square in } F_q, \\ \sharp \mathrm{O}(2m, Q_{2m}) = 2q_v^{m(m-1)}(q_v^m-(-1)^m)\prod_{i=1}^{m-1}(q_v^{2i}-1) & \text{if } -1 \text{ is not a square in } F_q. \end{cases}$$

If the characteristic of the finite field κ_v is 2, then the order of an orthogonal group is as follows:

$$\begin{cases} \sharp \mathrm{SO}(2m+1) = q_v^{m^2}\prod_{i=1}^{m}(q_v^{2i}-1), \\ \sharp \mathrm{SO}(2m) = q_v^{m(m-1)}(q_v^m-1)\prod_{i=1}^{m-1}(q_v^{2i}-1), \\ \sharp^2 \mathrm{SO}(2m) = q_v^{m(m-1)}(q_v^m+1)\prod_{i=1}^{m-1}(q_v^{2i}-1). \end{cases}$$

By combining these with Proposition 6.1, we have the following theorem.

Theorem 6.2. (1) *When $n=2m+1$,*

$$\mathrm{Mass}(L, Q_n) = \left(\left(\prod_{i=1}^{m}\frac{(2i-1)!}{(2\pi)^{2i}}\right)\cdot 2^{m+1}\right)^d \cdot d_k^{n(n-1)/4}\cdot D(L)\cdot\prod_{i=1}^{m}\zeta_k(2i), \tag{6.3}$$

$$D(L) = \begin{cases} \prod_{v\in\mathcal{I}}\frac{q_v^m+1}{2q_v^{m+1}} & \text{if } n\equiv \pm 1 \bmod 8, \\ \prod_{v\in\mathcal{I}_1}\frac{q_v^m-1}{2q_v^{m+1}}\cdot\prod_{v\in\mathcal{I}_2}\frac{q_v^m+1}{2q_v^{m+1}} & \text{if } n\equiv \pm 3 \bmod 8. \end{cases}$$

(2) *When $n=2m$,*

$$\mathrm{Mass}(L, Q_n) = \left(\left(\prod_{i=1}^{m-1}\frac{(2i-1)!}{(2\pi)^{2i}}\right)\cdot\frac{(m-1)!}{(2\pi)^m}\cdot 2^{2m}\right)^d \cdot d_k^{n(n-1)/4}\cdot D(L)\cdot L_k(m,\chi)\cdot\prod_{i=1}^{m-1}\zeta_k(2i), \tag{6.4}$$

$$D(L) = \begin{cases} \prod_{v\in\mathcal{I}}\frac{(q_v^{m-1}+1)(q_v^m-1)}{2q_v^{2m}} & \text{if } n\equiv 0 \bmod 8, \\ \prod_{v\in\mathcal{I}_1}\frac{(q_v^{m-1}-1)(q_v^m-1)}{2q_v^{2m}}\cdot\prod_{v\in\mathcal{I}_2}\frac{(q_v^{m-1}+1)(q_v^m-1)}{2q_v^{2m}} & \text{if } n\equiv 4 \bmod 8, \\ \prod_{v\in\mathcal{I}}\frac{1}{2q_v} & \text{if } n\equiv \pm 2 \bmod 8. \end{cases}$$

Here χ is the Galois character of $k(\sqrt{(-1)^m})$ over k.

Since the mass formula represents a rational number, we can rewrite the above formulas using the functional equations of the Dedekind zeta function and the Hecke L-series. For the functional equation of the Dedekind zeta function, we refer to [Neu99].

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Proposition 6.5 [Neu99, Corollary VII.5.10]. *Let $\zeta_k(s)$ be the Dedekind zeta function of the totally real number field k. The completed zeta function*

$$Z_k(s) = d_k^{s/2} \cdot (\pi^{-s/2}\Gamma(s/2))^d \cdot \zeta_k(s)$$

satisfies the functional equation

$$Z_k(s) = Z_k(1-s).$$

If $s = 2i$ for an integer i, the above proposition gives the following equation:

$$\zeta_k(1-2i) = d_k^{2i-1/2} \cdot \left(\pi^{-2i+1/2}\frac{\Gamma(i)}{\Gamma(\frac{1}{2}-i)}\right)^d \cdot \zeta_k(2i), \tag{6.6}$$

where $\Gamma(i) = (i-1)!$ and $\Gamma(\frac{1}{2}-i) = ((-4)^i \cdot i!)/(2i)!\sqrt{\pi}$. Therefore,

$$\frac{\Gamma(i)}{\Gamma(\frac{1}{2}-i)} = 2 \cdot (2i-1)! \cdot (-4)^{-i} \cdot \pi^{-1/2}.$$

