PART X

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A STATISTICAL MODEL FOR THE CLOUD STRUCTURE OF THE INTERSTELLAR MEDIUM

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Abstract. We have formulated and studied the properties of a mathematical model intended to span between the hydrodynamical and discontinuous regimes. Model parameters and predictions permit matching the observed mass spectrum of those interstellar clouds with masses between 10^3 and $10^6 M_\odot$. The central element of the theory is the specification of $P(m, m'; \mu, \mu', \ldots \mu'')$, the conditional probability that once a collision has occurred between clouds of mass m, m' the result is clouds of mass $\mu, \mu', \ldots \mu''$. For stable solutions to exist we require that the total mass in the system be conserved. Such simple models as total coalescence, geometric overlap and partition statistics were considered for P as well as several probabilities based on possible physical conditions that might prevail in cloud-cloud collisions. One immediate numerical result of these models is that nearly total coalescence must obtain in actual cloud-cloud collisions before one can build up a non-infinitesimal concentration of large mass couds. Field and coworkers (1965, 1968) indicate that total coalescence would produce a minimum in the mass spectrum and that as the position of the minimum went to infinity the spectrum itself more closely approximates a power law curve with index $-\frac{3}{2}$. In none of our calculations was anything but strict monotonic decrease observed and under the same physical assumptions as Field the curve we obtain (analytically) has no minimum, independent of the largest mass in the system. Both of these spectra are flatter than recent observational evidence indicate. True equilibrium solutions exist in our formulation with an e-folding time $\simeq 10^7$ yr for objects such as these. We are continuing work on the detailed time evolution of interstellar clouds in particular and formulating other astronomically interesting applications of the general theory.

1. Introduction

There are many physical, astronomical and chemical processes of interest wherein there is a reaction of the form $X+Y \rightarrow A+B+\cdots+C$ where the products of the reaction are not necessarily of the same generic type as the reactants. To describe the time evolution of the processes with particular application to the cloud structure of the interstellar medium (ISM), a mathematical formulation of such processes has been created in a very general way. This formulation and applications to the ISM will be presented here. The nature of the mathematical structure will be indicated without proof in Section 3. The results of some of our calculations as pertains to the mass distribution of clouds is contained in the last section. Further possible applications of the formalism to astronomical problems are molecular formation in the (interstellar medium) ISM, the mass structure of the asteroid belt and sporadic meteors, nucleogenesis and the production of large energy fluxes via stellar collisions in galactic nuclei.

2. Mathematical Formulation

We now consider the coalescence and disruption of clouds in the ISM. Since we are primarily interested in deriving the mass spectrum the velocity parameter has been

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averaged over. The simplest geometrical situation we invision for cloud-cloud collisions will have at most three fragments and further generalizations will not be included here. For complete details see Taff (1973). When the fraction of space occupied by the clouds is small compared to the total available volume only binary collisions will be of primary importance. We assume this to be the case. We shall formulate the problem in a discrete language for simplicity. Thus, we introduce for convenience the assumption that all cloud masses are integral multiples of some unit mass m, i.e. if the jth cloud (or cloud of type j) has mass m_j then

$$m_j = j \cdot m . \tag{1}$$

In the discussion of the equilibrium problem for the mass distribution we are concerned with closed systems and allow no unbalanced sources or sinks. This implies large objects cannot leave the system as in the original work by Field and coworkers (Field and Saslaw, 1965; Field and Hutchins, 1968). Finite computing machine memories also limit the largest mass, m_n, which we can consider in detail.

The specification of a collision between clouds of mass i,j involves two steps. First is the number of such collisions occurring per unit time per unit volume. We shall symbolize this as $<\sigma v>_{ij} N_i(t) N_j(t)$. $N_i(t)$ is the number of clouds of type i per unit volume at time t and $<\sigma v>_{ij}$ is the velocity averaged collison rate between clouds i and j. The time dependence will not always be explicitly exhibited. Once a collision has occurred we need the conditional probability that it will yield fragments of mass p, q, r in accordance with

$$m_i + m_j = m_p + m_q + m_r . \tag{2}$$

We denote this probability by $P(i, j; p, q, r)$. In any particular application of this statistical mechanical formulation the *mechanics* lies in the specification of the P's. To determine the differential equation governing the time evolution of $N_k(t)$ we need the production and destruction rates for clouds of type k. The processes are:

(I) Input to N_k.
- (a) When one reactant is a k.
  - (i) Only one is.
  - (ii) Both are.
- (b) Neither of the reactants is a k
  - (i) The reactants are different from each other.
  - (ii) The reactants are identical.

