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Topological Properties of a Class of Higher-dimensional Self-affine Tiles

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Abstract. We construct a family of self-affine tiles in $\mathbb{R}^d$ ($d \geqslant 2$) with noncollinear digit sets, which naturally generalizes a class studied originally by Q.-R. Deng and K.-S. Lau in $\mathbb{R}^2$, and its extension to $\mathbb{R}^3$ by the authors. We obtain necessary and sufficient conditions for the tiles to be connected and for their interiors to be contractible.

1 Introduction

Let $d \geq 1$ be an integer and A be a $d \times d$ expanding matrix, *i.e.*, all of its eigenvalues have moduli greater than 1. It is well known [8,12] that for any finite set $\mathcal{D} \subset \mathbb{R}^d$ there exists a unique nonempty compact set $T = T(A, \mathcal{D})$ such that $T = \bigcup_{d \in \mathcal{D}} A^{-1}(T + d)$. The equation can be rewritten as $AT = T + \mathcal{D}$ and T can be expressed as

$$T = \Big\{ \sum_{k \geq 1} A^{-k} d_k : d_k \in \mathcal{D} \Big\}. \tag{1.1}$$

We call $\mathcal{D}$ a *digit set*, $(A, \mathcal{D})$ a *self-affine pair*, and T a *self-affine set*. If $\#\mathcal{D} = |\det(A)|$ is an integer and the interior of T is nonempty, then T actually tiles $\mathbb{R}^d$ in the following sense: there exists a discrete set $\mathcal{L} \subset \mathbb{R}^d$ that satisfies $T + \mathcal{L} = \mathbb{R}^d$ and $(T^\circ + \iota_1) \cap (T^\circ + \iota_2) = \varnothing$ for all distinct $\iota_1, \iota_2 \in \mathcal{L}$. (Here T° denotes the interior of T). Such a set T is called a *self-affine tile* [12].

The connectedness of self-affine sets has been studied extensively by many authors (see [1,7,10,11,13,14] and the references therein). Disk-likeness of self-affine tiles has also been studied by many authors (see [2,13,16] and the references therein). Ball-likeness was investigated in [3,4,9]. For digit sets that are consecutive and collinear, Kirat and Lau [10] formulated an algebraic condition, known as the *height reducing property*, to determine whether the self-affine tiles are connected. This condition is not applicable if the digit set is nonconsecutive or noncollinear. Deng and Lau [5] initiated the study of self-affine tiles with noncollinear digit sets and obtained the following result concerning connectedness.

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Theorem 1.1 *Let $p, q \in \mathbb{Z}$ with $|p|, |q| \geq 2$, $a \in \mathbb{R}$, and for each $i \in \{0, 1, \dots, |p| - 1\}$, fix any $b_i \in \mathbb{R}$. Let*

$$A = \left(\begin{smallmatrix} p & 0 \\ -a & q \end{smallmatrix}\right), \qquad \mathcal{D} = \left\{ (i, j + b_i) : 0 \leq i \leq |p| - 1,\ 0 \leq j \leq |q| - 1 \right\}.$$

Then the self-affine set T is a tile, and it is connected if and only if

$$\left| \frac{b_{i+1} - b_i}{q} + \frac{\operatorname{sgn}(p)(b_0 - b_{|p|-1}) - a}{q(q - \operatorname{sgn}(p))} \right| \leq 1$$

for all i. Here and throughout this paper, $\operatorname{sgn}(p) = 1$ *if $p > 0$, and* $\operatorname{sgn}(p) = -1$ *if $p < 0$.*

Various extensions of this result in $\mathbb{R}^2$ have been obtained by a number of authors [15,17]. An extension of this result to $\mathbb{R}^3$ was studied in [4], where a family of ball-like self-affine tiles is constructed by the authors of the present paper. To summarize the results in [4], let

$$A := \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ -t & -s & r \end{pmatrix}, \tag{1.2}$$
$$\mathcal{D} := \left\{ (i, j, k + a_i + b_j) : 0 \leq i < |p|, 0 \leq j < |q|, 0 \leq k < |r| \right\},$$

where $a_i, b_j \in \mathbb{R}$. Define

$$\rho_1(i) := \frac{a_{p-1} - a_0 + t}{r(r-1)} + \frac{a_i - a_{i+1}}{r}, \tag{1.3}$$
$$\rho_2(j) := \frac{b_{q-1} - b_0 + s}{r(r-1)} + \frac{b_j - b_{j+1}}{r},$$

and

$$\delta_1(i) := \inf\left\{ \left| |\rho_1(i)| - r^{-n} |\rho_2(j)| \right| : 0 \leq j < q - 1, n \geq 1 \right\}, \tag{1.4}$$

$$\delta_2(j) := \inf\left\{ \left| |\rho_2(j)| - r^{-n} |\rho_1(i)| \right| : 0 \leq i < p - 1, n \geq 1 \right\}, \tag{1.5}$$

$$\delta_3(i, j) := \left| |\rho_1(i)| - |\rho_2(j)| \right|. \tag{1.6}$$

The following is the main result in [4].

Theorem 1.2 *Let $(A, \mathcal{D})$ be given as in (1.2) with $p, q, r \geq 2$ and let T be the corresponding self-affine set. Assume ρ_1, ρ_2 are defined as (1.3) and $\delta_1, \delta_2, \delta_3$ are given as in (1.4)–(1.6).*

- (i) *T is connected if one of the following statements holds.*
  - (a) *For all i, $\delta_1(i) \leq 1$, and for all j, either $\delta_2(j) \leq 1$ or there exists i (depending only on j) such that $\delta_3(i, j) \leq 1$.*
  - (b) *For all j, $\delta_2(j) \leq 1$, and for all i, either $\delta_1(i) \leq 1$ or there exists j (depending only on i) such that $\delta_3(i, j) \leq 1$.*

  Moreover, if a_i, b_j are zero for all i, j, then each of the above sufficient conditions is necessary, i.e., *if T is connected, then* (a) *or* (b) *holds (see Figure 1* (a)*).*
- (ii) *T° is connected if and only if $|\rho_1(i)| < 1$ and $|\rho_2(j)| < 1$ for all i, j (see Figure 1* (b)*).*
- (iii) *If T is homeomorphic to a ball, then $|\rho_1(i)| + |\rho_2(j)| < 1$ for all i, j. The converse holds if all a_i, b_j are zero and $st \geq 0$ (see Figure 1* (c)*).*