We first assume that $n = 2m+1$. Then (6.6) induces the following equation:

$$\prod_{i=1}^{m} \zeta_k(1-2i) = (d_k)^{m^2+m/2} \cdot 2^{md} \cdot (-1)^{m(m+1)d/2} \cdot \prod_{i=1}^{m}\left(\left(\frac{(2i-1)!}{(2\pi)^{2i}}\right)^d \cdot \zeta_k(2i)\right).$$

If we apply the above to (6.3), then we have the following Mass formula:

$$\text{Mass}\,(L, Q_n) = (-1)^{m(m+1)d/2} \cdot 2^d \cdot D(L) \cdot \prod_{i=1}^{m} \zeta_k(1-2i). \tag{6.7}$$

We next assume that $n = 2m$. Then (6.6) induces the following equation:

$$\prod_{i=1}^{m-1} \zeta_k(1-2i) = (d_k)^{m^2-m/2} \cdot 2^{md-d} \cdot (-1)^{m(m-1)d/2} \cdot \prod_{i=1}^{m-1}\left(\left(\frac{(2i-1)!}{(2\pi)^{2i}}\right)^d \cdot \zeta_k(2i)\right).$$

If we apply the above to (6.4), then we have the following Mass formula:

$$\text{Mass}(L, Q_n) = d_k^{m-1/2} \cdot \pi^{-md} \cdot (-1)^{m(m-1)d/2} \cdot ((m-1)!)^d \cdot 2^d \cdot D(L) \cdot L_k(m,\chi) \cdot \prod_{i=1}^{m-1} \zeta_k(1-2i). \tag{6.8}$$

Let us apply the functional equation of the Hecke L-series to (6.8). If m is even, equivalently $n \equiv 0$ or 4 mod 8, then the character χ is trivial. Let $m = 2m'$. If we put $i = m'$ in (6.6), we have the following:

$$\zeta_k(1-m) = d_k^{m-1/2} \cdot \left(\pi^{-m+1/2}\frac{\Gamma(m')}{\Gamma(1/2-m')}\right)^d \cdot \zeta_k(m).$$

Equivalently,

$$\zeta_k(1-m) = d_k^{m-1/2} \cdot 2^d \cdot \left(\frac{(m-1)!}{(-1)^{m'} \cdot (2\pi)^m}\right)^d \cdot \zeta_k(m). \tag{6.9}$$

We apply (6.9) to (6.8). Then we have the following Mass formula:

$$\text{Mass}(L, Q_n) = 2^{md} \cdot D(L) \cdot \zeta_k(1-m) \cdot \prod_{i=1}^{m-1} \zeta_k(1-2i). \tag{6.10}$$

Assume that m (=$2m'+1$) is odd, equivalently $n \equiv \pm 2$ mod 8. In this case, the Galois character χ is non-trivial. We state the functional equation for the Hecke L-series.

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Proposition 6.11 [Neu99, Corollary VII.8.6]. *The completed Hecke L-series*

$$\Lambda_k(s,\chi) = (d_k \cdot N_{k/\mathbb{Q}}\mathfrak{f}(\chi))^{s/2} \cdot \left(\pi^{-(s+1)/2} \cdot \Gamma\left(\frac{s+1}{2}\right)\right)^d \cdot L_k(s,\chi)$$

satisfies the functional equation

$$\Lambda_k(s,\chi) = \epsilon(\chi) \cdot \Lambda_k(1-s,\chi).$$

Here, $\mathfrak{f}(\chi)$ is the conductor of the Hecke character χ and $|\epsilon(\chi)| = 1$.

If $s = m$ in the above proposition, we have the following equation:

$$L_k(m,\chi) = \epsilon(\chi) \cdot L_k(1-m,\chi) \cdot (d_k \cdot N_{k/\mathbb{Q}}\mathfrak{f}(\chi))^{(1-n)/2} \cdot \pi^{md} \cdot \frac{(-4)^{m'd}}{((m-1)!)^d}. \tag{6.12}$$

Let us apply (6.12) to (6.8). Then we have the following Mass formula:

$$\mathrm{Mass}(L,Q_n) = 2^{md} \cdot D(L) \cdot \epsilon(\chi) \cdot (N_{k/\mathbb{Q}}\mathfrak{f}(\chi))^{(1-n)/2} \cdot L_k(1-m,\chi) \cdot \prod_{i=1}^{m-1} \zeta_k(1-2i). \tag{6.13}$$

By combining (6.7), (6.10) and (6.13), we finally have the following theorem.