(II) Output from N_k.
- (a) One reactant is a k but none of the products is.
- (b) Both of the reactants are k's.
  - (i) None of the products is a k.
  - (ii) One of the products is a k.

Processes not listed above (apart from redistribution discussed below and spontaneous input-output terms) produce no net change in N. Let us write $<\sigma v>_{ij}$ as $A_{\cdot j}$.

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Summing all of the above terms we have apart from the matter of masses $>n$ the equation

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = (\tfrac{1}{2}) \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} N_i N_j H_{ij}^k \tag{3}$$

with

$$\begin{aligned} H_{ij}^k = H_{ji}^k = \sum_{p=0}^{2n} &(P(i,j;p,k,i+j-p-k) + P(i,j;i+j-p-k,p,k) + \\ &+ P(i,j;k,i+j-p-k,p)) - \delta_{ik} - \delta_{jk} \\ = \; &p_{ij}^k - \delta_{ik} - \delta_{jk}, \end{aligned} \tag{4}$$

where, as an example, the *Ibi* term can be seen to be equivalent to

$$\begin{aligned} &(\tfrac{1}{2}) \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} N_i N_j (1-\delta_{ij})(1-\delta_{ik})(1-\delta_{jk}) . \\ &\sum_{p,q,r=0}^{2n} P(i,j;p,q,r) \delta_{i+j,\, p+q+r} (\delta_{pk} + \delta_{qk} + \delta_{rk}) . \end{aligned} \tag{5}$$

The massive clouds formed from clouds with masses less than or equal to n are assumed to disrupt because of gravitational instability and then be redistributed into clouds of smaller mass. Although a spectrum of redistribution is more probable then redistribution into clouds of one particular mass we shall follow the procedure of Field and coworkers, based on a suggestion due to Oort (1954), and assume all of this mass return as clouds of unit mass. H_{ij}^1 will now be assumed to be corrected for this.

We have postulated a closed system for the equilibrium discussion and must not only take into account redistribution but insure that the total mass is conserved, i.e.

$$\sum_{k=1}^{n} k \, \mathrm{d}N_k / \mathrm{d}t = 0 , \tag{6}$$

which, in turn, is a condition on the type of probability function P that we are using since it implies that

$$\sum_{k=1}^{2n} k H_{ij}^k = 0 , \quad \text{for all} \quad i \text{ and } j. \tag{7}$$

This constraint will be called the k sum. For simple probability functions one can show that this is the same statement as the one that P is a true probability. The mathematical implications of the k sum are such to insure the existence of a mathematically nontrivial solution for the equilibrium problem and its stability relative to perturbations.

Once the set of H matrices, which are real, symmetric, non-definite and non-pairwise commuting*, along with $N_k(0)$ are specified the time evolution of the system can

* This means the commutator

$$\sum_{j=1}^{n} (H^r{}_{ij} H^s{}_{jk} - H^s{}_{kj} H^r{}_{ji}) \neq 0, \quad r \neq s.$$

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be studied. We shall instead study the equilibrium problem here to indicate the kind of result the time-dependent problem will yield. The statement of the equilibrium problem is the solution of

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} N_i N_j H_{ij}^k = 0, \quad k = 1,2,\ldots,n, \tag{8}$$

with

$$\sum_{k=1}^{n} k N_k = M/(mV) = \text{constant}, \tag{9}$$

where M is the total mass of clouds contained in the fixed volume V.

3. Mathematical Properties

The above problem can be solved analytically only for the non-physical case of $n = 2$, for all n, if there is total coalescence and A_{ij} is independent of i, $_j$ (Taff and Savedoff, 1972). Fortunately, the theorems below are not n dependent. The proofs can be found in Taff (1973). The $A_{ij} H_{ij}^k$ will be assumed to have the general properties listed above.

Theorem 1. For a given set of inital conditions $N_i(0)$ there is one and only continuous solution to Equations (3). This is true with or without the k sum. (Equation (9)).