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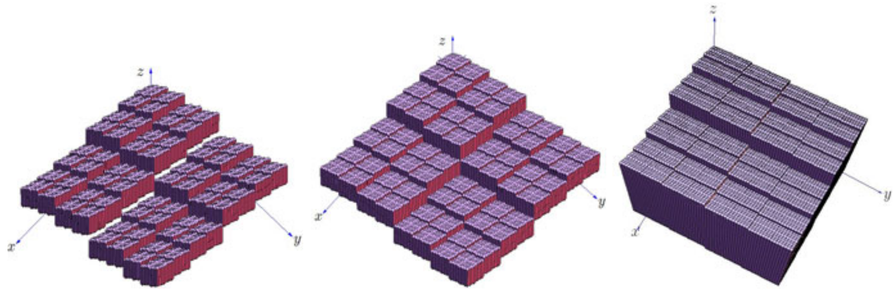

(a) (b) (c)

Figure 1: The sixth iterations of three kinds of self-affine tiles T. The figures are drawn with $(A, \mathcal{D})$ as in (1.7) and t, s as follows: $t = 2, s = 3$ for (a), $t = s = 1.95$ for (b), and $t = 0.5, s = 0.1$ for (c). The tile T in (a) is connected, but T° is not; for the one in (b), T° is connected, but T is not homeomorphic to a ball; the tile T in (c) is homeomorphic to a ball.

Figure 1 illustrates Theorem 1.2 with

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -t & -s & 2 \end{pmatrix}, \quad \mathcal{D} = \{(i, j, k) : 0 \le i, j, k \le 1\}. \tag{1.7}$$

In the special case $A = 3I$, where I is the identity matrix, Kamae, Luo, and Tan [9] proved the interesting result that a class of such tiles T are homeomorphic to the cube $[0,1]^d$, $d \ge 3$.

The main purpose of this paper is to further extend some of the results in [4] to $\mathbb{R}^d$, $d \ge 4$. We obtain the following main result on the connectedness of T and its interior.

Theorem 1.3 *For $d \ge 2$, let $p_1, \dots, p_d$ be integers with $|p_i| \ge 2$ for $1 \le i \le d$. Let A_1 be the diagonal matrix* $\operatorname{diag}(p_1, \dots, p_{d-1})$ *and $\mathbf{s}_1 = (s_1, \dots, s_{d-1}) \in \mathbb{R}^{d-1}$. Suppose the self-affine pair $(A, \mathcal{D})$ satisfies*

$$A = \begin{pmatrix} A_1 & -\mathbf{s}_1 \\ 0 & p_d \end{pmatrix}, \quad \mathcal{D} = \left\{(i_1, \dots, i_d) : 0 \le i_j < |p_j|, 1 \le j \le d\right\}. \tag{1.8}$$

Then

(i) *T is a self-affine tile.*
(ii) *T is connected if and only if*

$$\max_{1 \le j < d} \left\{ \left| \frac{s_j}{p_j(p_j - \operatorname{sgn}(p_d))} \right| \right\} \le 1. \tag{1.9}$$

(iii) *T° is connected if and only if the inequality in (1.9) is strict. Moreover, if T° is connected, then T° is contractible.*

The rest of this paper is organized as follows. In Section 2, we establish some preliminary results needed in the proof of Theorem 1.3 and also prove a connectedness theorem. Theorem 1.3 is proved in Section 3. Finally we state a conjecture in Section 4.

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2 Preliminaries and a Connectedness Theorem

In this section, we establish some results that will be used in the proof of Theorem 1.3. For an integer $m \geq 2$, denote

$$\Sigma_m^k := \{0,1,\ldots,m-1\}^k, \quad \Sigma_m^* := \bigcup_{k\geq 0} \Sigma_m^k, \quad \Sigma_m^\infty := \{0,1,\ldots,m-1\}^\infty,$$

where $\Sigma_m^0 := \{\varnothing\}$ and $\varnothing$ is the *empty word.*

We call $\mathbf{i} \in \Sigma_m^k$ an *m-adic word* of length k, and denote its length by $|\mathbf{i}|$. For $\mathbf{i} = i_1\cdots i_k \in \Sigma_m^k$ and $\mathbf{j} = j_1 j_2\cdots \in \Sigma_m^* \cup \Sigma_m^\infty$, let $\mathbf{ij} = i_1\cdots i_k j_1 j_2\cdots$, $\mathbf{j} \mid n = j_1\cdots j_n$. For $\mathbf{i} \in \Sigma_m^*$, we denote the infinite word $\mathbf{ii}\cdots$ by $\bar{\mathbf{i}}$.

It is well known that the map $\varphi_m : \Sigma_m^\infty \to [0,1]$ defined as

$$\varphi_m(\mathbf{i}) := \sum_{n\geq 1} \frac{i_n}{m^n}, \quad \mathbf{i} = i_1 i_2\cdots,$$

is surjective. We call $\mathbf{i} \in \Sigma_m^\infty$ an *m-adic expansion* (or simple *expansion*) of $x \in [0,1]$ if $x = \varphi_m(\mathbf{i})$. A real number x can have one or two m-adic expansions. It has two expansions if and only if $x = \sum_{n=1}^{|\mathbf{i}|} i_n m^{-n}$ for some $\mathbf{i} \in \Sigma_m^*$ with $i_{|\mathbf{i}|} \neq 0$; in this case the two expansions are

$$i_1\cdots(i_{|\mathbf{i}|}-1)\overline{(m-1)} \quad \text{and} \quad i_1\cdots i_{|\mathbf{i}|}\bar{0},$$

where $0 < i_{|\mathbf{i}|} < m$. We also define $\varphi_m(\mathbf{i}) := \sum_{n=1}^{|\mathbf{i}|} m^{-n} i_n$ for $\mathbf{i} \in \Sigma_m^*$.

The following theorem will be used in establishing connectedness.

Theorem 2.1 (Hata [6]) *Let $\{\psi_j\}_{j=1}^N$ be a family of contractions on $\mathbb{R}^d$ and let K be its attractor. Then K is connected if and only if for any $i \neq j \in \{1,\ldots,N\}$, there exists a finite sequence of indices $j_1,\ldots,j_n$ in $\{1,\ldots,N\}$, with $j_1 = i$ and $j_n = j$, such that $\psi_{j_k}(K) \cap \psi_{j_{k+1}}(K) \neq \varnothing$ for all $1 \leq k < n$.*

We first introduce some notation. For $\alpha \in \mathbb{R}$, let

$$\pi_\alpha := \{(x_1,\ldots,x_{d-1},\alpha) : x_i \in \mathbb{R} \text{ for } 1 \leq i < d\}$$

denote the hyperplane in $\mathbb{R}^d$ whose last coordinate is α. We assume that all edges of any hypercube are parallel to the corresponding coordinate axes.