Theorem 6.14. (1) *When $n = 2m+1$,*

$$\mathrm{Mass}(L,Q_n) = (-1)^{m(m+1)d/2} \cdot d_k^m \cdot 2^d \cdot D(L) \cdot \prod_{i=1}^{m} \zeta_k(1-2i).$$

(2) *When $n = 2m$,*

$$\mathrm{Mass}(L,Q_n) = 2^{md} \cdot D(L) \cdot \epsilon(\chi) \cdot (N_{k/\mathbb{Q}}\mathfrak{f}(\chi))^{(1-n)/2} \cdot L_k(1-m,\chi) \cdot \prod_{i=1}^{m-1} \zeta_k(1-2i).$$

Here, $|\epsilon(\chi)| = 1$ and $D(L)$ is defined in Theorem 6.2.

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Appendix A. The proof of Lemma 4.2

Proof. Recall that $\tilde{M}$ is the special fiber of $\underline{M}^*$. Similar to the construction of φ_i explained at the beginning of § 4.1, the morphism φ_i is extended to the morphism

$$\tilde{\varphi}_i : \tilde{M} \longrightarrow \mathrm{Aut}_\kappa(\bar{V}_i)$$

such that $\tilde{\varphi}_i|_{\tilde{G}} = \varphi_i$. Here, $\bar{V}_i = B_i/Z_i$. We define

$$\tilde{\varphi} = \prod_i \tilde{\varphi}_i : \tilde{M} \longrightarrow \prod_i \mathrm{Aut}_\kappa(\bar{V}_i).$$

Then $\tilde{\varphi}|_{\tilde{G}} = \varphi$.

Before describing the equations defining $\mathrm{Ker}\,\tilde{\varphi}$, we state notation here. We use $a_{i-1}, b_{i-1}, c_{i-1}, d_{i-1}, e_{i-1}, f_{i-1}, g_{i-1}, h_{i-1}, i_{i-1}$ (respectively $a'_{i-1}, b'_{i-1}, c'_{i-1}, d'_{i-1}, e'_{i-1}, f'_{i-1}, g'_{i-1}, h'_{i-1}, i'_{i-1}$) to denote a block in $m_{i-1,i}$ (respectively $m_{i,i-1}$).

Recall that we have represented the given quadratic form q by a symmetric matrix $(2^i \cdot \delta_i)$ with $2^i \cdot \delta_i$ for the (i,i)-block and 0 for remaining blocks. Assume that L_i is *of type I*. Let $\delta_i = \begin{pmatrix} \delta'_i & 0 \\ 0 & \delta''_i \end{pmatrix}$, where δ'_i is an $(n_i - 1) \times (n_i - 1)$-matrix (respectively an $(n_i - 2) \times (n_i - 2)$-matrix) if L_i is *of type I^o* (respectively *of type I^e*). In particular, if L_i is *of type I^e*, $\delta''_i = \begin{pmatrix} 1 & 1 \\ 1 & 2\gamma_i \end{pmatrix}$ by Theorem 2.4. We denote γ_i mod 2 by $\widetilde{\gamma}_i (\in \kappa)$. In addition, we denote the solution of the equation $x^2 - \widetilde{\gamma}_i = 0$ by $\sqrt{\widetilde{\gamma}_i} (\in \kappa)$.

We now describe the equations defining $\mathrm{Ker}\,\tilde{\varphi}$. There are the following nine cases according to each type of L_{i-1}, L_i, L_{i+1}.

(i) Assume that L_i is *free*:

- (a) if L_i is *of type II*, set $m_{i,i} = \mathrm{id}$;
- (b) if L_i is *of type I^o*, set $s_i = \mathrm{id}$;
- (c) if L_i is *of type I^e_1*, set $s_i = \mathrm{id}$ and $v_i = 0$;
- (d) if L_i is *of type I^e_2*, set $s_i = \mathrm{id}$.