Theorem 2. If the $A_{ij} H_{ij}^k$ approach, uniformly, a nonzero constant as t goes to infinity there is a non-trivial solution of Equations (8) subject to the k sum.

Theorem 3. Under the time dependence assumptions of Theorem 2 if a solution vector N exists to Equations (8) and (9) such that each of its components N_i is real, bounded below by 0 and above by $M/(miV)$ then it is unique.

The stability of the equilibrium solution relative to perturbations depends on the eigenvalues of the matrix $\boldsymbol{H}$ given by

$$\mathbf{H}_{ki} = 2 \sum_{j=1}^{n} A_{ij} H_{ij}^k N_j^0 \tag{10}$$

and we have introduced the superscript zero to indicate the equilibrium solution. The properties of $\boldsymbol{H}$ are given by Theorem 4.

Theorem 4 The matrix $\boldsymbol{H}$ defined by Equation (10) when N^0 is the equilibrium solution vector has the following properties:

(i) $\det(\boldsymbol{H}) = 0$

(ii) The rank of $\boldsymbol{H}$ is $n-1$ and it therefore possesses only one zero eigenvalue whose eigenvector is N^0.

(iii) If λ is an eigenvalue of $\boldsymbol{H}$ then $\text{Re}(\lambda) < 0$ unless $\lambda = 0$.

With these characteristics for $\boldsymbol{H}$ one may prove Theorem 5.

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Theorem 5. Every solution $N(t, N(0))$ of Equations (3) and (9) approaches N^0 as t approaches infinity. Thus the solution N^0 is absolutely stable relative to perturbations and N^0 is independent of $N(0)$.

4. Cloud Models

It is known that under the assumptions

(i) $p_{ij}^k = \delta_{i+j,\,k}$ (ie. total coalescence)

(ii) $n = \infty$

(iii) A_{ij} is independent of i, j.

that

$$N_k = \text{constant} \cdot k^{-3/2} \tag{11}$$

independent of the time dependence (independence) or type of redistribution (Taff and Savedoff, 1972). If one modifies the third assumption to one of geometric cross-section the mass spectrum is still a power law with index $-\frac{5}{3}$ (Taff, 1973). These mass distributions are lower limits to those of more realistic models. For simplicity let us assume all clouds are spheres with the same mean interior density. Then

$$A_{ij} = (3m/4\pi\varrho)^{2/3}(i^{1/3} + j^{1/3})^2 \langle v \rangle_{ij}, \tag{12}$$

because gravitation is negligible. There are two reasonable forms for $\langle v \rangle_{ij}$. The first is that it is uncorrelated with i, j and the second that there is kinetic energy equipartition (KEE) in which case $\langle v \rangle_{ij} \propto ((i+j)/ij)^{\frac{1}{2}}$.

There are two simple geometrical models for $P(i, j; p, q, r)$ which we have considered. In the first when two clouds of mass i, j $(j > i)$ collide the result is (for impact parameters greater than that of internal tangency $= c_{\text{int}}$) the part of i swept out by j

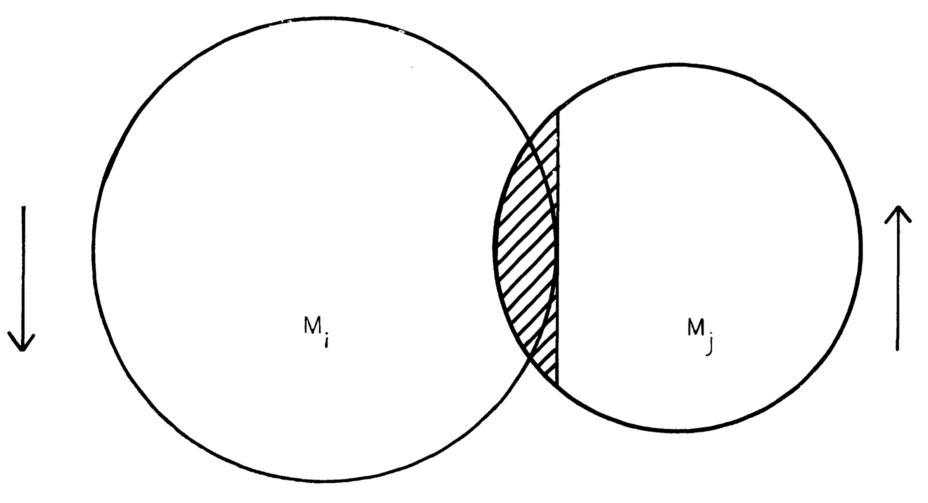

Fig. 1. The two fragment model ($j > i$). The shaded part of i sticks to j and the remainder goes free. Arrows indicate directions of relative velocity.