For $1 \leq n \leq d$, let $\mathcal{L}_n$ be the restriction of the integer lattice $\mathbb{Z}^n$ to $\prod_{j=1}^n [0,|p_j|-1]$ and say $\boldsymbol{\iota} = (\iota_1,\ldots,\iota_n)$, $\boldsymbol{\iota}' = (\iota_1',\ldots,\iota_n') \in \mathbb{Z}^n$ are *neighbors* if $\sum_{k=1}^n |\iota_k - \iota_k'| = 1$. Let $\Sigma_m^n, \Sigma_m^*, \Sigma_m^\infty$ be defined as in Section 2. An element in $\Sigma_{|p_j|}^* \cup \Sigma_{|p_j|}^\infty$ will be denoted by $\mathbf{i}_j = i_1^{(j)}\cdots i_{|\mathbf{i}_j|}^{(j)}$ or $\mathbf{i}_j = i_1^{(j)} i_2^{(j)}\cdots$. For convenience, we use r to denote the integer p_d. For each integer $k \in [0,|r|-1]$, we add a perturbation $a_k^{(j)}$ to the j-th coordinate of each digit in $\mathcal{D}$ in Theorem 1.3. Then the digit set can be put in a more general form:

$$\mathcal{D} = \{(\boldsymbol{\iota} + (a_k^{(1)},\ldots,a_k^{(d-1)}), k) : 0 \leq k < |r|,\ \boldsymbol{\iota} \in \mathcal{L}_{d-1}\}. \tag{2.1}$$

The necessary and sufficient condition for connectedness in Theorem 1.3 can be adjusted as follows.

Theorem 2.2 *Let A and $\mathcal{D}$ be as in (1.8) and (2.1), respectively. Then $T = T(A, \mathcal{D})$ is connected if and only if*

$$\max_{\substack{0\le k<|r|-1\\ 1\le j<d}} \left\{ \left| \frac{a_{k+1}^{(j)} - a_k^{(j)}}{p_j} + \frac{\operatorname{sgn}(r)(a_0^{(j)} - a_{|r|-1}^{(j)}) - s_j}{p_j(p_j - \operatorname{sgn}(r))} \right| \right\} \le 1. \tag{2.2}$$

The proof of this theorem will be given near the end of this section.

The following lemma shows that by using this perturbation strategy, we can assume that the diagonal entries of A are positive. Let

$$\widetilde{\mathcal{L}}_d = \mathbb{Z}^d \cap \prod_{j=1}^{d} [0, p_j^2 - 1].$$

Lemma 2.3 *Assume the hypotheses of Theorem 2.2. Let $\widetilde{s}_j = (p_j + r)s_j$, $1 \le j < d$. Then there exists another digit set*

$$\widetilde{\mathcal{D}} = \left\{ (\boldsymbol{\iota} + (\widetilde{a}_k^{(1)}, \dots, \widetilde{a}_k^{(d-1)}), k) : 0 \le k < r^2, \boldsymbol{\iota} \in \widetilde{\mathcal{L}}_{d-1} \right\}$$

such that $(A, \mathcal{D})$ satisfies inequality (2.2) if and only if $(A^2, \widetilde{\mathcal{D}})$ satisfies the following inequality:

$$\max_{\substack{0\le k<r^2-1\\ 1\le j<d}} \left\{ \left| \frac{\widetilde{a}_{k+1}^{(j)} - \widetilde{a}_k^{(j)}}{p_j^2} + \frac{\operatorname{sgn}(r^2)(\widetilde{a}_0^{(j)} - \widetilde{a}_{r^2-1}^{(j)}) - \widetilde{s}_j}{p_j^2(p_j^2 - \operatorname{sgn}(r^2))} \right| \right\} \le 1,$$

where, of course, $\operatorname{sgn}(r^2) = 1$.

Proof We notice that for any $\mathbf{x} \in \mathbb{R}^d$, $T(A^2, A\mathcal{D} + \mathcal{D} + \mathbf{x})$ is just a translation of T. Now we will find a suitable $\mathbf{x} \in \mathbb{R}^d$ so that $\widetilde{D} = A\mathcal{D} + \mathcal{D} + \mathbf{x}$ fulfills the requirements. Recall that $r = p_d$. By letting $\mathbf{s}_2 = (\widetilde{s}_1, \dots, \widetilde{s}_{d-1})^t$, we get

$$A^2 = \begin{pmatrix} A_1^2 & -\mathbf{s}_2 \\ 0 & r^2 \end{pmatrix}.$$

Notice that

$$\begin{aligned} A\mathcal{D} + \mathcal{D} = \Big\{ \big(& a_k^{(1)} + a_{k'}^{(1)} p_1 - k' s_1 + p_1 i_1' + i_1, \dots, \\ & a_k^{(d-1)} + a_{k'}^{(d-1)} p_{d-1} - k' s_{d-1} + p_{d-1} i_{d-1}' + i_{d-1}, k + k' r \big) : \\ & \qquad 0 \le i_j, i_j' < |p_j|,\ 1 \le j < d,\ 0 \le k, k' < |r| \Big\}. \end{aligned}$$

Let $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, where $x_j = 2^{-1}(p_j^2 + p_j)(1 - \operatorname{sgn}(p_j))$. It follows that $x_j + p_j i_j' + i_j$, $1 \le j < d$, ranges through $0, \dots, p_j^2 - 1$ when i_j, i_j' increase from 0 to $|p_j| - 1$. Also, we have $\{x_d + k + k' r\}_{k,k'=0}^{|r|-1} = \{0, 1, \dots, r^2 - 1\}$. It follows that the mapping

$$r'' = r''(k, k') = x_d + k + k' r \tag{2.3}$$

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is a bijection from $\{0,1,\ldots,|r|-1\}^2$ onto $\{0,1,\ldots,r^2-1\}$. When $k=|r|-1$, (2.3) implies that

$$\begin{cases} k' < r-1 & \operatorname{sgn}(r) > 0, \\ k' > 0 & \operatorname{sgn}(r) < 0. \end{cases} \tag{2.4}$$