(ii) Assume that L_{i-1} is *of type I^o* and L_{i+1} is *of type II*:

- (a) if L_i is *of type II*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} \\ 2b_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1)\times(n_i)$-matrix, etc. Set $m_{i,i} = \mathrm{id}$ and $b_{i-1} = 0$;
- (b) if L_i is *of type I^o*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2c_{i-1} \\ 2b_{i-1} & 4f_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1) \times (n_i - 1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b_{i-1} = 0$;
- (c) if L_i is *of type I^e*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2c_{i-1} & 2e_{i-1} \\ 2b_{i-1} & 2d_{i-1} & 4f_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1} - 1) \times (n_i - 2)$-matrix, etc. If L_i is *of type I^e*, set $s_i = \mathrm{id}$ and $b_{i-1} = \sqrt{\widetilde{\gamma}_i} \cdot v_i$.

(iii) Assume that L_{i-1} is *of type I^e* and L_{i+1} is *of type II*:

- (a) if L_i is *of type II*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} \\ 2b_{i-1} \\ 2c_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1} - 2) \times (n_i)$-matrix, etc. Set $m_{i,i} = \mathrm{id}$ and $b_{i-1} = 0$;
- (b) if L_i is *of type I^o*, the matrix form of $2m_{i-1i}$ is $\begin{pmatrix} 2a_{i-1} & 2d_{i-1} \\ 2b_{i-1} & 4f_{i-1} \\ 2c_{i-1} & 2e_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1} - 2) \times (n_i - 1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b_{i-1} = 0$;
- (c) if L_i is *of type I^e*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2i_{i-1} & 2h_{i-1} \\ 2b_{i-1} & 2d_{i-1} & 4f_{i-1} \\ 2c_{i-1} & 2g_{i-1} & 2e_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1} - 2) \times (n_i - 2)$-matrix, etc. If L_i is *of type I^e*, set $s_i = \mathrm{id}$ and $b_{i-1} = \sqrt{\widetilde{\gamma}_i} \cdot v_i$.

(iv) Assume that L_{i-1} is *of type II* and L_{i+1} is *of type I^o*:

- (a) if L_i is *of type II*, the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i \\ b'_i \end{pmatrix}$, where a'_i is an $(n_{i+1} - 1) \times (n_i)$-matrix, etc. Set $m_{i,i} = \mathrm{id}$ and $b'_i = 0$;

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(b) if L_i is *of type* I^o, the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & c'_i \\ b'_i & 2f'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i-1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b'_i = 0$;

(c) if L_i is *of type* I^e, the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & c'_i & e'_i \\ b'_i & d'_i & 2f'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i-2)$-matrix, etc. If L_i is *of type* I^e, set $s_i = \mathrm{id}$ and $b'_i = \sqrt{\widetilde{\gamma}_i} \cdot v_i$.

(v) Assume that L_{i-1} is *of type II* and L_{i+1} is *of type* I^e:

(a) if L_i is *of type II*, the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i \\ b'_i \\ c'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-2) \times (n_i)$-matrix, etc. Set $m_{ii} = \mathrm{id}$ and $b'_i = 0$;

(b) if L_i is *of type* I^o, the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & d'_i \\ b'_i & 2f'_i \\ c'_i & e'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-2) \times (n_i-1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b'_i = 0$;

(c) if L_i is *of type* I^e, the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & i'_i & h'_i \\ b'_i & d'_i & 2f'_i \\ c'_i & g'_i & e'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-2) \times (n_i-2)$-matrix, etc. If L_i is *of type* I^e, set $s_i = \mathrm{id}$ and $b'_i = \sqrt{\widetilde{\gamma}_i} \cdot v_i$.

(vi) Assume that L_{i-1} is *of type* I^o and L_{i+1} is *of type* I^o:

(a) if L_i is *of type II*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} \\ 2b_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1) \times (n_i)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i \\ b'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i)$-matrix, etc. Set $m_{i,i} = \mathrm{id}$ and $b_{i-1} = b'_i$;

(b) if L_i is *of type* I^o, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2c_{i-1} \\ 2b_{i-1} & 4f_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1) \times (n_i-1)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & c'_i \\ b'_i & 2f'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i-1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b_{i-1} = b'_i$;

(c) if L_i is *of type* I^e, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2c_{i-1} & 2e_{i-1} \\ 2b_{i-1} & 2d_{i-1} & 4f_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1) \times (n_i-2)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & c'_i & e'_i \\ b'_i & d'_i & 2f'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i-2)$-matrix, etc. If L_i is *of type* I^e, set $s_i = \mathrm{id}$ and $b_{i-1} + \sqrt{\widetilde{\gamma}_i} \cdot v_i + b'_i = 0$.