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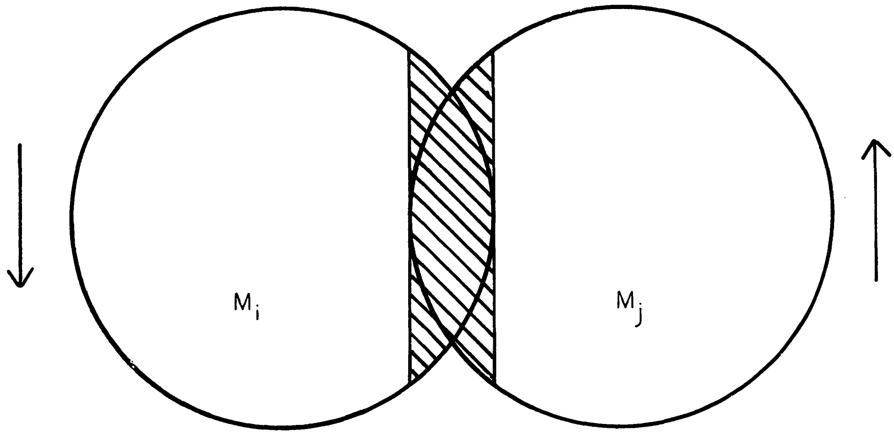

Fig. 2. The two fragment model ($i = j$). The shaded parts of i and j coalesce and both remaining outer pieces go free. Arrows indicate directions of relative velocity.

sticks to j, the remainder of i (the 'ear') going free. When $i = j$ one obtains three fragments, the central region of coalescence and the two ears. See Figures 1 and 2. In the remaining case of $j > i$ if the impact parameter is less than or equal to c_{int} we assume total coalescence. It is clear that this is the next most sophisticated model favoring growth as compared to total coalescence. A less favorable assumption to mass accretion would be to always allow three fragments (two ears plus the central region of coalescence) for impact parameters larger than c_{int}, the case of collisions closer than c_{int} remains unchanged.

We have numerically solved these two models (ie. solved Equations (8) and (9)) for various values of n. Both of the above mentioned possibilities for the velocity dependence were included. In every case the resulting values for the N_k were least squares fit to a power law and the results, along with the correlation coeficients are contained in Table I. The correlation coefficient, or alternatively the Student t statistic,

TABLE I

Results of fit of theoretical mass spectrum to a power law $x_k = a \cdot k^{-p}$ and the correlation coefficient r

Three Fragment Model					Two Fragment Model				
n	No KEE		KEE		n	No KEE		KEE	
	p	r	p	r		p	r	p	r
3	4.53	0.96	4.33	0.96	5	2.83	0.99	2.61	0.99
5	5.51	0.97	5.35	0.97	10	2.45	0.98	2.25	0.98
10	7.40	0.97	7.30	0.97	20	2.11	0.98	1.92	0.97
20	9.51	0.98	9.47	0.98	40	1.82	0.97	1.65	0.97

can be used as a measure of the signifiance of the fit. It is clear that as n approaches infinity in the three fragment model the index of the power law approaches $-\infty$. In the case of the two fragment model the power law index is approximated by

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$$p(n) = 1.68 + 12.62\left(\frac{\ln(n)}{n^{3/2}}\right) - 30.82\left(\frac{\ln(n)}{n^{3/2}}\right)^2 \tag{13a}$$

for no KEE and by

$$p(n) = 1.51 + 12.31\left(\frac{\ln(n)}{n^{3/2}}\right) - 30.83\left(\frac{\ln(n)}{n^{3/2}}\right)^2 \tag{13b}$$

when there is KEE. Thus there is little difference between these and the total coalescence model with geometric cross-section with/without KEE. Complete details can be found in Taff (1973).

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