For $0 \le r'' < r^2$, let $k, k' \in \{0,\ldots,|r|-1\}$ such that $r'' = r''(k,k')$, and let

$$\widetilde{a}_{r''}^{(j)} = a_k^{(j)} + a_{k'}^{(j)} p_j - k' s_j.$$

We claim that the digit set

$$\widetilde{\mathcal{D}} = \left\{ (\boldsymbol{\iota} + (\widetilde{a}_k^{(1)},\ldots,\widetilde{a}_k^{(d-1)}), k) : 0 \le k < r^2, \boldsymbol{\iota} \in \widetilde{\mathcal{L}}_{d-1} \right\}$$

is a desired one. In fact, for each $1 \le j < d$, each $0 \le r'' < r^2-1$, and $0 \le k, k' < |r|$ satisfying $r'' = r''(k,k')$, we have

$$\begin{aligned} &\left| \frac{\widetilde{a}_{r''+1}^{(j)} - \widetilde{a}_{r''}^{(j)}}{p_j^2} + \frac{\operatorname{sgn}(r^2)(\widetilde{a}_0^{(j)} - \widetilde{a}_{r^2-1}^{(j)}) - \widetilde{s}_j}{p_j^2(p_j^2 - \operatorname{sgn}(r^2))} \right| \\ &= \begin{cases} \left| \dfrac{1}{p_j} \left(\dfrac{a_{k+1}^{(j)} - a_k^{(j)}}{p_j} + \dfrac{\operatorname{sgn}(r)(a_0^{(j)} - a_{|r|-1}^{(j)}) - s_j}{p_j(p_j - \operatorname{sgn}(r))} \right) \right| & k < |r|-1, \\ \left| \dfrac{\operatorname{sgn}(r)(a_{k'+\operatorname{sgn}(r)}^{(j)} - a_{k'}^{(j)})}{p_j} + \dfrac{\operatorname{sgn}(r)(a_0^{(j)} - a_{|r|-1}^{(j)}) - s_j}{p_j(p_j - \operatorname{sgn}(r))} \right| & k = |r|-1, \end{cases} \end{aligned}$$

with the maximum being attained in the case $k = |r|-1$. Now the claim follows from (2.4) and the proof is complete. ∎

Remark 2.4 In view of Lemma 2.3, we assume throughout the rest of this section that the matrix A in a self-affine pair $(A, \mathcal{D})$ satisfies (1.8) with positive eigenvalues, and the digit set $\mathcal{D}$ has the form (2.1).

It follows from (1.8) that

$$A^{-n} = \begin{pmatrix} A_1^{-n} & \mathbf{s}'_n \\ 0 & r^{-n} \end{pmatrix}, \quad n \ge 1,$$

where $\mathbf{s}'_n = (s_1 s_n^{(1)}, \ldots, s_{d-1} s_n^{(d-1)})^t$ and

$$s_n^{(j)} := \begin{cases} \dfrac{p_j^{-n} - r^{-n}}{r - p_j}, & p_j \ne r \\ \dfrac{n}{r^{n+1}} & p_j = r. \end{cases} \tag{2.5}$$

For $\mathbf{k} = (k_1, \ldots, k_{|\mathbf{k}|}) \in \Sigma_r^* \cup \Sigma_r^\infty$, let

$$a_j(\mathbf{k}) := \sum_n \frac{a_{k_n}^{(j)}}{p_j^n}, \qquad s_j(\mathbf{k}) := \sum_n s_n^{(j)} k_n, \quad 1 \le j < d, \tag{2.6}$$

and

$$\mathbf{b}(\mathbf{k}) := \left(a_1(\mathbf{k}) + s_1 \cdot s_1(\mathbf{k}), \ldots, a_{d-1}(\mathbf{k}) + s_{d-1} \cdot s_{d-1}(\mathbf{k}) \right).$$

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The numbers $a_j(\mathbf{k})$ and $s_j(\mathbf{k})$ are used to define the vectors $\mathbf{b}(\mathbf{k})$, which are in turn used to describe the iterated function system (IFS) generating T, or to describe the tile T itself (see the first two lines in (2.7)). It follows from definitions that T is the attractor of the IFS

$$S_{\iota,k}(\mathbf{x}) = A^{-1}\Big(\mathbf{x} + (\boldsymbol{\iota} + (a_k^{(1)}, \dots, a_k^{(d-1)}), k)\Big), \quad \boldsymbol{\iota} \in \mathcal{L}_{d-1}, 0 \le k < r.$$

Let $G_k := \bigcup_{\iota\in\mathcal{L}_{d-1}} S_{\iota,k}(T)$, $0 \le k < r$, and hence we can view T as a union of the r parallel pieces $G_0, \dots, G_{r-1}$ along the $x^{(r)}$-axis (see Figure 3). From equations (1.1), (2.1), and the above formulas, we see that for $\boldsymbol{\iota} = (\iota_1, \dots, \iota_{d-1})$ and $k \in \Sigma_r^1$,

(2.7)

$$\begin{aligned} T &= \Big\{\Big(\mathbf{b}(\mathbf{k}) + (\varphi_{p_1}(\mathbf{i}_1), \dots, \varphi_{p_{d-1}}(\mathbf{i}_{d-1})), \varphi_r(\mathbf{k})\Big) : \mathbf{i}_j \in \Sigma_{p_j}^\infty, \mathbf{k} \in \Sigma_r^\infty\Big\}, \\ S_{\iota,k}(T) &= \Big\{\Big(\mathbf{b}(k\mathbf{k}) + (\varphi_{p_1}(\iota_1\mathbf{i}_1), \dots, \varphi_{p_{d-1}}(\iota_{d-1}\mathbf{i}_{d-1})), \varphi_r(k\mathbf{k})\Big) : \mathbf{i}_j \in \Sigma_{p_j}^\infty, \mathbf{k} \in \Sigma_r^\infty\Big\}, \\ G_k = G_k(T) &= \Big\{\Big(\mathbf{b}(k\mathbf{k}) + (\varphi_{p_1}(\mathbf{i}_1), \dots, \varphi_{p_{d-1}}(\mathbf{i}_{d-1})), \varphi_r(k\mathbf{k})\Big) : \mathbf{i}_j \in \Sigma_{p_j}^\infty, \mathbf{k} \in \Sigma_r^\infty\Big\}. \end{aligned}$$

The following lemma is an analogue of [5, Proposition 2.2]. We omit its proof.