(vii) Assume that L_{i-1} is *of type* I^e and L_{i+1} is *of type* I^o:

(a) if L_i is *of type II*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} \\ 2b_{i-1} \\ 2c_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-2) \times (n_i)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i \\ b'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i)$-matrix, etc. Set $m_{ii} = \mathrm{id}$ and $b_{i-1} = b'_i$;

(b) if L_i is *of type* I^o, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2d_{i-1} \\ 2b_{i-1} & 4f_{i-1} \\ 2c_{i-1} & 2e_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-2) \times (n_i-1)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & c'_i \\ b'_i & 2f'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i-1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b_{i-1} = b'_i$;

(c) if L_i is *of type* I^e, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2i_{i-1} & 2h_{i-1} \\ 2b_{i-1} & 2d_{i-1} & 4f_{i-1} \\ 2c_{i-1} & 2g_{i-1} & 2e_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-2) \times (n_i-2)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a'_i & c'_i & e'_i \\ b'_i & d'_i & 2f'_i \end{pmatrix}$, where a'_i is an $(n_{i+1}-1) \times (n_i-2)$-matrix, etc. If L_i is *of type* I^e, set $s_i = \mathrm{id}$ and $b_{i-1} + \sqrt{\widetilde{\gamma}_i} \cdot v_i + b'_i = 0$.

(viii) Assume that L_{i-1} is *of type* I^o and L_{i+1} is *of type* I^e:

(a) if L_i is *of type II*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} \\ 2b_{i-1} \end{pmatrix}$, where a_i' is an $(n_{i-1}-1)\times(n_i)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a_i' \\ b_i' \\ c_i' \end{pmatrix}$, where a_i' is an $(n_{i+1}-2)\times(n_i)$-matrix. Set $m_{i,i} = \mathrm{id}$ and $b_{i-1} = b_i'$;

(b) if L_i is *of type* I^o, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2c_{i-1} \\ 2b_{i-1} & 4f_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1)\times(n_i-1)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a_i' & d_i' \\ b_i' & 2f_i' \\ c_i' & e_i' \end{pmatrix}$, where a_i' is an $(n_{i+1}-2)\times(n_i-1)$-matrix. Set $s_i = \mathrm{id}$ and $b_{i-1} = b_i'$;

(c) if L_i is *of type* I^e, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2c_{i-1} & 2e_{i-1} \\ 2b_{i-1} & 2d_{i-1} & 4f_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-1)\times(n_i-2)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a_i' & i_i' & h_i' \\ b_i' & d_i' & 2f_i' \\ c_i' & g_i' & e_i' \end{pmatrix}$, where a_i' is an $(n_{i+1}-2)\times(n_i-2)$-matrix. If L_i is *of type* I^e, set $s_i = \mathrm{id}$ and $b_{i-1} + \sqrt{\tilde{\gamma}_i}\cdot v_i + b_i' = 0$.

(ix) Assume that L_{i-1} is *of type* I^e and L_{i+1} is *of type* I^e:

(a) if L_i is *of type II*, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} \\ 2b_{i-1} \\ 2c_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-2)\times(n_i)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a_i' \\ b_i' \\ c_i' \end{pmatrix}$, where a_i' is an $(n_{i+1}-2)\times(n_i)$-matrix, etc. Set $m_{i,i} = \mathrm{id}$ and $b_{i-1} = b_i'$;

(b) if L_i is *of type* I^o, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2d_{i-1} \\ 2b_{i-1} & 4f_{i-1} \\ 2c_{i-1} & 2e_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-2)\times(n_i-1)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a_i' & d_i' \\ b_i' & 2f_i' \\ c_i' & e_i' \end{pmatrix}$, where a_i' is an $(n_{i+1}-2)\times(n_i-1)$-matrix, etc. Set $s_i = \mathrm{id}$ and $b_{i-1} = b_i'$;

(c) if L_i is *of type* I^e, the matrix form of $2m_{i-1,i}$ is $\begin{pmatrix} 2a_{i-1} & 2i_{i-1} & 2h_{i-1} \\ 2b_{i-1} & 2d_{i-1} & 4f_{i-1} \\ 2c_{i-1} & 2g_{i-1} & 2e_{i-1} \end{pmatrix}$, where a_{i-1} is an $(n_{i-1}-2)\times(n_i-2)$-matrix, and the matrix form of $m_{i+1,i}$ is $\begin{pmatrix} a_i' & i_i' & h_i' \\ b_i' & d_i' & 2f_i' \\ c_i' & g_i' & e_i' \end{pmatrix}$, where a_i' is an $(n_{i+1}-2)\times(n_i-2)$-matrix, etc. If L_i is *of type* I^e, set $s_i = \mathrm{id}$ and $b_{i-1}+\sqrt{\tilde{\gamma}_i}\cdot v_i+b_i' = 0$.