Lemma 2.5 *Let T be the attractor of the self-affine pair $(A, \mathcal{D})$.*

(i) *The set T is a tile with Lebesgue measure 1. Moreover, for any sequence of $(d-1)$-dimensional vectors $\{\boldsymbol{\beta}_n\}_{n\in\mathbb{Z}}$, the set*

$$\mathcal{L} = \big\{(\boldsymbol{\iota} + \boldsymbol{\beta}_n, n) : \boldsymbol{\iota} \in \mathbb{Z}^{d-1}, n \in \mathbb{Z}\big\}$$

is a tiling set for T in $\mathbb{R}^d$.

(ii) *For any two neighboring integer vectors $\boldsymbol{\iota}, \boldsymbol{\iota}' \in \mathcal{L}_{d-1}$, $S_{\iota,k}(T) \cap S_{\iota',k}(T) \neq \varnothing$.*

For $0 \le y \le 1$ and $1 \le j < d$, define

$$\begin{aligned} d_{\min}^{(j)}(y) &:= \min\big\{ a_j(\mathbf{k}) + s_j \cdot s_j(\mathbf{k}) : \mathbf{k} \in \Sigma_r^\infty,\ \varphi_r(\mathbf{k}) = y\big\}, \\ d_{\max}^{(j)}(y) &:= \max\big\{ a_j(\mathbf{k}) + s_j \cdot s_j(\mathbf{k}) : \mathbf{k} \in \Sigma_r^\infty,\ \varphi_r(\mathbf{k}) = y\big\}. \end{aligned}$$

To visualize $d_{\min}^{(j)}(y)$ and $d_{\max}^{(j)}(y)$ geometrically, we project the restriction of T on the plane π_y to the $x^{(j)}$-axis and denote the resulting set by ℓ, which is a line segment or a union of two line segments. Then $d_{\min}^{(j)}(y)$ is the left endpoint of ℓ and $d_{\max}^{(j)}(y)$ is the right endpoint of ℓ minus one (see an illustration in Figure 2). The following equality can be derived by a simple calculation and will be used later: for $y = \varphi_r(\mathbf{k}(k+1))$ with $\mathbf{k} \in \Sigma_r^{n-1}$ and $0 \le k < r-1$,

$$d_{\max}^{(j)}(y) - d_{\min}^{(j)}(y) = \left| \frac{a_{k+1}^{(j)} - a_k^{(j)}}{p_j^n} + \frac{a_0^{(j)} - a_{r-1}^{(j)} - s_j}{p_j^n(p_j - 1)} \right|. \tag{2.8}$$

Moreover, the following holds.

Lemma 2.6 *Let $P = (x_1, \dots, x_{d-1}, y) \in T$ with $0 < y < 1$. Then $P \in T^\circ$ if and only if*

$$d_{\max}^{(j)}(y) < x_j < d_{\min}^{(j)}(y) + 1 \textit{ for each } j \in \{1, \dots, d-1\}. \tag{2.9}$$

Proof By Lemma 2.5, we let $\mathcal{L} = \mathbb{Z}^d$ be a tiling set for T. Assume (2.9) holds. If $P \in T + \mathbf{t}$ for some $\mathbf{t} = (t_1, \dots, t_d) \in \mathcal{L}$, then $t_d = 0$ since y and $y - t_d$ both belong

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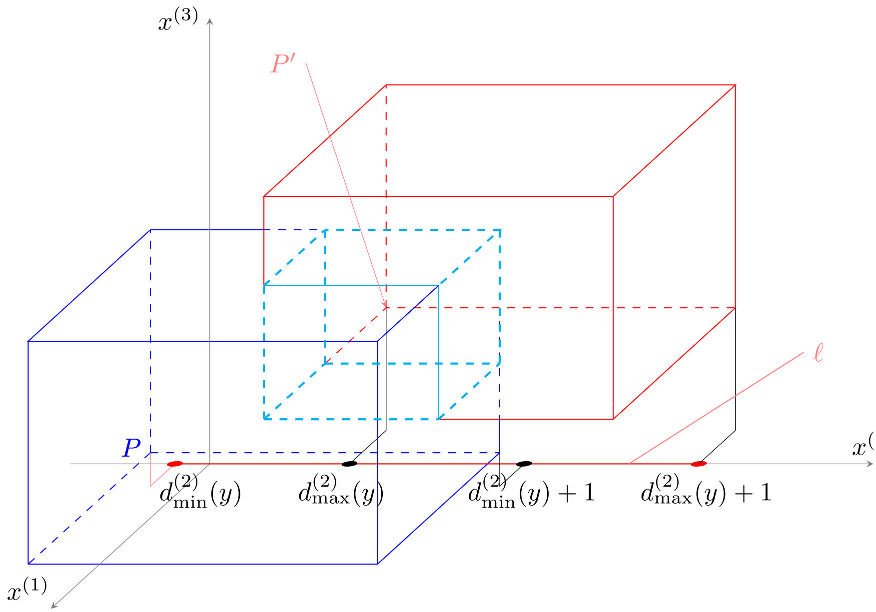

Figure 2: An illustration of the cross-section of T on the hyperplane π_y and the definitions of $d^{(j)}_{\min}$ and $d^{(j)}_{\max}$. Each big hypercube has side length 1 and

$$P' = P + (d^{(1)}_{\max}(y) - d^{(1)}_{\min}(y), \ldots, d^{(d-1)}_{\max}(y) - d^{(d-1)}_{\min}(y), 0).$$

This figure is drawn with $d = 4$ and $j = 2$.

to $(0,1)$. In this case, for each $j \in \{1,\ldots,d-1\}$, we have $x_j - t_j \in [d^{(j)}_{\min}(y), d^{(j)}_{\max}(y)+1]$, namely,

$$d^{(j)}_{\min}(y) + t_j \le x_j \le d^{(j)}_{\max}(y) + 1 + t_j.$$

Comparing this with (2.9), we see that $t_j = 0$. So P is an interior point of T.

If (2.9) fails, then there exist $j \in \{1,\ldots,d-1\}$ and $k \neq 0$ such that

$$x_j \in [0,1] + d^{(j)}_{\max}(y) + k \quad \text{or} \quad x_j \in [0,1] + d^{(j)}_{\min}(y) + k.$$

So, if we take $\mathbf{t}_0 = (t_1,\ldots,t_d)$ with $t_i = 0$ if $i \neq j$ and $t_j = k$, then $P \in T \cap (T + \mathbf{t}_0)$. This implies $P \notin T^\circ$ and completes the proof. ■

We will use the following fact to prove Lemma 2.8. We point out here that all hypercubes in question are closed.