Before investigating the equations defining $\operatorname{Ker}\varphi$, let us introduce the notation $\delta_i'(b_i)$ in this paragraph. Recall that we say $\delta_i = \begin{pmatrix} \delta_i' & 0 \\ 0 & \delta_i'' \end{pmatrix}$ at the beginning of Appendix A. Then the symmetric matrix δ_i' defines a quadratic form which is 0 modulo 2. We define

$$\delta_i'(b_i) := \tfrac{1}{2} b_i \cdot \delta_i' \cdot {}^t b_i$$

as matrix multiplication, where b_i is a $1\times(n_i-1)$-low vector (respectively $1\times(n_i-2)$-low vector) if L_i is *of type* I^o (respectively *of type* I^e). If L_i is *of type II*, we define $\delta_i'(b_i) = \frac{1}{2} b_i \cdot \delta_i \cdot {}^t b_i$, where b_i is a $(1\times n_i)$-low vector.

We are ready to state the equations defining $\operatorname{Ker}\varphi$. They are obtained by the matrix equation ${}^t m q m = q$, where m is an element of $\operatorname{Ker}\tilde{\varphi}(R)$ for a κ-algebra R.

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By observing the diagonal (i,i)-blocks of ${}^tmqm=q$, we have the following matrix equation:

$$\begin{aligned}&{}^tm_{i,i}\delta_i m_{i,i}+2({}^tm_{i-1,i}\delta_{i-1}m_{i-1,i}+{}^tm_{i+1,i}\delta_{i+1}m_{i+1,i})\\&\quad+4({}^tm_{i-2,i}\delta_{i-2}m_{i-2,i}+{}^tm_{i+2,i}\delta_{i+2}m_{i+2,i})=(\delta_i),\end{aligned}\tag{A1}$$

where $0\leqslant i<N$.

By observing the $(i,i+1)$-blocks of ${}^tmqm=q$, we have the following matrix equation:

$${}^tm_{i,i}\delta_i m_{i,i+1}+{}^tm_{i+1,i}\delta_{i+1}m_{i+1,i+1}+2({}^tm_{i-1,i}\delta_{i-1}m_{i-1,i+1}+{}^tm_{i+2,i}\delta_{i+2}m_{i+2,i+1})=0,\tag{A2}$$

where $0\leqslant i<N-1$.

By observing the (i,j)-blocks of ${}^tmqm=q$, where $i+2\leqslant j$, we have the following matrix equation:

$$\sum_{i\leqslant k\leqslant j}{}^tm_{k,i}\delta_k m_{k,j}=0,\tag{A3}$$

where $0\leqslant i,j<N$.

We first state the equations $\mathcal{F}_i$ and $\mathcal{E}_i$. These equations determine the connected components of $\operatorname{Ker}\varphi$. Assume that L_i is *of type I*. By computing the (2×2)-block (if L_i is *of type* I^o) or the (3×3)-block (if L_i is *of type* I^e) of (A1), we have the equation $\mathcal{F}_i$:

$$\mathcal{F}_i: z_i+z_i^2+\delta'_{i-1}(b'_{i-1})+\delta'_{i+1}(b_i)+\widetilde{\gamma_{i-1}}\cdot e_{i-1}^2+\widetilde{\gamma_{i+1}}\cdot d_i^2+x_{i-2}^2+x_i^2=0.\tag{A4}$$

Here:

- b'_{i-1} and b_i are the blocks in $m_{i,i-1}$ and $m_{i,i+1}$ respectively, as defined above;
- e_{i-1} is the (3×2)-block (if L_i is *of type* I^o) or the (3×3)-block (if L_i is *of type* I^e) of $m_{i-1,i}$;
- d_i is the (2×2)-block of $m_{i,i+1}$;
- x_{i-2} is the (2×2)-block (respectively the (2×3)-block) of $m_{i-2,i}$ when L_i is *of type* I^o (respectively *of type* I^e) and L_{i-2} is *of type I*;
- x_i is the (2×2)-block or the (2×3)-block of $m_{i,i+2}$ when L_{i+2} is *of type* I^o or *of type* I^e, respectively. If L_{i-2} (respectively L_{i+2}) is not *of type I*, we remove x_{i-2}^2 (respectively x_i^2) in the equation $\mathcal{F}_i$.