Lemma 2.7 *Let H_1 and H_2 be m-dimensional unit hypercubes in $\mathbb{R}^m$ with centers $(x_1,\ldots,x_m)$ and $(y_1,\ldots,y_m)$, respectively. Then $H_1 \cap H_2 \neq \varnothing$ if and only if*

$$|x_i - y_i| \le 1, \quad 1 \le i \le m. \tag{2.10}$$

Moreover, $H_1 \cap H_2$ is an $(m-n)$-dimensional hypercube if and only if there are n equalities in (2.10).

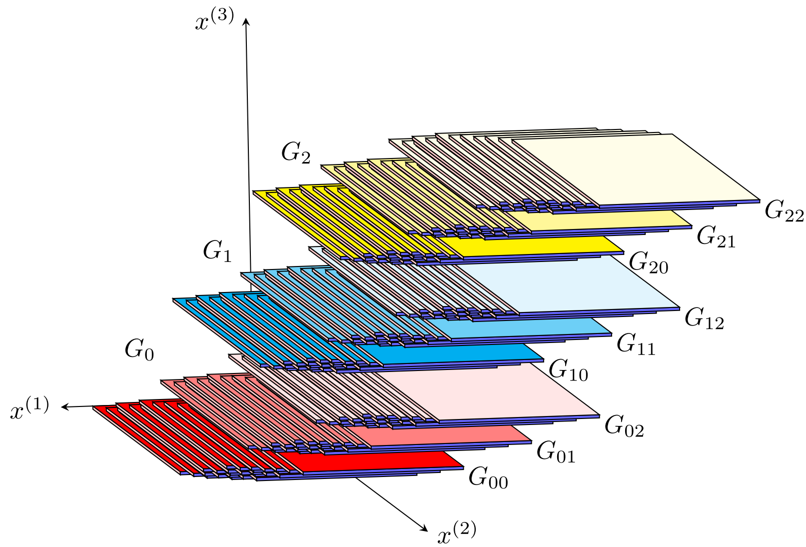

Figure 3: (Colour online) A figure illustrating G_k and $G_{\mathbf{k}}$ for $\mathbf{k} \in \Sigma_r^2$, using the fifth iteration of $T \subset \mathbb{R}^3$. The figure is drawn with $d = 3$, and $(A, \mathcal{D})$ as in (3.1). G_k's and $G_{\mathbf{k}}$'s are drawn in different colors, each being parallel to the $x^{(1)}x^{(2)}$-plane.

Lemma 2.8 *Let $(A, \mathcal{D})$ be the self-affine pair described as in Remark 2.4.*

(i) *If $G_k(T) \cap G_{k'}(T) \neq \varnothing$, then $|k - k'| \leq 1$.*

(ii) *The intersection $G_k(T) \cap G_{k+1}(T)$ is a $(d-1)$-dimensional hypercube if and only if for $1 \leq j < d$,*

$$d_{\max}^{(j)}\Big(\frac{k+1}{r}\Big) - d_{\min}^{(j)}\Big(\frac{k+1}{r}\Big) < 1, \tag{2.11}$$

and is a $(d-1-n)$-dimensional hypercube if exactly n of the inequalities in (2.11) are replaced by equalities.

Proof (i) By (2.7), G_k is the part of T between the $(d-1)$-dimensional hyperplanes $\pi_{k/r}$ and $\pi_{(k+1)/r}$. Hence (i) is true.

(ii) By (2.7) again, the restriction of G_k to $\pi_{k/r}$ or to $\pi_{(k+1)/r}$ is a $(d-1)$-dimensional unit hypercube since each $\varphi_{p_j}(\mathbf{i}_j)$ forms a unit interval $[0,1]$ and $\mathbf{k} = \overline{r-1}$ or $\mathbf{k} = \overline{0}$. Hence, we take $G_k \cap G_{k+1}$, which lies in $\pi_{(k+1)/r}$ if it is not empty, as the intersection of two $(d-1)$-dimensional hypercubes H_1, H_2, where $H_1 = G_k \cap \pi_{(k+1)/r}$ and $H_2 = G_{k+1} \cap \pi_{(k+1)/r}$. The j-th coordinates of all elements in H_1 form a unit interval $[0,1] + a_j(k\overline{r-1}) + s_j s_j(k\overline{r-1})$. The center of such an interval is

$$\alpha_j = \frac{a_{r-1}^{(j)} + s_j}{p_j(p_j-1)} + \frac{(k+1)s_j}{p_j r} + \frac{a_k^{(j)}}{p_j} + \frac{1}{2},$$

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by the formulas of $a_j(\mathbf{k})$, $s_j(\mathbf{k})$, and the fact (see (2.5))

$$\sum_{k=2}^{\infty} s_k^{(j)} = \frac{p_j + r - 1}{p_j r(p_j - 1)(r-1)}.$$

Similarly, all the j-th coordinates of elements in H_2 form a unit interval with center

$$\alpha'_j = \frac{a_0^{(j)}}{p_j(p_j-1)} + \frac{(k+1)s_j}{p_j r} + \frac{a_{k+1}^{(j)}}{p_j} + \frac{1}{2}.$$

Note that (2.8) yields $|\alpha_j - \alpha'_j| = d_{\max}^{(j)}(r^{-1}(k+1)) - d_{\min}^{(j)}(r^{-1}(k+1))$. Thus the results follow from Lemma 2.7. ∎

The following lemma can be obtained easily; we omit the proof.

Lemma 2.9 *Let $\boldsymbol{\iota}$, $\boldsymbol{\iota}'$ be two elements in $\mathcal{L}_{d-1}$. Then there exists a finite ordered sequence $(\boldsymbol{\iota}_1, \dots, \boldsymbol{\iota}_N)$ in $\mathcal{L}_{d-1}$, with $\boldsymbol{\iota}_1 = \boldsymbol{\iota}$ and $\boldsymbol{\iota}_N = \boldsymbol{\iota}'$, such that all vectors in $\mathcal{L}_{d-1}$ appear in the sequence and any two adjacent vectors in the sequence are neighbors.*

We now prove Theorem 2.2.

Proof of Theorem 2.2 Assume that T is connected. Notice that $T = \bigcup_k G_k$ and each G_k is nonempty and compact. Thus for any k, G_k must intersect some $G_{k'}$, $k' \neq k$. The necessity follows from Lemma 2.8 and (2.8). The sufficiency follows immediately from Theorem 2.1, and Lemmas 2.8, 2.9, and 2.5(ii). ∎

If T is disconnected, we have a precise count of the connected components. We omit the proof, since it is similar to that of [5, Theorem 1.2].

Theorem 2.10 *Let T be a tile as in Theorem 1.3. Suppose that for each $j \in \{1, \dots, d-1\}$, there exists some positive integer m_j such that $|p_j|^{m_j - 1} < |s_j| \, |p_j(p_j - \operatorname{sgn}(r))|^{-1} \le |p_j|^{m_j}$. Then T is disjoint and has $\prod_{j=1}^{d-1} |p_j|^{m_j}$ connected components.*

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We consider the connectedness of T and T°. For $\mathbf{k} \in \Sigma_r^n$, let $I_{\mathbf{k}} := [0, r^{-n}] + \varphi_r(\mathbf{k})$ and

$$G_{\mathbf{k}} := \left\{ \big(\mathbf{b}(\mathbf{k}\mathbf{k}') + (\varphi_{p_1}(\mathbf{i}_1), \dots, \varphi_{p_{d-1}}(\mathbf{i}_d)), \varphi_r(\mathbf{k}\mathbf{k}') \big) : \mathbf{i}_j \in \Sigma_{p_j}^\infty, \mathbf{k}' \in \Sigma_r^\infty \right\}.$$

Recall that T is viewed as a union of the r parallel pieces $G_0, \dots, G_{r-1}$ (defined by the third line in (2.7)) along the $x^{(r)}$-axis. We iterate $T = \bigcup_{\iota,k} S_{\iota,k}(T)$ n times; then view T as a union of the r^n parallel pieces of $G_{\mathbf{k}}$, $\mathbf{k} \in \Sigma_r^n$ along the $x^{(r)}$-axis.

We illustrate G_k and $G_{\mathbf{k}}$ in Figure 3 by using

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \left\{ (i,j,k) : 0 \le i, j \le 1, 0 \le k \le 2 \right\}. \tag{3.1}$$

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Order the elements of Σ_r^n as $\mathbf{k}_1, \ldots, \mathbf{k}_{r^n}$, using the lexicographic order. For each $1 \le m \le r^n$, let $P_{\mathbf{k}_m}$ be the center of $G_{\mathbf{k}_m}$ and let $P'_{\mathbf{k}_m}$ be the center of $G_{\mathbf{k}_m} \cap G_{\mathbf{k}_{m+1}}$ if $m < r^n$. More precisely,

$$P_{\mathbf{k}_m} = (x_1(y_0), \ldots, x_{d-1}(y_0), y_0), \quad 1 \le m \le r^n,$$
$$P'_{\mathbf{k}_m} = (x_1(y_1), \ldots, x_{d-1}(y_1), y_1), \quad 1 \le m < r^n,$$

where $y_0 = y_0(\mathbf{k}_m) (= 2^{-1} r^{-n} + \varphi_r(\mathbf{k}_m))$ is the center of the interval $I_{\mathbf{k}_m}$, $y_1 = y_1(\mathbf{k}_m) = \varphi_r(\mathbf{k}_m) + r^{-n}$, and $x_j(y) = 2^{-1}(d_{\min}^{(j)}(y) + d_{\max}^{(j)}(y) + 1)$. We also denote by $P'_{\mathbf{k}_0}$ and $P'_{\mathbf{k}_{r^n}}$ the centers of the restrictions of T to π_0 and π_1, respectively, *i.e.*,

$$P'_{\mathbf{k}_0} = (x_1(0), \ldots, x_{d-1}(0), 0), \qquad P'_{\mathbf{k}_{r^n}} = (x_1(1), \ldots, x_{d-1}(1), 1).$$

Let ℓ be the piecewise linear curve defined as

$$\ell = P'_{\mathbf{k}_0} P_{\mathbf{k}_1} P'_{\mathbf{k}_1} P_{\mathbf{k}_2} P'_{\mathbf{k}_2} \cdots P_{\mathbf{k}_{r^n}} P'_{\mathbf{k}_{r^n}}. \tag{3.2}$$

Clearly, for $y \in [0,1]$, $P(y) = (x_1(y), \ldots, x_{d-1}(y), y)$ is the unique point lying in ℓ.

Lemma 3.1 *Suppose* (2.11) *holds for all* $0 \le k \le r-1$ *and* $1 \le j < d$. *Then there exists* n *such that, except for the end-points, the piecewise linear curve* ℓ *in* (3.2) *is contained in* T°.

Proof Denote

$$\lambda = \frac{1}{8} \min\Big\{ \Big(d_{\min}^{(j)}\Big(\frac{k+1}{r}\Big) + 1 - d_{\max}^{(j)}\Big(\frac{k+1}{r}\Big)\Big) : 0 \le k < r, 0 \le j < d \Big\}.$$

Then $0 < \lambda < 1/8$, where the first inequality follows from (2.11). Choose n large enough such that, for $\mathbf{k} \in \Sigma_r^n$, $y, y' \in I_{\mathbf{k}}^\circ$, and $1 \le j < d$, we have

$$\big| d_{\max}^{(j)}(y) - d_{\min}^{(j)}(y') \big| \le \lambda.$$

Such n does exist by the fact that the first n terms of any expansions of y and y' are the same and the series $a_j(\cdot)$ and $s_j(\cdot)$ in (2.6) converge absolutely. We show the stronger conclusion that each point $P = (x_1(y), \ldots, x_{d-1}(y), y) \in \ell$ satisfies

$$d_{\max}^{(j)}(y) + \lambda < x_j(y) < d_{\min}^{(j)}(y) + 1 - \lambda, \quad 0 \le j < d.$$