If L_i is *of type* I^e, the (2×2)-block of (A1) induces the equation $\mathcal{E}_i$:

$$\mathcal{E}_i: u_i+\widetilde{\gamma}_i\cdot u_i^2+\delta'_i(v_i)+d_{i-1}^2+e_i^2=0.\tag{A5}$$

Here d_{i-1} (respectively e_i) appears only when L_{i-1} (respectively L_{i+1}) is *of type I*.

We now choose a non-negative integer j such that L_j is *of type I* and L_{j+2} is *of type II*. For such a j, there is a non-negative integer m_j such that L_{j-2l} is *of type I* for every l with $0\leqslant l\leqslant m_j$ and $L_{j-2(m_j+1)}$ is *of type II*. Then the sum of equations

$$\sum_{l=0}^{m_j}\mathcal{F}_{j-2l}+\sum_l{}'\,\widetilde{\gamma_{j+1-2l}}\cdot\mathcal{E}_{j+1-2l}$$

becomes

$$\sum_{l=0}^{m}(z_{j-2l}+z_{j-2l}^2)+\sum_l{}'(\widetilde{\gamma_{j+1-2l}}\cdot u_{j+1-2l}+\widetilde{\gamma_{j+1-2l}}^{\,2}\cdot u_{j+1-2l}^2)=0.\tag{A6}$$

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Here $\sum_l' \widetilde{\gamma_{j+1-2l}} \cdot \mathcal{E}_{j+1-2l}$ is the sum of the equations $\widetilde{\gamma_{j+1-2l}} \cdot \mathcal{E}_{j+1-2l}$ such that $0 \leqslant l \leqslant m_j + 1$ and L_{j+1-2l} is *of type* I^e. On the other hand, if L_i is *free of type* I^e_1 so that v_i is 0, the equation $\widetilde{\gamma}_i \cdot \mathcal{E}_i$ becomes

$$\widetilde{\gamma}_i \cdot u_i + \widetilde{\gamma}_i^{\,2} \cdot u_i^2 = 0. \tag{A7}$$

We next state the remaining equations defining $\operatorname{Ker}\varphi$. In addition to the equations $\mathcal{F}_i$ and $\mathcal{E}_i$, (A1) induces the following.

– If L_i is *of type* I^o, the (1×2)-block of (A1) is

$$\delta'_i y_i + {}^t v_i + \mathcal{P}^i_{1,2} = 0. \tag{A8}$$

Here, $\mathcal{P}^i_{1,2}$ is a polynomial with variables $m_{i-1,i}, m_{i+1,i}$.

– If L_i is *of type* I^e, the (1×2)-block of (A1) is

$$\delta'_i r_i + {}^t v_i = 0. \tag{A9}$$

– If L_i is *of type* I^e, the (1×3)-block of (A1) is

$$\delta'_i t_i + {}^t y_i + {}^t v_i z_i + \widetilde{\gamma}_i \cdot {}^t v_i + \mathcal{P}^i_{1,3} = 0. \tag{A10}$$

Here, $\mathcal{P}^i_{1,3}$ is a polynomial with variables $m_{i-1,i}, m_{i+1,i}$.

– If L_i is *of type* I^e, the (2×3)-block of (A1) is

$$x_i + w_i + {}^t r_i \delta'_i t_i + z_i + u_i z_i + \widetilde{\gamma}_i \cdot u_i + \mathcal{P}^i_{2,3} = 0. \tag{A11}$$

Here, $\mathcal{P}^i_{2,3}$ is a polynomial with variables $m_{i-1,i}, m_{i+1,i}$.

Equation (A2) induces the following.

– If either L_i or L_{i+1} is *of type II*, (A2) becomes

$${}^t m_{i,i} \delta_i m_{i,i+1} + {}^t m_{i+1,i} \delta_{i+1} m_{i+1,i+1} = 0. \tag{A12}$$

– If both L_i and L_{i+1} are *of type I*, (A2) consists of two parts:

$$\left\{\begin{matrix} {}^t m_{i,i} \delta_i m_{i,i+1} + {}^t m_{i+1,i} \delta_{i+1} m_{i+1,i+1} = 0 & \text{except for the } (n_i \times n_{i+1})\text{-entry}, \\ f_i + f'_i + \mathcal{P}'_i = 0 & \text{for the } (n_i \times n_{i+1})\text{-entry}. \end{matrix}\right\} \tag{A13}$$

Here, $\mathcal{P}'_i$ is a polynomial with variables $m_{i,i}, m_{i,i+1}, m_{i+1,i}, m_{i+1,i+1}, m_{i-1,i}, m_{i-1,i+1}, m_{i+2,i}$ and $m_{i+2,i+1}$. Notice that the polynomial $\mathcal{P}'_i$ does not include the variables f_i, f'_i. We recall that f_i (respectively f'_i) is such that $2f_i$ (respectively $2f'_i$) is the entry of $m_{i,i+1}$ (respectively $m_{i+1,i}$).

Finally, we observe (A3)–(A13). The closed subscheme $\operatorname{Ker}\varphi$ of $\operatorname{Ker}\tilde{\varphi}$ is determined by these equations. It is easily seen that $\operatorname{Ker}\varphi$ is a disconnected affine space with $2^{\alpha+\beta}$ components. Moreover, the dimension of $\operatorname{Ker}\varphi$ can be computed by observing these equations or by the following lemma, and it is l. This completes the proof. □

The dimension of $\operatorname{Ker}\varphi$ is also computed from the following lemma easily.

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Lemma A14. *The dimension of* Ker φ *is* l.

Proof. Since the dimension of Ker φ is at least l, it suffices to show that the dimension of the tangent space of Ker φ at the identity e is l. It is enough to check the statement over the algebraic closure $\bar{\kappa}$ of κ. Recall the morphism

$$\tilde{\varphi} : \tilde{M} \to \prod_i \mathrm{Aut}_\kappa(\bar{V}_i).$$

Then we have

$$\mathrm{Ker}\,\varphi = \tilde{G} \cap \mathrm{Ker}\,\tilde{\varphi}$$

as closed subgroup schemes of $\tilde{M}$. Recall that we have defined the map $\rho_{*,m} : T_m \to T_{\rho(m)}$ and we identified T_m and $T_{\rho(m)}$ with $T_1(\bar{\kappa})$ and $T_2(\bar{\kappa})$ respectively, in the proof of Lemma 3.6. Based on these, the tangent space of $\tilde{G}$ at e is the kernel of the map $\rho_{*,e}$. In addition, the tangent space of Ker $\tilde{\varphi}$ at e is identified with the subspace of $T_1(\bar{\kappa})$, satisfying with nine cases described at the beginning of Appendix A if we change $s_i = \mathrm{id}$ and $m_{i,i} = \mathrm{id}$ to $s_i = 0$ and $m_{i,i} = 0$, respectively. We denote this subspace by $T_0(\bar{\kappa})$.

The tangent space of Ker φ at e is the intersection of Ker $\rho_{*,e}$ and $T_0(\bar{\kappa})$ as subspaces of $T_1(\bar{\kappa})$. Thus it suffices to show that Ker $\rho_{*,e} \cap T_0(\bar{\kappa})$ has dimension l.

Since $X \mapsto q \cdot X$ is a bijection $T_1(\bar{\kappa}) \to T_3(\bar{\kappa})$ as explained in the proof of Lemma 3.6, it suffices to show that $q \cdot (\mathrm{Ker}\,\rho_{*,e} \cap T_0(\bar{\kappa}))$ as a subspace of $T_3(\bar{\kappa})$ has dimension l. We have

$$q \cdot (\mathrm{Ker}\,\rho_{*,e} \cap T_0(\bar{\kappa})) = q \cdot \mathrm{Ker}\,\rho_{*,e} \cap q \cdot T_0(\bar{\kappa}).$$

The space $q \cdot \mathrm{Ker}\,\rho_{*,e}$ is the kernel of the map $T_3(\bar{\kappa}) \to T_2(\bar{\kappa}), Y \mapsto {}^tY + Y$. Thus it is the subspace of $T_3(\bar{\kappa})$ consisting of symmetric matrices Y whose diagonal entries are 0. Notice that this fact implies that the dimension of $q \cdot \mathrm{Ker}\,\rho_{*,e}$ is $n(n-1)/2$, which is the dimension of $\tilde{G}$. By considering $q \cdot T_0(\bar{\kappa})$, it is easily seen that the dimension of $q \cdot \mathrm{Ker}\,\rho_{*,e} \cap q \cdot T_0(\bar{\kappa})$ is exactly l. □

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Sungmun Cho sungmuncho12@gmail.com
Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

Current address: Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada

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