The conclusion is true for all $P = P'_{\mathbf{k}_m}$, $0 \le m \le r^n$. Now suppose $P = (x_1(y), \ldots, x_{d-1}(y), y) \in G_{\mathbf{k}}^\circ$ for some $\mathbf{k}$. For symmetry, we suppose $y \ge y_0 = y_0(\mathbf{k})$. If $x_j(y) \ge x_j(y_0)$, then

$$\begin{aligned} x_j(y) - d_{\max}^{(j)}(y) &\ge x_j(y_0) - d_{\max}^{(j)}(y_0) - |d_{\max}^{(j)}(y_0) - d_{\max}^{(j)}(y)| \\ &\ge \frac{1}{2}(d_{\min}^{(j)}(y_0) - d_{\max}^{(j)}(y_0) + 1) - \lambda \\ &\ge \frac{1}{2} - |d_{\max}^{(j)}(y_0) - d_{\min}^{(j)}(y_0)| - \lambda \ge 2\lambda, \end{aligned}$$

where we have used the inequality $4\lambda < 1/2$. If $x_j(y) < x_j(y_0)$, Lemma 2.6 implies that the point $Q' = P'_{\mathbf{k}} - 3(\lambda, \ldots, \lambda, 0) := (x'_1, \ldots, x'_{d-1}, y_1)$ belongs to T°, where $y_1 = \varphi_r(\mathbf{k}) + r^{-n}$. Therefore, there exists a point $Q'' = (x''_1, \ldots, x''_{d-1}, y') \in G_{\mathbf{k}}^\circ$ with $\|Q' - Q''\| < \lambda$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. In other words, $|x'_j - x''_j| < \lambda$

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for $1 \le j < d$. Since P lies in the line segment $P_k P_k'$, we have $x_j(y_1) \le x_j(y) < x_j(y_0)$. If $x_j'' \ge d_{\max}^{(j)}(y)$, then

$$x_j(y) - d_{\max}^{(j)}(y) \ge x_j(y_1) - x_j'' \ge x_j(y_1) - x_j' - |x_j' - x_j''| \ge 3\lambda - \lambda > \lambda.$$

If $x_j'' < d_{\max}^{(j)}(y)$, then $d_{\min}^{(j)}(y') \le x_j'' < d_{\max}^{(j)}(y)$, and hence

$$\begin{aligned} x_j(y) - d_{\max}^{(j)}(y) &\ge x_j(y_1) - x_j' + (x_j' - x_j'') + (x_j'' - d_{\max}^{(j)}(y)) \\ &\ge x_j(y_1) - x_j' - |x_j' - x_j''| - (d_{\max}^{(j)}(y) - d_{\min}^{(j)}(y')) \\ &> 3\lambda - \lambda - \lambda = \lambda. \end{aligned}$$

Similarly, we get $d_{\min}^{(j)} + 1 - x_j(y) > \lambda$, and the proof is complete. ∎

Lemma 3.2 *Suppose* (2.11) *holds for all* $0 \le k \le r-1$ *and* $1 \le j < d$. *Then both* T *and* T° *are contractible. In particular,* T^o *and* T *are simply connected.*

Proof To show the simple connectedness of T, we will prove that T is contractible (to a single point). Let ℓ be defined as in (3.2) and satisfy Lemma 3.1. For $t \in [0,1]$, define a continuous function $h_t : \mathbb{R}^{d-1} \times [0,1] \to \mathbb{R}^{d-1} \times [0,1]$ as follows:

$$h_t(P) = (1-t)P + tP(y), \quad P = (x_1, \dots, x_{d-1}, y).$$

For each $P \in T$ with the last coordinate being y, there exists $\mathbf{k} \in \Sigma_r^\infty$ with $\varphi_r(\mathbf{k}) = y$ such that $P - (\mathbf{b}(\mathbf{k}), y) \in [0,1]^{d-1} \times \{0\}$. From the proof of Lemma 3.1, $d_{\max}^{(j)}(y) < x_j(y) < d_{\min}^{(j)}(y) + 1$. That is, $P(y) - (\mathbf{b}(\mathbf{k}), y) \in (0,1)^{d-1} \times \{0\}$. Hence the j-th coordinate of $(1-t)P + tP(y) - (\mathbf{b}(\mathbf{k}), y)$ is in $[0,1]$ for $1 \le j < d$ and is zero if $j = d$. So, for each $t \in [0,1]$, h_t maps T into T. Clearly, h_0 is the identity map and $h_1(T)$ is the piecewise linear curve ℓ. Thus T is contractible to the piecewise linear curve and hence contractible to a single point. In particular, T is simply connected.

Notice that by Lemma 3.1 and Lemma 2.6, each function h_t, $t \in [0,1]$, also maps T° into T°. An analogous argument as above implies that T° is contractible (to a single point), and is thus simply connected. ∎

Theorem 3.3 *Let* $(A, \mathcal{D})$ *be as in Theorem* 2.2. *Then* T° *is connected if and only if* (2.11) *holds for all* $0 \le k \le r-1$ *and* $1 \le j < d$. *Consequently,* T° *is contractible if it is connected.*

Proof The sufficiency is shown in Lemma 3.2, since contractibility implies connectedness.

For the necessity suppose T° is connected. If T is disconnected, then Theorem 2.2 implies that there exist some k and j such that

$$\Big| \frac{a_{k+1}^{(j)} - a_k^{(j)}}{p_j} + \frac{a_0^{(j)} - a_{r-1}^{(j)} - s_j}{p_j(p_j - 1)} \Big| > 1.$$

Lemma 2.8 says that $G_k \cap G_{k+1} = \varnothing$. Denote $T_1 := \bigcup_{l=0}^{k} G_l$ and $T_2 := \bigcup_{l>k}^{r-1} G_l$. Since T_1 and T_2 are closed, the fact $T^\circ = T_1^\circ \cup T_2^\circ$ implies that T° is disconnected, which is impossible. Hence T is connected. In other words, (2.11) holds for all k if "<" is

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replaced by "≤". To show that the inequality is strict, we suppose on the contrary that equality holds for some j, k. Then the last conclusion of Lemma 2.8 says that $G_k \cap G_{k+1}$ contains only an n-dimensional hypercube, with $n < d - 1$. Thus Lemma 2.8 implies $\bigcup_{j=0}^{k} G_j$ and $\bigcup_{j=k+1}^{r-1} G_j$ are connected by an n-dimensional hypercube. However, $T = \bigcup_{k=0}^{r-1} G_k$, and thus T° cannot be connected, a contradiction. The necessity follows.

The last conclusion is a direct consequence of the necessity and Lemma 3.2. ■

Proof of Theorem 1.3 Part (i) is proved in Lemma 2.5(i). Part (ii) is a special case of Theorem 2.2. Finally, part (iii) is proved in Theorem 3.3. ■

4 Final comment

Theorem 1.3 gives a necessary and sufficient condition for T° to be contractible. We conjecture that T is homeomorphic to a d-ball under the same condition.

Conjecture 4.1 *Assume the same hypotheses as Theorem 1.3. Then T is homeomorphic to a d-ball if and only if T° is connected.*

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